Unidexterous versus ambidexterous gravities

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(Received 3 April 1996)

The process of soldering two unidexterous gravities of opposite chiralities is considered at a quantum level, by using the field-antifield formalism with a Pauli-Villars regularization scheme. The resulting effective theory gives rise to a diffeomorphism anomaly which is compared with the original $\mathcal{W}2$ anomalies. [S0556-2821(96)04118-5]

PACS number(s): 11.25.Hf, 04.50.+h

The research in two-dimensional quantum field theories has been one of the principal arenas for the development of physical ideas. The inescapable features of field theories in two dimensions are those related to the chiral splitting of space-time [1–5]. Interesting questions come, for example, from the study of chiral scalars in a gravitational background. In a recent work [6] it has been shown, in a classical way, that two Siegel bosons [1] or equivalently two $\mathcal{W}2$ gravities [4] of opposite chiralities can be coupled in a nontrivial way such that the resulting effective theory is essentially one of a scalar boson in a gravitational background. This implies that the effective theory so obtained must be invariant under diffeomorphisms, which cannot be the mere direct sum of two unidexterous diffeomorphisms. Actually, the fundamental quantities appearing in the original $\mathcal{W}$ gravities become combined in a nontrivial way in the effective theory. This has been considered by Wotzasek and the authors at a classical level [6]. Here we extend our previous results to the quantum level, by using the field-antifield formalism [7,8]. This procedure is chosen since it can be applied in a natural way to theories with open gauge algebras, which is just the case we are considering. As the action studied here is invariant under diffeomorphism, there arises naturally the question of quantum anomalies. This implies that the Becchi-Roult-Stora-Tyutin (BRST) variation of the quantum action has then to be regularized. We adopt here the Pauli-Villars (PV) regularization scheme in the field-antifield formalism [8,9] since it leads to consistent anomalies. It might be interesting to mention here that in the process of soldering, the resulting theory appears naturally with a Beltrami parametrization [10]. The Beltrami parameters can then be seen as sources of energy-momentum tensors which define two decoupled Virasoro algebras [3]. However the anomalies that come from BRST variation of the induced action are not just a trivial composition of the Virasoro anomalies presented by the energy-momentum tensors of the original theories [5,9,11].

We begin with the gauge invariant action presented in Ref. [6] and derive some of its classical features, with emphasis on its gauge symmetries and algebra. The field-antifield formalism for the specific model is then developed where the quantization is carried out by using the Pauli-Villars regularization scheme. We show that the process of soldering leads naturally to a Beltrami parametrization [10] of the 2D metric. The anomalous Ward identity is evaluated and further aspects of the model are discussed along with some concluding remarks.

The Euclidean version of the action presented in Ref. [6] can be written in complex coordinates as

$$\mathcal{S}_0=\int dz d\bar{z}[D\varphi\bar{D}\varphi+\lambda(\bar{D}\varphi)^2+D\rho\bar{D}\rho+\bar{\lambda}(D\rho)^2 -(\varphi-\rho)E], \tag{1}$$

where the "electric field" is defined as

$$E=\partial\bar{A}-\bar{\partial}A. \tag{2}$$

In Eq. (1), $\partial=\partial/\partial z, \bar{\partial}=\partial/\partial\bar{z}$, where z and $\bar{z}$ are complex coordinates. $D\varphi=\partial\varphi+A$ and $\bar{D}\varphi=\bar{\partial}\varphi+\bar{A}$, with similar expressions for ρ. If the fields A, $\bar{A}$ are taken to be zero, the action described in Eq. (1) represents two decoupled $\mathcal{W}2$ gravities of opposite chiralities. In that case, φ, ρ, λ, and $\bar{\lambda}$ represent chiral fields which implement the Siegel symmetry in each one of the unidexterous sectors of space-time. As we are going to see, the introduction of the gauge fields A, $\bar{A}$ promotes in an effective way the original Siegel invariances to a true diffeomorphism.

In Ref. [6] we have basically done an extensive analysis of all the symmetries of action (1). Instead, we will start here by considering the equations of motion that come from action (1). They can be written as

$$\frac{\delta\mathcal{S}_0}{\delta\varphi}=-2\bar{\partial}(D\varphi+\lambda\bar{D}\varphi)-2E=0\,, \tag{3}$$

$$\frac{\delta\mathcal{S}_0}{\delta\rho}=-2\partial(\bar{D}\rho+\bar{\lambda}D\rho)+2E=0\,, \tag{4}$$

$$\frac{\delta\mathcal{S}_0}{\delta\bar{A}}=2(D\varphi+\lambda\bar{D}\varphi)=0\,, \tag{5}$$

$$\frac{\delta\mathcal{S}_0}{\delta A}=2(\bar{D}\rho+\bar{\lambda}D\rho)=0\,, \tag{6}$$

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$$\frac{\delta\mathcal{S}_0}{\delta\lambda}=(\bar{D}\varphi)^2=0\,, \tag{7}$$

$$\frac{\delta\mathcal{S}_0}{\delta\bar\lambda}=(D\rho)^2=0\,, \tag{8}$$

defining the stationary surface Σ. This is an interesting system of coupled equations, consistency of which implies that E must vanish. This is a nice feature because in the decoupled theory the equations of motion for φ and ρ give just this. Thus we can think of E as just the ''anomaly'' resulting from the gauging procedure. Equations (5) and (6) can be solved for A, $\bar{A}$ as

$$A=\frac{1}{\Delta}[\partial\varphi+\lambda\bar\partial(\varphi-\rho)-\lambda\bar\lambda\partial\rho],$$

$$\bar{A}=\frac{1}{\Delta}[\bar\partial\rho+\bar\lambda\partial(\rho-\varphi)-\lambda\bar\lambda\bar\partial\varphi]. \tag{9}$$

where $\Delta=\lambda\bar\lambda-1$. When the expressions (9) are inserted in Eq. (2), the condition of vanishing E gives

$$\bar\partial\left[\frac{1}{\Delta}(\partial\phi+\lambda\bar\partial\phi)\right]+\partial\left[\frac{1}{\Delta}(\bar\partial\phi+\bar\lambda\partial\phi)\right]+\partial\bar\partial\phi=0\,, \tag{10}$$

where we have defined

$$\phi=\frac{1}{2}(\varphi-\rho). \tag{11}$$

We note that Eq. (10) can be rewritten in a covariant way as

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\phi)=0\,, \tag{12}$$

if we define the metric tensorial density

$$\sqrt{-g}g^{zz}=\frac{2}{\Delta}\bar\lambda,$$

$$\sqrt{-g}g^{\bar z\bar z}=\frac{2}{\Delta}\lambda,$$

$$\sqrt{-g}g^{z\bar z}=1+\frac{2}{\Delta}. \tag{13}$$

We note that due to conformal symmetry, it is not possible to determine the metric tensor itself. However, as it should be, $\det(\sqrt{-g}g^{\alpha\beta})=-1$. The above expression displays the physical content, at the classical level, of the theory described by action (1): two coupled $\mathcal{W}2$-gravities belonging respectively to the holomorphic and antiholomorphic sectors of a two-dimensional (2D) Euclidean space-time can be effectively soldered as a scalar boson in a gravitational background.

It is interesting to observe that Eqs. (13) give just the Beltrami parametrization [10] of the two-dimensional Riemannian metric. The last two equations of motion (7) and (8) are the constraints of the theory. When one uses the expressions (9), we can see that these constraints read

$$\bar{\mathcal{T}}=\tfrac{1}{2}(\pi-\bar\partial\phi)^2\approx 0,$$

$$\mathcal{T}=\tfrac{1}{2}(\pi+\partial\phi)^2\approx 0\,, \tag{14}$$

which are generators of the diffeomorphism group. In the above expressions, π represents the momentum conjugate to ϕ. As it is well known, $\mathcal{T}$ and $\bar{\mathcal{T}}$ define two decoupled Virasoro algebras once one uses the commutation relations for ϕ [see expression (11)] and π [6].

To perform the functional quantization of the model it is first necessary to do an analysis of the symmetries of $\mathcal{S}_0$. In addition to the (hidden) conformal symmetry, action (1) is invariant under the α symmetry

$$\delta\varphi=\alpha,\quad \delta\rho=\alpha,$$

$$\delta\lambda=0\,,\quad \delta\bar\lambda=0\,,$$

$$\delta A=-\partial\alpha,\quad \delta\bar{A}=-\bar\partial\alpha, \tag{15}$$

and under the diffeomorphism

$$\delta\varphi=(\eta\partial+\bar\eta\bar\partial)\varphi,$$

$$\delta\rho=(\eta\partial+\bar\eta\bar\partial)\rho$$

$$\delta\lambda=-\partial\bar\eta+\lambda^2\bar\partial\eta+[(\partial\eta)-(\bar\partial\bar\eta)+\bar\eta\bar\partial+\eta\partial]\lambda,$$

$$\delta\bar\lambda=-\bar\partial\eta+\bar\lambda^2\partial\bar\eta+[(\bar\partial\bar\eta)-(\partial\eta)+\bar\eta\bar\partial+\eta\partial]\bar\lambda,$$

$$\delta A=\partial(\eta A)+\bar\eta\bar\partial A+\bar{A}\partial\bar\eta-\frac{1}{4}\frac{\delta\mathcal{S}_0}{\delta\bar{A}}\partial\bar\eta,$$

$$\delta\bar{A}=\bar\partial(\bar\eta\bar{A})+\eta\partial\bar{A}+A\bar\partial\eta-\frac{1}{4}\frac{\delta\mathcal{S}_0}{\delta A}\bar\partial\eta. \tag{16}$$

It is not difficult to see that under the above transformations $\sqrt{-g}g^{\alpha\beta}$ transforms as

$$\delta(\sqrt{-g}g^{\alpha\beta})=\partial_\gamma(\sqrt{-g}g^{\alpha\beta}\eta^\gamma)-\sqrt{-g}(g^{\alpha\gamma}\partial_\gamma\eta^\beta+g^{\gamma\beta}\partial_\gamma\eta^\alpha), \tag{17}$$

as expected, where we have identified $\eta^z=\eta$; $\eta^{\bar z}=\bar\eta$. φ and ρ transform as scalars and A and $\bar{A}$ transform as the two components of a vector plus terms that vanish on-shell. If one truncates the theory, taking for instance $\rho=\bar\lambda=A=\bar{A}=\bar\eta=0$, the diffeomorphism for the surviving quantities (here φ and λ) reduces to the Siegel symmetry. A remarkable feature of the set of variations (16) is that they are invariant under the duality symmetry $\lambda\to 1/\bar\lambda, \bar\lambda\to 1/\lambda$, which is related to the symmetry under the interchange of the right and left movers [6].

The α symmetry (15) gives an Abelian algebra. Diffeomorphism algebra closes on all the fields of the theory, except for $A,\bar{A}$, where it is open. We get

$$[\delta_1,\delta_2]\Phi=\delta_3\Phi \tag{18}$$

for $\Phi=\varphi,\rho,\lambda,\bar\lambda$, where the composition rule for the group parameters is given by

$$\eta_3=(\eta_2\partial+\overline{\eta_2}\,\overline{\partial})\eta_1-(\eta_1\partial+\overline{\eta_1}\,\overline{\partial})\eta_2,$$

$$\overline{\eta_3}=(\eta_2\partial+\overline{\eta_2}\,\overline{\partial})\overline{\eta_1}-(\eta_1\partial+\overline{\eta_1}\,\overline{\partial})\overline{\eta_2}. \tag{19}$$

For $A,\overline{A}$ we get

$$[\delta_1,\delta_2]A=\delta_3A+V\frac{\delta\mathcal{S}}{\delta A},$$

$$[\delta_1,\delta_2]\overline{A}=\delta_3\overline{A}-V\frac{\delta\mathcal{S}}{\delta\overline{A}}, \tag{20}$$

where $V=\frac{1}{4}(\partial\overline{\eta_1}\,\overline{\partial}\eta_2-\partial\overline{\eta_2}\,\overline{\partial}\eta_1)$. The structure of Eq. (20) is that of an open gauge algebra. The term that vanishes on-shell represents a trivial gauge transformation [7]. From the inspection of the above equations one sees that all the lower rank structure functions are field independent and that they satisfy Jacobi identities only by symmetry, without the appearance of higher rank structure functions.

As the algebraic structure of the gauge invariance of action (1) has been displayed, one can quantize the theory using a standard method. Because of the existence of an open algebra, we have chosen the field-antifield formalism [6] as the quantization procedure. To do so it is necessary to introduce a ghost C corresponding to parameter α and the ghosts $c,\overline{c}$ corresponding to diffeomorphism parameters $\eta,\overline{\eta}$. The ghosts have odd Grassmannian parity and ghost number $g=1$. Corresponding to each one of the fields $\varphi,\rho\lambda,\overline{\lambda},A,\overline{A},C,c,\overline{c}$ there is an antifield, written as the corresponding field, but with an asterisk. The Grassmannian parity of any antifield is opposite of that of the corresponding field, and its ghost number is $g^*=-g-1$. In the field-antifield formalism, the classical action $\mathcal{S}_0$ is extended to the minimal action

$$\begin{aligned}\mathcal{S}=\mathcal{S}_0+\int dzd\overline{z}\,[&\varphi^*[C+(\overline{c}\,\overline{\partial}+c\partial)\varphi]+\rho^*[C+(\overline{c}\,\overline{\partial}+c\partial)\rho]+\lambda^*[-\partial\overline{c}+\lambda^2\overline{\partial}c+(\partial c-\overline{\partial}\,\overline{c}+\overline{c}\,\overline{\partial}+c\partial)\lambda]+\overline{\lambda}^*[-\overline{\partial}c+\overline{\lambda}^2\partial\overline{c}\\&+(\overline{\partial}\,\overline{c}-\partial c+\overline{c}\,\overline{\partial}+c\partial)\overline{\lambda}]+\overline{A}^*(-\overline{\partial}C-\tfrac{1}{2}(D\varphi+\lambda\overline{D}\varphi)\overline{\partial}c+\overline{\partial}(\overline{c}\overline{A})+c\partial\overline{A}+A\overline{\partial}c)+A^*(-\partial C-\tfrac{1}{2}(\overline{D}\rho+\overline{\lambda}D\rho)\partial\overline{c}+\partial(cA)\\&+\overline{c}\,\overline{\partial}A+\overline{A}\,\overline{\partial}\overline{c})-\overline{c}^*(c\partial+\overline{c}\,\overline{\partial})\overline{c}-c^*(c\partial+\overline{c}\,\overline{\partial})c+\tfrac{1}{4}A^*\overline{A}^*\partial\overline{c}\,\overline{\partial}c].\end{aligned} \tag{21}$$

We note here that the antifield C^* does not appear in the action because the corresponding symmetry is Abelian and irreducible. The antibrackets[1] of two variables X and Y,

$$(X,Y)=\frac{\partial_rX}{\partial\phi^A}\frac{\partial_lY}{\partial\phi_A^*}-\frac{\partial_rX}{\partial\phi_A^*}\frac{\partial_lY}{\partial\phi^A}, \tag{22}$$

are such that $\mathcal{S}$ must be a proper solution of the classical master equation

$$(\mathcal{S},\mathcal{S})=0\,. \tag{23}$$

This is also equivalent to saying that $\mathcal{S}$ is classically BRST invariant, once we recognize that the BRST transformation of any quantity X depending on fields and antifields is given by

$$\delta X=(X,\mathcal{S}). \tag{24}$$

From action (21) and the above expressions, we get the classical BRST transformations for the fields and antifields of the theory. To quantize the model in the path integral approach, it is necessary to fix the gauge degrees of freedom corresponding to the gauge parameters α, η, and $\overline{\eta}$. As the theory is irreducible, three independent gauge-fixing functions have to be used: Ξ, χ, and $\overline{\chi}$. They are written in terms of a unique gauge-fixing fermion as

$$\Psi=\widetilde{C}\,\Xi+\widetilde{c}\chi+\widetilde{\overline{c}}\,\overline{\chi}. \tag{25}$$

In the above expression we have introduced the Grassmannian variables $\widetilde{C}$, $\widetilde{c}$, and $\widetilde{\overline{c}}$, which have ghost number -1. Introducing also the bosonic and 0 ghost number quantities B, b, and $\overline{b}$, we can extend the minimal action (21) to the nonminimal one

$$\mathcal{S}_{\rm nm}=\mathcal{S}+\int dzd\overline{z}\,(B\widetilde{C}^*+b\widetilde{c}^*+\overline{b}\,\widetilde{\overline{c}}^*). \tag{26}$$

It is easy to see that the master equation is not modified by the nonminimal extension (26). In the space of all fields and antifields, the gauge-fixed surface is defined through the relation

$$\Phi_A^*=\frac{\partial\Psi}{\partial\Phi^A}. \tag{27}$$

As we have seen, the linear combination $\varphi+\rho$ classically does not appear as a dynamical quantity, and hence can be gauge fixed to zero. We also constrain the fields λ and $\overline{\lambda}$ to be identical to some arbitrary functions h and $\overline{h}$, respectively. As a consequence, the gauge-fixing fermion can be chosen to be [see (25)]

$$\Psi=(\varphi+\rho)\widetilde{C}+(\lambda-h)\widetilde{c}+(\overline{\lambda}-\overline{h})\widetilde{\overline{c}}. \tag{28}$$

Now we can define the induced action $\Gamma(h,\overline{h})$ through

[1] $\partial_{r(l)}/\partial\Phi^A$ stands for a right (left) derivative with respect to Φ^A.

$$e^{-\Gamma(h,\bar h)}=\int[d\Phi^A][d\Phi^*_A]\,\delta\left[\Phi^*_A-\frac{\partial\Psi}{\partial\Phi^A}\right]e^{-\mathcal{S}_{\rm nm}}. \quad (29)$$

Because of the δ function, integrations over the antifields become trivial. The integrations over $A,\bar A$ are Gaussian. These variables can be shifted in the usual way by the solutions of their equations of motion [see Eq. (9)]. Integrations over B,b, and $\bar b$ give δ functionals which can be used to perform the integrations over ρ, λ, and $\bar\lambda$. Integrations in $\tilde C,C$ are also trivial. At last we get

$$e^{-\Gamma(h,\bar h)}=\int[d\phi][dc][d\tilde c][d\bar c][d\tilde{\bar c}]\,e^{\mathcal{S}_{\rm eff}}, \quad (30)$$

where

$$\begin{aligned}\mathcal{S}_{\rm eff}=-\frac{1}{2}\int dz d\bar z\{&\sqrt{-g(h,\bar h)}\,g^{\alpha\beta}(h,\bar h)\partial_\alpha\phi\partial_\beta\phi\\&+\tilde c[-\partial\bar c+h^2\bar\partial c+(\partial c-\bar\partial\bar c+\bar c\bar\partial+c\partial)h]\\&+\tilde{\bar c}[-\bar\partial c+\bar h^2\partial\bar c+(\bar\partial\bar c-\partial c+\bar c\bar\partial+c\partial)\bar h]\}. \quad (31)\end{aligned}$$

By now it is quite obvious that Eq. (30) can be interpreted as the definition of the induced action for a boson ϕ in an external gravitational field with metric given by the tensor $g_{\alpha\beta}$ parametrized by h and $\bar h$. In what follows we are going to avoid the technicalities of being in a curved space-time and consider the problem in a somewhat naive way. At least it is always possible to think of our problem as some conformal field theory in flat space-time, having the Beltrami parameters as external sources of the components of the energy-momentum tensor. This kind of procedure, for instance, can be found in Ref. [12].

It is easy to show that the effective action (31) is manifestly BRST invariant. To extract physical results from the induced action, however, it is necessary to use some regularization procedure, since further integrations can only be done in a formal way, as they are divergent. We adopt here the Pauli-Villars regularization scheme [7–9] which gives by construction a consistent expression for the anomaly. To implement this regularizing scheme, a PV field χ^A corresponding to each one of the fields Φ^A appearing in the measure of Eq. (30) is introduced. They have the same Grassmannian character as the corresponding original fields, but have the formal property of contributing to the path integral with determinants which are the inverses of those ones that come from the Φ^A integrations [8,9]. Following the PV scheme, and using the heat-kernel method for evaluating determinants, the BRST variation of the regularized induced action is found to be

$$\mathcal{A}=-\frac{25}{24\pi}\int dz d\bar z(\partial\bar c+\bar\partial c)R_{z\bar z z\bar z}[g], \quad (32)$$

where $R_{z\bar z z\bar z}[g]$ is the $z\bar z z\bar z$ component of Riemann tensor calculated with the metric tensor written in terms of the Beltrami parameters h and $\bar h$. In the above expression, it is quite obvious that both sectors of the originally decoupled space-time appear in a nontrivial combination. However, as can be verified, its $\mathcal{W}2$ limits are quite obvious. By taking, for instance, $\bar h=c=0$, we get

$$\mathcal{A}=-\frac{25}{24\pi}\int dz d\bar z\,\bar c\,\partial^3 h, \quad (33)$$

which is essentially the same as the expression derived by Polyakov [5]. The complementary limit is also obvious.

Concluding, in this work we have considered, at the quantum level, the question of soldering two Siegel bosons, or equivalently two $\mathcal{W}2$ gravities, in a fully diffeomorphism invariant manner. We are led to an effective action which describes a full boson living in a two-dimensional space-time with nontrivial Riemannian curvature. We have considered in detail the algebraic structure of the theory and performed its quantization, by using the field-antifield formalism with a Pauli-Villars regularization scheme. Although it is straightforward to take a fully covariant theory and restrict to chiral sectors, we have addressed the opposite problem here. As it is clear, this is nontrivial and leads to many interesting features.

This work has been supported in part by U.S. Department of Energy, Grant No. DE-FG-02-91ER 40685, and by CNPq, Brazilian research agency, Brasilia, Brazil.

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