Chiral symmetry at finite temperature: Linear versus nonlinear σ models

Alexander Bochkarev*
Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455

Joseph Kapusta
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
(Received 27 February 1996)

The linear O(N) σ model undergoes a symmetry-restoring phase transition at finite temperature. We show that the nonlinear O(N) σ model also undergoes a symmetry-restoring phase transition; the critical temperatures are the same when the linear model is treated in the mean field approximation and the nonlinear model is treated to leading plus subleading order in the $1/N$ expansion. We also carefully define and study the behavior of f_π and the scalar condensate at low temperatures in both models, showing that they are independent of field redefinition. [S0556-2821(96)00718-7]

PACS number(s): 11.10.Wx, 11.30.Rd, 12.38.Mh

I. INTRODUCTION

The O(N) model as a quantum field theory in $d+1$ dimensions [1] is a basis or prototype for many interesting physical systems. The bosonic field $\mathbf{\Phi}$ has N components. When the Lagrangian is such that the vacuum state exhibits spontaneous symmetry breaking it is known as a σ model. This is the case of interest to us here. In $d=3$ space dimensions the linear σ model has the potential

$$\frac{\lambda}{4}(\mathbf{\Phi}^2-f_\pi^2)^2,$$

where λ is a positive coupling constant and f_π is the pion decay constant. The model is renormalizable. In the limit that $\lambda\to\infty$ the potential goes over to a δ-function constraint on the length of the field vector and is then known as a nonlinear σ model.

The classical limit of the field theory is obtained by neglecting or freezing out the time variable, leaving a field theory in d dimensions. In this limit, only the zero Matsubara frequency of the full $(d+1)$-dimensional theory contributes to the partition function, and the temperature acts like a coupling constant. One then has a description of an O(N) Heisenberg magnet in d dimensions which is a model for real material systems. This subject has a vast literature [2,3].

When $N=4$ one has a model for the low energy dynamics of quantum chromodynamics (QCD). More explicitly, it is essentially the unique description of the dynamics of very soft pions. This is basically because of the isomorphism between the groups O(4) and SU(2)$\times$SU(2), the latter being the appropriate group for two flavors of massless quarks in QCD. The linear σ model, including the nucleon, goes back to the work of Gell-Mann and Levy [4]. This subject also has a vast literature. In the last decade much work has been done on *chiral perturbation theory* which starts with the nonlinear σ model and adds higher order, nonrenormalizable terms to the Lagrangian, ordered by the dimensionality of the coefficients or field derivatives [5]. The goal is to construct an effective Lagrangian which describes the low energy properties of QCD to the desired accuracy. This whole program really has its origins in the classic works of Weinberg [6,7].

Finally, the standard model of the electroweak interactions of Weinberg, Salam, and Glashow has an SU(2) doublet scalar Higgs boson field responsible for spontaneous symmetry breaking. If one neglects spin-1 gauge fields the Higgs boson sector is also an O(4) field theory.

All of these limits are interesting to study at finite temperature. Magnetic materials typically undergo a phase transition from an ordered to a disordered state. If quarks are massless, QCD is expected to undergo a chiral symmetry-restoring phase transition [8,9]. This may have implications for high energy nucleus-nucleus collisions; see especially [10,11] in this respect. The electroweak theory is expected to have a symmetry-restoring phase transition, too, at which point the baryon number of the early Universe would have been finally determined [12].

The linear σ model was studied in the classic papers on relativistic quantum field theories at finite temperature [13–15]. The usefulness of leaving N as a parameter arises from the fact that there is only one other parameter in the problem, the quartic coupling constant λ (f_π just sets the scale and is held fixed in our considerations). For QCD at least, and perhaps for electroweak theory too (this is not known, it is related to the Higgs boson mass), the appropriate limit seems to be $\lambda\gg 1$, possibly even infinity. This limit is the nonlinear σ model. The only proposed expansion parameter for this model is $1/N$. For $N=4$ the first few terms in this expansion may not be quantitatively reliable but it is a good start. At least in the theoretical world we can imagine N as large as we wish. Presumably, the physics does not change qualitatively with N as long as it is greater than one.

Our basic physical interest in this paper is QCD. Among the questions that are routinely asked are: Does QCD have a finite temperature phase transition? If so, is it associated with color deconfinement, or with the restoration of the spontaneously broken chiral symmetry, or are they inextricably intertwined? What would be the order of this phase transition?

*On leave from INR, Russian Academy of Sciences, Moscow 117312, Russia.

What would be its critical temperature? How do pions, the Goldstone bosons of QCD, decouple as the temperature is raised? How do the quark and gluon condensates behave as functions of temperature? Restricting our attention to N_f flavors of massless quarks, there are strong arguments [8,9] and numerical lattice computations [16] which say that for $N_f=2$ there is a second order phase transition and for $N_f\geqslant 3$ there is a first order phase transition. Lattice calculations also suggest strongly that chiral symmetry is restored and color is deconfined at the same temperature [17].

The QCD Lagrangian is invariant under $\mathrm{U}(N_f)\times\mathrm{U}(N_f)$ transformations. This is isomorphic to $\mathrm{SU}_L(N_f)\times\mathrm{SU}_R(N_f)\times\mathrm{U}_{\mathrm{baryon}}(1)\times\mathrm{U}_{\mathrm{axial}}(1)$; that is, left- and right-handed chirality transformations of the quark fields, baryon number conservation, and the famous axial U(1) symmetry. The axial U(1) is broken by quantum effects, particularly instantons. Baryons can be added to the σ model if desired, but we shall not do so. The O(N) σ model has no vestige of the axial U(1), although there do exist versions of the σ model based on other groups in which it can be incorporated. In general, $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ is isomorphic to $\mathrm{O}(N_f^2)$ only for $N_f=2$. Otherwise, the interactions are different. This restricts any potential quantitative results of our analysis of the O(N) model to QCD to $N=4$. The very interesting issue of whether the axial U(1) symmetry is restored at high temperatures or not cannot be addressed here [18].

One must be careful in understanding how σ models are being applied to the study of QCD at finite temperature. At very low temperatures one may argue that the only degrees of freedom which are excited are pions. One may then use chiral perturbation theory to study the thermodynamic properties, as in the classic works of Leutwyler [19]. At the very lowest energies this is just the nonlinear σ model. Note that in this domain one is really studying a full (3+1)-dimensional quantum field theory. On the other hand, near the critical temperature one may argue [8,9] that the soft, long wavelength modes of QCD are in the same universality class as the O(4) Heisenberg magnet in three spatial directions. Now, one is studying a classical field theory. The parameters of the effective free energy functional near the critical temperature may not be simply related to the parameters of the σ field theory model at a very low temperature. Support for this point of view comes from lattice QCD computations which show that critical exponents are consistent with those of the O(4) Heisenberg magnet in three spatial dimensions [20]. Conversely, it has been pointed out that in certain models with composite bosons, these arguments might break down [21] because the compositeness is important near a second order phase transition. A particularly useful observation is that long range correlation functions may be dominated by the soft modes (pions, σ meson, . . .) near the critical temperature but the equation of state itself is not [22,18]; it is dominated by the myriad of other degrees of freedom (Hagedorn or Particle Data Book compendium of mesons and baryons or all of the colored quarks and gluons).

We would like to shed some light on just a few of the issues relating to the above discussion. *First, from the outset we restrict our attention to the linear and nonlinear σ models in 3+1 dimensions* [23]. Within these confines we address three specific topics: the existence and nature of a chiral symmetry-restoring phase transition, the low temperature dependence of the "pion decay constant," and the low temperature dependence of the "quark condensate." The first of these involves folklore, and the answer, at least in the nonlinear σ model, is either obvious to the reader or else very surprising. For the latter two we point out some popular misconceptions and reproduce the existing results (when $N=4$) while showing that they are invariant under field redefinition.

First consider the linear σ model. At zero temperature the effective potential has a shape similar to the bottom of a wine bottle. It is minimized by a nonzero value of the field; this is the condensate. As the temperature is increased, the radius of bottom of the potential shrinks, and goes to zero at a critical temperature of $T_c=\sqrt{12/(N+2)}f_\pi$. It is a second-order symmetry-restoring phase transition. In the nonlinear σ model $|\mathbf{\Phi}|^2$ is fixed at the value f_π^2. Therefore, it would seem, chiral symmetry breaking is built into the Lagrangian and there is no possibility of restoring it at finite temperature. Another way of saying this is that there is no order parameter which can go to zero at finite temperature. At least this is the folklore in much of the nuclear and particle physics community. On the other hand, the critical temperature in the linear model is independent of λ in the mean field approximation, so one can take the limit $\lambda\to\infty$ and still have a phase transition. The counter argument to this is that the phase transition can go away in the limit and so nothing special happens at the aforementioned value of T_c in the nonlinear model. We shall study the nonlinear model directly in Sec. II at finite temperature in the large N approximation. To leading order in N we shall show quite straightforwardly that despite the constraint the nonlinear model has a second order phase transition at a critical temperature equal to that of the linear model. We shall show that this persists to the next-to-leading order in N, although here we must make an additional high energy approximation. The order parameter is identified as is the nature of the two phases.

Next, we study what is meant by "the pion decay constant on finite temperature." At zero temperature a common definition is

$$\langle 0|\mathcal{A}_\mu^a|\pi^b(p)\rangle = i f_\pi p_\mu \delta^{ab}, \tag{1}$$

which relates it to the matrix element of the axial vector current of QCD between the vacuum state and a one-pion state of momentum p. It is difficult, though perhaps not impossible, to generalize this definition to finite temperature. Within the linear σ model one sometimes sees in the literature $f_\pi(T)$ identified with the thermal average of the σ field, $v=\langle\sigma\rangle=|\langle\mathbf{\Phi}\rangle|$, which is the radius of the bottom of the effective potential. In the nonlinear σ model this radius is necessarily fixed at f_π. How then can one understand the result of Gasser and Leutwyler [24],

$$f_\pi(T)=f_\pi\left[1-\frac{N_f}{2}\left(\frac{T^2}{12f_\pi^2}\right)+\cdots\right], \tag{2}$$

which was obtained at low temperature in chiral perturbation theory? (At this order chiral perturbation theory and the nonlinear σ model are the same.) This issue is addressed carefully in Sec. III.

Finally, a quantity of much interest, especially for the application of QCD sum rules at finite temperature, is the temperature dependence of the quark condensate. Gasser and Leutwyler [24] and also Gerber and Leutwyler [25] computed this quantity at low temperature to be

$$\langle\overline{q}q\rangle=\langle 0|\overline{q}q|0\rangle\left[1-\frac{N_f^2-1}{N_f}\left(\frac{T^2}{12f_\pi^2}\right)-\frac{N_f^2-1}{2N_f^2}\left(\frac{T^2}{12f_\pi^2}\right)^2+\cdots\right]. \tag{3}$$

This is obviously a different temperature dependence than that of $f_\pi(T)$. In addition, one wonders how is it possible to obtain information on quark condensates from a theory which has no explicit reference to quarks? These and related topics are studied in Sec. IV.

Before beginning the technical part we remark that we will deal with vacuum loop divergences in a simple way: we ignore them. Put another way, it is known that the partition function can be expressed in terms of the vacuum-scattering amplitudes, or S-matrix elements, for arbitrary reactions involving n particles going in and m particles coming out [26]. This is the relativistic virial expansion. Consider, for example, a real scalar field with a quartic interaction [27]. The one-loop contribution to the partition function is just the free Bose gas expression. The two-loop contribution corresponds to two-particle scattering with the amplitude evaluated at the tree level. The three-loop contribution corresponds to two particles in and four particles out, plus three particles in and three particles out, plus four particles in and two particles out, with all scattering amplitudes evaluated at the tree level. The three-loop contribution also has a part which corresponds to a vacuum one-loop correction to the two-particle scattering amplitude. *By dropping all vacuum loop divergencies we are doing a virial expansion with the* S *matrix evaluated at the tree level.* It is well known that a relativistic quantum field theory is not defined until a prescription is given for dealing with divergences. This is our way of defining a nonrenormalizable field theory.

The linear σ model is renormalizable and vacuum loops can be computed. However, they do not change any of the principal results at finite temperature since the physics is dominated by long wavelengths. The nonlinear model is not renormalizable, and although one may consider it to be the $\lambda\to\infty$ limit of the linear model, we do not know how to do a strong coupling expansion in λ anyway. The short distance physics of these models in the context of QCD is not correct in any case.

An early systematic study of various approximation schemes for the linear σ model at finite temperature is [28]. The nonlinear sigma model in 3+1 dimensions has been studied at *low* temperatures in the leading order of the $1/N$ expansion in [29] using an ultraviolet cutoff $700<\Lambda<1000$ MeV to regulate divergences. An overall introduction to relativistic quantum field theory at finite temperature is [30]. A somewhat analogous study of the lattice O(N) Heisenberg model is [31].

II. CHIRAL PHASE TRANSITION

It is well known that the linear O(N) σ model in 3+1 dimensions has a second-order phase transition when treated in the mean field approximation. We briefly repeat that analysis here as a warm-up and precursor to the study of the nonlinear σ model which is not nearly as well studied. We must emphasize that the direct quantitative applicability to QCD is limited by at least two factors. The first is that these σ models do not have quark and gluon degrees of freedom and so one can never describe high temperature quark-gluon plasma with them. In addition, the compositeness of the bosons may even influence the phase transition itself [21]. The second is that the group $SU(N_f)\times SU(N_f)$ is isomorphic to $O(N_f^2)$ only for $N_f=2$ and this limits the analogy to two flavors of massless quarks. Indeed, lattice computations of the nonlinear σ model in three dimensions based on other groups show first order behavior [32].

The conventions and notation used here are consistent with those of [30].

A. Linear σ model

The linear σ model Lagrangian is

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\mathbf{\Phi})^2-\frac{\lambda}{4}(\mathbf{\Phi}^2-f_\pi^2)^2, \tag{4}$$

where λ is a positive coupling constant. The bosonic field $\mathbf{\Phi}$ has N components. Rather arbitrarily, we define the first $N-1$ components to represent a pion field $\boldsymbol{\pi}$ and the last Nth component to represent the σ field. Since the O(N) symmetry is broken to an O($N-1$) symmetry at low temperatures, we immediately allow for a σ condensate v whose value is temperature dependent and yet to be determined. We write

$$\Phi_i(\mathbf{x},t)=\pi_i(\mathbf{x},t),\quad i=1,\ldots N-1,$$
$$\Phi_N(\mathbf{x},t)=v+\sigma(\mathbf{x},t). \tag{5}$$

In terms of these fields the Lagrangian is

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\boldsymbol{\pi})^2+\frac{1}{2}(\partial_\mu\sigma)^2-\frac{\lambda}{4}(v^2-f_\pi^2+2v\sigma+\sigma^2+\boldsymbol{\pi}^2)^2. \tag{6}$$

The action at finite temperature is obtained by rotating to imaginary time, $\tau=it$, and integrating τ from 0 to $\beta=1/T$. (However, we keep the Minkowski metric; hence, $\partial_\mu=\partial/\partial x^\mu$ with $\partial_0=\partial/\partial t=i\partial/\partial\tau$.) The action is defined as

$$S=-\frac{\lambda}{4}(f_\pi^2-v^2)^2\beta V+\int_0^\beta d\tau\int_V d^3x\left\{\frac{1}{2}[(\partial_\mu\boldsymbol{\pi})^2-\overline{m_\pi^2}\boldsymbol{\pi}^2+(\partial_\mu\sigma)^2-\overline{m_\sigma^2}\sigma^2]+\frac{\lambda}{2}v(v^2-f_\pi^2)\sigma-\lambda v\sigma(\boldsymbol{\pi}^2+\sigma^2)-\frac{\lambda}{4}(\sigma^2+\boldsymbol{\pi}^2)^2\right\}, \tag{7}$$

where the effective masses are

$$\overline{m_\pi^2}=\lambda(v^2-f_\pi^2),$$
$$\overline{m_\sigma^2}=\lambda(3v^2-f_\pi^2). \tag{8}$$

At any temperature v is chosen such that $\langle\sigma\rangle=0$. This eliminates any one-particle reducible (1PR) diagrams in perturbation theory, leaving only one-particle irreducible (1PI) diagrams.

At zero temperature the potential is minimized when $v=f_\pi$. The pion is massless and the σ particle has a mass of $\sqrt{2\lambda}f_\pi$. The Goldstone theorem is satisfied. Lin and Serot [33] have argued that the σ meson should not be identified with the attractive s-wave interaction in the $\pi-\pi$ interaction, which is responsible for nuclear attraction. Rather, they argue that the σ meson should have a mass that is at least 1 GeV if not more. This means that λ is on the order of 50 or greater.

The simplest approximation at finite temperature is the mean field approximation. One allows for v to be temperature dependent; hence the effective masses are temperature dependent as well. However, interactions among the particles or collective excitations are neglected. The pressure includes only the contribution of the condensate and of the thermal motion of the independently moving particles. Thus,

$$P=\frac{T}{V}\ln Z=-\frac{\lambda}{4}(f_\pi^2-v^2)^2+P_0(T,m_\sigma)+(N-1)P_0(T,m_\pi). \tag{9}$$

The pressure of a free relativistic boson gas can be written in several ways:

$$P_0=-T\int\frac{d^3p}{(2\pi)^3}\ln(1-e^{-\beta\omega})$$
$$=\int\frac{d^3p}{(2\pi)^3}\frac{p^2}{3\omega}\frac{1}{e^{\beta\omega}-1}. \tag{10}$$

This is a relatively simple but surprisingly powerful first approximation that allows one to gain much insight into the behavior of relativistic quantum field theories at high temperature. It was used in all the pioneering papers.

One expects that as the temperature is raised, thermal fluctuations will tend to disorder the condensate field v, and at sufficiently high temperature it may even disappear. If there is a second-order phase transition, then the correlation length should go to infinity, which is equivalent to the effective σ mass going to zero. With such an expectation one may expand the free boson gas pressure about zero mass to obtain

$$P_0(T,m)=\frac{\pi^2}{90}T^4-\frac{m^2T^2}{24}+\frac{m^3T}{12\pi}+\cdots. \tag{11}$$

Since the masses are proportional to the square root of λ, it is generally inconsistent to retain the cubic term in m because there exist loop diagrams that are not included in the mean field approximation but that contribute to the same order in λ. Therefore, we take

$$P(T,v)=N\frac{\pi^2}{90}T^4+\frac{\lambda}{2}v^2\left[f_\pi^2-\frac{N+2}{12}T^2\right]-\frac{\lambda}{4}v^4, \tag{12}$$

where the pion and σ masses have been expressed in terms of λ, v, and f_π. Maximizing the pressure with respect to v gives

$$v^2=f_\pi^2-\frac{N+2}{12}T^2. \tag{13}$$

This result is easily understood. Going back to Eq. (7) we can differentiate ln$Z$ with respect to v with the result that

$$v^2=f_\pi^2-3\langle\sigma^2\rangle-\langle\pi^2\rangle \tag{14}$$

as long as we choose $\langle\sigma\rangle=0$. For any free bosonic field ϕ with mass m

$$\langle\phi^2\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1}, \tag{15}$$

where $\omega=\sqrt{p^2+m^2}$. In the limit that the temperature is greater than the mass $\langle\phi^2\rangle\to T^2/12$. This yields directly Eq. (13).

The condensate goes to zero at a critical temperature given by

$$T_c^2=\frac{12}{N+2}f_\pi^2. \tag{16}$$

Above this temperature thermal fluctuations are too large to allow a nonzero condensate. It is a straightforward exercise to show that the pressure and its first derivative is continuous at T_c but that the second derivative is discontinuous. This is therefore a second-order phase transition.

There are two major problems with the mean field approximation as described. The first is that the pion has a negative mass squared at every temperature greater than zero. Not only is the Goldstone theorem not satisfied, but there are tachyons as well. The σ particle also gets a negative mass squared at temperatures above $\sqrt{8/(N+2)}f_\pi<T_c$. This violation of basic physical principles is resolved by recognizing that the finite temperature corrections to the squared masses are proportional to λT^2, and that one-loop self-energy corrections, not included in the mean field analysis, are of the same order. This can be understood with the following analysis.

At high temperatures, when the masses can be neglected in the loops, the mean field result is obtained by combining Eqs. (8) and (13):

$$\overline{m_\pi^2}=-\frac{N+2}{12}\lambda T^2,$$
$$\overline{m_\sigma^2}=2\lambda f_\pi^2-\frac{N+2}{4}\lambda T^2. \tag{17}$$

The full one-loop self-energies for pions and the σ meson are drawn in Figs. 1 and 2. If one chooses $\langle\sigma\rangle=0$ then there are no 1PR diagrams and the tadpoles should not be included; they are already included in the temperature dependence of v. One may check this by fixing $v=f_\pi$ and then computing the tadpole contributions to the effective masses. One gets precisely Eq. (17). The diagrams involving the four-point vertices contribute an amount $(N+2)\lambda T^2/12$ to both the pion and σ meson self-energies. When evaluated in the high temperature approximation and at low frequency and momentum the 1PI diagrams involving the three-point

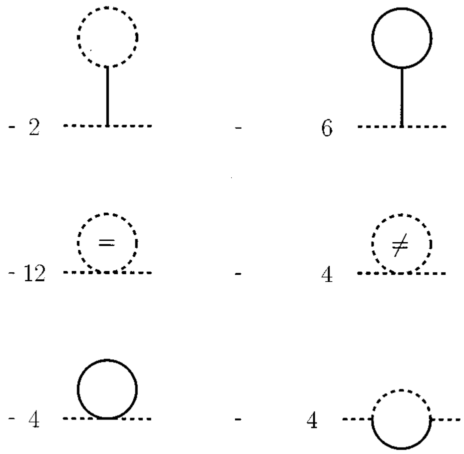

FIG. 1. One-loop self-energy diagrams for the pion in the linear σ model. The dashed lines represent pions and the solid lines represent σ mesons. The three-point vertices are $-\lambda v$ and the four-point vertices are $-\lambda/4$. The $=$ and $\neq$ indicate that the pion in the loop has the same or different quantum number than the external pion, respectively. If v is fixed at its vacuum value of f_π then the two tadpoles contribute. If v is allowed to vary with temperature by maximizing the pressure then the tadpoles are not to be included in the self-energy; their effect is already included in the mean field mass via $v(T)$.

vertices may be neglected. (This follows from power counting. These diagrams involve two propagators instead of one, and so are only logarithmically divergent in the UV in the vacuum. The other diagrams are quadratically divergent, which leads to a T^2 behavior at finite temperature.) When all contributions of order λT^2 are included, the pole positions of the pion and σ propagators move with the result that, below T_c,

$$m_\pi^2 = \overline{m_\pi^2} + \Pi_\pi = 0,$$

$$m_\sigma^2 = \overline{m_\sigma^2} + \Pi_\sigma = 2\lambda f_\pi^2 (1 - T^2/T_c^2), \tag{18}$$

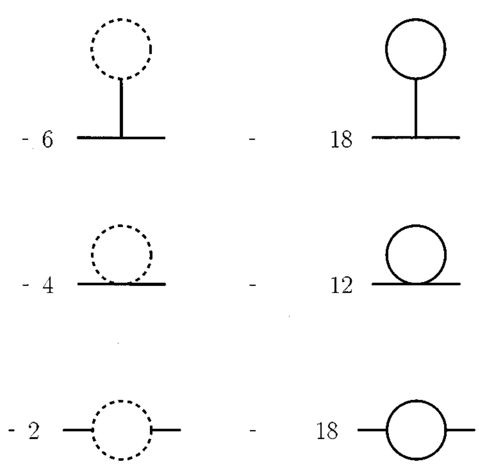

FIG. 2. One-loop self-energy diagrams for the σ meson in the linear σ model. See Fig. 1 for remarks.

and above T_c,

$$m_\pi^2 = m_\sigma^2 = m_\Phi^2 = -\lambda f_\pi^2 + \Pi_\Phi = \frac{N+2}{12}\lambda (T^2 - T_c^2). \tag{19}$$

The Goldstone theorem is satisfied, there are no tachyons, and restoration of the full symmetry of the Lagrangian above T_c is evident.

It must be recognized that the results (17)–(19) are valid to order λ and cannot be extrapolated to $\lambda \to \infty$. At low temperature, where pions scatter from each other sequentially and there is essentially no propagation off mass shell between scatterings because of the low particle density, one may take the point of view that λ is a parameter to be adjusted to fit π-π scattering data and it doesn't matter how large λ is. This point of view cannot be taken at high temperature where the pion number density is large, for then multiple scatterings will occur and they cannot be factorized into independent scatterings. This means that multiloop self-energy diagrams will be important at high temperature if λ is not perturbatively small.

The second major problem is that long wavelength fluctuations very near the phase transition cannot be treated with perturbation theory because the self-interacting boson fields become massless just at the transition. Although this is a well-known problem in the statistical mechanics of second-order phase transitions, exactly how it affects the critical temperature is not known for the linear σ model in 3 + 1 dimensions. This is a topic for further study. The result presented here must be accepted for what it is: a one-loop estimate of the critical temperature.

B. Nonlinear σ model

The nonlinear σ model may be defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} (\partial_\mu \mathbf{\Phi})^2 \tag{20}$$

together with the constraint

$$f_\pi^2 = \mathbf{\Phi}^2(\mathbf{x}, t). \tag{21}$$

The partition function is

$$Z = \int [d\mathbf{\Phi}] \delta(f_\pi^2 - \mathbf{\Phi}^2) \exp\left\{ \int_0^\beta d\tau \int_V d^3x \mathcal{L} \right\}. \tag{22}$$

Because the length of the chiral field is fixed and cannot be changed by thermal fluctuations it is often said that chiral symmetry breaking is built into this model and, therefore, there can be no chiral symmetry-restoring phase transition. On the other hand, the linear σ model does undergo a symmetry-restoring phase transition. Taking the quartic coupling constant λ to infinity essentially constrains the length of the chiral field to be f_π just as in the nonlinear model. The critical temperature, however, is independent of λ at least in the mean field approximation. So it would seem that the phase transition survives. If this is true, then one ought to be able to derive it entirely within the context of the nonlinear model. That is what we shall do, although it involves a lot more effort than treatment of the linear model in the mean field approximation. Since the only parameter in the model is

f_π, and we are interested in temperatures comparable to it, we cannot do an expansion in powers of T/f_π. The only other parameter is N, the number of field components. This suggests an expansion in $1/N$.

Begin by representing the field-constraining δ function by an integral.

$$Z=\int [d\mathbf{\Phi}][db']\exp\left\{\int_0^\beta d\tau\int_V d^3x[\mathcal{L}+ib'(\mathbf{\Phi}^2-f_\pi^2)]\right\}. \tag{23}$$

As with the linear model, we define the first $N-1$ components of $\mathbf{\Phi}$ to be the pion field and the last component to be the σ field. We allow for a zero frequency and zero-momentum condensate of the σ field referred to as v. Following Polyakov [34], we also separate out explicitly the zero frequency and zero-momentum mode of the auxiliary field b'. Integrating over all the other modes will give us an effective action involving the constant part of the fields. We will then minimize the free energy with respect to these constant parts, which is a saddle-point approximation. Integrating over fluctuations about the saddle point is a finite volume correction and of no consequence in the thermodynamic limit. The Fourier expansions are

$$\Phi_i(\mathbf{x},\tau)=\pi_i(\mathbf{x},\tau)=\sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} e^{i(\mathbf{x}\cdot\mathbf{p}+\omega_n\tau)}\tilde{\pi}_i(\mathbf{p},n),$$

$$\Phi_N(\mathbf{x},\tau)=v+\sigma(\mathbf{x},\tau)$$
$$=v+\sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} e^{i(\mathbf{x}\cdot\mathbf{p}+\omega_n\tau)}\tilde{\sigma}(\mathbf{p},n),$$

$$b'(\mathbf{x},\tau)=i\frac{m^2}{2}+b(\mathbf{x},\tau)$$
$$=i\frac{m^2}{2}+T\sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} e^{i(\mathbf{x}\cdot\mathbf{p}+\nu_n\tau)}\tilde{b}(\mathbf{p},n). \tag{24}$$

One must remember to exclude the zero frequency and zero-momentum mode from the summations. The field $\mathbf{\Phi}$ must be periodic in imaginary time for the usual reasons, but there is no such requirement on the auxiliary field b, hence $\omega_n=2\pi nT$ and $\nu_n=\pi nT$. Since the field b has dimensions of inverse length squared we inserted another factor of T so as to make its Fourier amplitude dimensionless, as they are for the other fields. The action then becomes

$$S=\int_0^\beta d\tau\int_V d^3x\left\{\frac{1}{2}[(\partial_\mu\boldsymbol{\pi})^2-m^2\boldsymbol{\pi}^2+(\partial_\mu\sigma)^2-m^2\sigma^2]\right.$$
$$\left.-ib(2v\sigma+\boldsymbol{\pi}^2+\sigma^2)\right\}+\frac{1}{2}m^2(f_\pi^2-v^2)\beta V. \tag{25}$$

Note that terms linear in the fields integrate to zero because $\langle\pi_i\rangle=\langle\sigma\rangle=\langle b\rangle=0$.

An effective action is derived by expanding $\exp(S)$ in powers of b and integrating over the pion and σ fields. The term linear in b vanishes on account of $\tilde{b}(\mathbf{0},0)\propto\langle b\rangle=0$. The

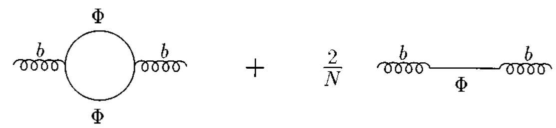

FIG. 3. Contribution to the effective action at finite temperature in the nonlinear σ model corresponding to Eq. (26); the wavy lines represent the Lagrange multiplier field b.

term proportional to b^2 is not zero and is exponentiated, thus summing a whole series of contributions. The term proportional to b^3 is not zero either and it too may be exponentiated, summing an infinite series of higher-order terms left out of the order b^2 exponentiation. After making the scaling $b\to b/\sqrt{2N}$, the effective action becomes

$$S_{\rm eff}=-\frac{1}{2}\sum_n\sum_{\mathbf{p}}(\omega_n^2+p^2+m^2)[\tilde{\boldsymbol{\pi}}(\mathbf{p},n)\cdot\tilde{\boldsymbol{\pi}}(-\mathbf{p},-n)$$
$$+\tilde{\sigma}(\mathbf{p},n)\tilde{\sigma}(-\mathbf{p},-n)]-\frac{1}{2}\sum_n\sum_{\mathbf{p}}\left[\Pi(p,\omega_n,T,m)\right.$$
$$\left.+\frac{2}{N}\frac{v^2}{\omega_n^2+p^2+m^2}\right]\tilde{b}(\mathbf{p},2n)\tilde{b}(-\mathbf{p},-2n)$$
$$+\frac{1}{2}m^2(f_\pi^2-v^2)\beta V+O(\tilde{b}^3/\sqrt{N}). \tag{26}$$

Note that only even Matsubara frequencies contribute in the b field: $\nu_n=2\pi nT$. This may have been anticipated. There appears the one-loop function

$$\Pi(p,\omega_n,T,m)=T\sum_l\int\frac{d^3k}{(2\pi)^3}\frac{1}{(\omega_n-\omega_l)^2+(\mathbf{p}-\mathbf{k})^2+m^2}$$
$$\times\frac{1}{\omega_l^2+k^2+m^2}. \tag{27}$$

The effective action is an infinite series in b. The coefficients are frequency and momentum dependent, arising from one-loop diagrams. The coefficient of the term quadratic in b in $S_{\rm eff}$ is illustrated schematically in Fig. 3. In addition, each successive term is suppressed by $1/\sqrt{N}$ compared to the previous one. This is the large N expansion.

The propagators for the pion and σ fields are of the usual form

$$D_0^{-1}(p,\omega_n,m)=\omega_n^2+p^2+m^2 \tag{28}$$

with an effective mass m yet to be determined. The propagator for the b field is more complicated, being

$$D_b^{-1}(p,\omega_n,m)=\Pi(p,\omega_n,T,m)+\frac{2}{N}\frac{v^2}{\omega_n^2+p^2+m^2}. \tag{29}$$

The value of the condensate v is not yet determined either.

Keeping only the terms up to order b^2 in $S_{\rm eff}$ (the rest vanish in the limit $N\to\infty$) allows us to obtain an explicit expression for the partition function and the pressure. This includes the next-to-leading order in N:

$$P=\frac{T}{V}\ln Z=\frac{1}{2}m^2(f_\pi^2-v^2)-\frac{N}{2}T\sum_n\int\frac{d^3p}{(2\pi)^3}\times\ln[\beta^2(\omega_n^2+p^2+m^2)]-\frac{1}{2}T\sum_n\int\frac{d^3p}{(2\pi)^3}\ln\left[\Pi(p,\omega_n,T,m)+\frac{2}{N}\frac{v^2}{\omega_n^2+p^2+m^2}\right]. \quad (30)$$

The second term under the last logarithm should and will be set to zero at this order. It may be needed at higher order in the large N expansion to regulate infrared divergences.

The pressure is extremized with respect to the mass parameter m. Therefore, $\partial P/\partial m^2=0$. From the initial expression for Z this is seen to be equivalent to the thermal average of the constraint:

$$f_\pi^2=\langle\mathbf{\Phi}^2\rangle=v^2+\langle\boldsymbol{\pi}^2\rangle+\langle\sigma^2\rangle. \quad (31)$$

If an approximation to the exact partition function is made, such as the large N expansion, this constraint should still be satisfied. It may, in fact, single out a preferred value of m.

To leading order in N we may neglect the term involving Π entirely. The pressure is then

$$P=\tfrac{1}{2}m^2(f_\pi^2-v^2)+NP_0(T,m). \quad (32)$$

The pressure must be a maximum with respect to variations in the condensate v. This means that

$$\partial P/\partial v=-m^2v=0, \quad (33)$$

which is equivalent to the condition that $\langle\sigma\rangle=0$. There are two possibilities.

(1) $m=0$: There exist massless particles, or Goldstone bosons, and the value of the condensate is determined by the thermally averaged constraint. This is the symmetry-broken phase.

(2) $v=0$: The thermally averaged constraint is satisfied by a nonzero temperature-dependent mass. There are no Goldstone bosons. This is the symmetry-restored phase.

Evidently, there is a chiral symmetry-restoring phase transition.

In the leading order of the large N approximation the particles are represented by free fields with a potentially temperature-dependent mass m. For any free bosonic field ϕ,

$$\partial P_0(T,m)/\partial m^2=\langle\phi^2\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1} \quad (34)$$

with $\omega=\sqrt{p^2+m^2}$. Thus, extremizing the pressure with respect to m^2 is equivalent to satisfying the thermally averaged constraint:

$$f_\pi^2=v^2+\langle\boldsymbol{\pi}^2\rangle+\langle\sigma^2\rangle. \quad (35)$$

Note, however, that the pion and σ fields have the same mass and therefore $\langle\boldsymbol{\pi}^2\rangle=(N-1)\langle\sigma^2\rangle$. Consider now the two different phases.

In the asymmetric phase the mass is zero. The constraint is satisfied by a temperature-dependent condensate:

$$v^2(T)=f_\pi^2-\frac{NT^2}{12}. \quad (36)$$

This condensate goes to zero at a critical temperature

$$T_c^2=\frac{12}{N}f_\pi^2 \quad \text{(leading } N \text{ approximation)}. \quad (37)$$

At exactly T_c the thermally averaged constraint is satisfied by the fluctuations of N massless degrees of freedom without the help of a condensate.

In the symmetric phase the condensate is zero. The constraint is satisfied by thermal fluctuations alone:

$$f_\pi^2=N\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1}. \quad (38)$$

Thermal fluctuations decrease with increasing mass at fixed temperature. The constraint is only satisfied by massless excitations at one temperature, namely, T_c. At temperatures $T>T_c$ the mass must be greater than zero. Near the critical temperature the mass should be small, and the flucutations may be expanded about $m=0$ as

$$f_\pi^2=NT^2\left[\frac{1}{12}-\frac{m}{4\pi T}-\frac{m^2}{8\pi^2T^2}\ln\left(\frac{m}{4\pi T}\right)-\frac{m^2}{16\pi^2T^2}+\cdots\right]. \quad (39)$$

As T approaches T_c from above, the mass approaches zero like:

$$m(T)=\frac{\pi}{3T}(T^2-T_c^2)+\cdots. \quad (40)$$

This is a second-order phase transition since there is no possibility of metastable supercooled or superheated states.

The mass must grow faster than the temperature at very high temperatures in order to keep the field fluctuations fixed and equal to f_π^2. Asymptotically, the particles move nonrelativistically. This allows us to compute the fluctuations analytically. We get

$$f_\pi^2=N\left(\frac{T}{2\pi}\right)^{3/2}\sqrt{m}e^{-m/T}. \quad (41)$$

This is a transcendental equation for $m(T)$. It can also be written as

$$m=T\ln\left(\frac{NT}{2\pi f_\pi}\sqrt{\frac{mT}{2\pi f_\pi^2}}\right). \quad (42)$$

Roughly, the solution behaves as

$$m\sim T\ln(T^2/T_c^2). \quad (43)$$

It is rather amusing that, in leading order of the large N approximation, the elementary excitations are massless below T_c, become massive above T_c, and at asymptotically high temperatures move nonrelativistically.

The result to first order of the large N expansion provides good insight into the nature of the two-phase structure of the nonlinear σ model, but it is not quite satisfactory for two reasons. First, it predicts N massless Goldstone bosons in the broken symmetry phase when in fact we know there ought to be only $N-1$. Second, the square of the critical temperature is $12f_\pi^2/N$ whereas it is $12f_\pi^2/(N+2)$ in the linear σ model in the mean field approximation; we expect them to be the same in the limit $\lambda\to\infty$. Both these problems can be rectified by inclusion of the next-to-leading order term in N, namely, the contribution of the b field.

It is natural to expect that the b field will contribute essentially one negative degree of freedom to the T^4 term in the pressure so as to give $N-1$ Goldstone bosons in the low temperature phase. Therefore, we move one of the N degrees of freedom and put it together with the b contribution as

$$P=\frac{1}{2}m^2(f_\pi^2-v^2)-\frac{N-1}{2}T\sum_n\int\frac{d^3p}{(2\pi)^3}\ln[\beta^2(\omega_n^2+p^2+m^2)]-\frac{1}{2}T\sum_n\int\frac{d^3p}{(2\pi)^3}\ln[\beta^2(\omega_n^2+p^2+m^2)\Pi]. \tag{44}$$

The function $\Pi(p,\omega_n,T,m)$ can be reduced to a one-dimensional integral

$$\Pi=\frac{1}{8\pi^2 p}\int_0^\infty\frac{dkk}{\omega}\ln\left[\frac{k^2+pk+\Lambda^2}{k^2-pk+\Lambda^2}\right]\frac{1}{e^{\beta\omega}-1}, \tag{45}$$

where

$$\Lambda^2=\Lambda^2(p,\omega_n,m)=\frac{(\omega_n^2+p^2)^2+4m^2\omega_n^2}{4(\omega_n^2+p^2)} \tag{46}$$

but unfortunately cannot be simplified any further. In any case, to the order in N to which we are working, the pressure is

$$P=\tfrac{1}{2}m^2(f_\pi^2-v^2)+(N-1)P_0(T,m)+P_{\rm I}(T,m). \tag{47}$$

This can be thought of, in the low temperature phase, as $N-1$ Goldstone bosons with an interaction term $P_{\rm I}$.

Because of the logarithm the main contribution to the interaction pressure will come when Π is very small compared to one. This corresponds to very large values of the parameter Λ; in other words, to very high momentum, Matsubara frequency, or mass. In this limit

$$\Pi\to\frac{1}{4\pi^2\Lambda^2}\int_0^\infty\frac{dkk^2}{\omega}\frac{1}{e^{\beta\omega}-1}=\frac{h_3(m/T)}{4\pi^2}\frac{T^2}{\Lambda^2}. \tag{48}$$

This may be thought of as a kind of high energy approximation, and we shall henceforth refer to it as such. Then

$$P_{\rm I}=\frac{1}{2}T\sum_n\int\frac{d^3p}{(2\pi)^3}\ln[\beta^2(\omega_n^2+p^2+m^2)\Pi]\approx-\frac{1}{2}T\sum_n\int\frac{d^3p}{(2\pi)^3}\ln\left[\frac{h_3}{\pi^2}\frac{(\omega_n^2+p^2)(\omega_n^2+p^2+m^2)}{(\omega_n^2+\omega_+^2)(\omega_n^2+\omega_-^2)}\right], \tag{49}$$

with the dispersion relations

$$\omega_\pm^2=p^2+2m^2\pm2m\sqrt{p^2+m^2}. \tag{50}$$

The interaction pressure can now be determined in the usual way to be

$$P_{\rm I}=-T\int\frac{d^3p}{(2\pi)^3}\{\ln[1-e^{-\beta p}]+\ln[1-e^{-\beta\omega(p)}]-\ln[1-e^{-\beta\omega_+(p)}]-\ln[1-e^{-\beta\omega_-(p)}]\}. \tag{51}$$

Note that $h_3(m/T)$ has no effect within this approximation. Note also that in the broken symmetry phase where $m=0$ the contribution of the b field cancels one of the massless degrees of freedom to give $N-1$ Goldstone bosons.

Now we are prepared to examine the behavior of the system near the critical temperature with the inclusion of next-to-leading terms in N. We do an expansion in m/T as before. The pressure is, up to and including order m^3,

$$P=(N-1)\frac{\pi^2}{90}T^4-\frac{N+2}{24}m^2T^2+\frac{1}{2}m^2(f_\pi^2-v^2)+\frac{N}{12\pi}m^3T. \tag{52}$$

In the high temperature phase where $v=0$, maximization with respect to m yields

$$f_\pi^2=T^2\left[\frac{N+2}{12}-\frac{N}{4\pi}\frac{m}{T}\right]. \tag{53}$$

This gives the same critical temperature as in the mean field treatment of the linear σ model:

$$T_{\rm c}^2=\frac{12}{N+2}f_\pi^2\quad(\text{subleading } N \text{ approximation}). \tag{54}$$

The mass approaches zero from above like

$$m(T)=\frac{\pi(N+2)}{3NT}(T^2-T_{\rm c}^2). \tag{55}$$

We leave it as an exercise for the reader to compute the asymptotic behavior of the mass with the inclusion of the subleading terms in N.

The results obtained immediately above used an approximation for Π which we referred to as a high energy approximation. Relaxing this approximation can be done albeit at the cost of a numerical calculation. We do not attempt that in this paper. Of course, one should also go beyond the mean field approximation in the linear model.

III. f_π AT LOW TEMPERATURE

Consideration of correlation functions at finite temperature is more involved than at zero temperature. Lorentz invariance is not manifest because there is a preferred frame of reference, the frame in which the matter is at rest. Thus, spectral densities and other functions may depend on energy and momentum separately and not just on their invariant s. Also, the number of Lorentz tensors is greater because there is a new vector available, namely, the vector $u_\mu = (1,0,0,0)$ which specifies the rest frame of the matter.

For a given four-momentum q it is useful to define two projection tensors. The first one $P_T^{\mu\nu}$ is both three- and four-dimensionally transverse,

$$P_T^{ij} \equiv \delta^{ij} - \frac{q^i q^j}{\mathbf{q}^2}, \tag{56}$$

with all other components zero. The second one $P_L^{\mu\nu}$ is only four-dimensionally transverse:

$$P_L^{\mu\nu} \equiv -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} + P_T^{\mu\nu} \right). \tag{57}$$

The notation is L for longitudinal and T for transverse with respect to $\mathbf{q}$. There are no other symmetric second-rank tensors that are four-dimensionally transverse.

In the usual fashion [35,36], one may construct a Green function for the axial vector current

$$G_{ab}^{\mu\nu}(z,\mathbf{q}) = \int_{-\infty}^{\infty} \frac{d\omega}{\omega - z} \rho_{ab}^{\mu\nu}(\omega,\mathbf{q}), \tag{58}$$

where the spectral density tensor is

$$\begin{aligned}\rho_{ab}^{\mu\nu}(\omega,\mathbf{q}) = \frac{1}{Z}\sum_{m,n} &(2\pi)^3 \delta(\omega - E_m + E_n)\,\delta(\mathbf{q} - \mathbf{p_m} + \mathbf{p_n}) \\ &\times (e^{-E_n/T} - e^{-E_m/T}) \langle n | \mathcal{A}_a^\mu(0) | m \rangle \\ &\times \langle m | \mathcal{A}_b^\nu(0) | n \rangle. \end{aligned}\tag{59}$$

The summation is over a complete set of energy eigenstates. The retarded, advanced, and Matsubara Green functions are

$$G_{ab}^{\mathrm{R}\mu\nu}(q_0,\mathbf{q}) = G_{ab}^{\mu\nu}(q_0 + i\epsilon,\mathbf{q}), \tag{60}$$

$$G_{ab}^{\mathrm{A}\mu\nu}(q_0,\mathbf{q}) = G_{ab}^{\mu\nu}(q_0 - i\epsilon,\mathbf{q}), \tag{61}$$

$$G_{ab}^{\mathrm{T}\mu\nu}(\omega_n,\mathbf{q}) = G_{ab}^{\mu\nu}(i\omega_n,\mathbf{q}), \tag{62}$$

where $\epsilon \to 0^+$.

Because of current conservation, the spectral density tensor can be decomposed into longitudinal and transverse pieces [37]:

$$\rho_{ab}^{\mu\nu}(q) = \delta_{ab}[\rho_A^L(q) P_L^{\mu\nu} + \rho_A^T(q) P_T^{\mu\nu}]. \tag{63}$$

In general, the spectral densities depend on q^0 and $\mathbf{q}$ separately as well as on the temperature. In the vacuum we can always go to the rest frame of a massive particle, and in that frame there can be no difference between longitudinal and transverse polarizations, so that $\rho_L = \rho_T = \rho$. We also observe that $P_L^{\mu\nu} + P_T^{\mu\nu} = -(g^{\mu\nu} - q^\mu q^\nu / q^2)$. The pion, being a massless Goldstone boson, is special. It contributes to the longitudinal axial spectral density and not to the transverse one. In vacuum

$$\rho^{\mu\nu}(q) = \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} \right) \rho_A(q^2) + f_\pi^2 \delta(q^2) q^\mu q^\nu. \tag{64}$$

This may be taken to be the definition of the pion decay constant at zero temperature. In fact, one can write the pion's contribution as

$$f_\pi^2 \delta(q^2) q^\mu q^\nu = f_\pi^2 q^2 \delta(q^2) P_L^{\mu\nu}. \tag{65}$$

This cannot be taken as the definition of the pion decay constant at *finite temperature* because the contribution of the pion to the longitudinal spectral density cannot be assumed to be a δ function in q^2. In general, the pion's dispersion relation will be more complicated and will develop a width at nonzero momentum. This smears out the δ function into something like a relativistic Breit-Wigner distribution. Fortunately, the Goldstone theorem [38] requires that there be a zero frequency excitation when the momentum is zero. (For a proof applicable to relativistic quantum field theories at finite temperature see [30].) This implies that the width must go to zero at $\mathbf{q} = \mathbf{0}$, which results in a δ function at zero frequency. Explicit calculations support this assertion [39–41]. Therefore, it seems to make sense to define

$$f_\pi^2(T) \equiv 2 \lim_{\epsilon \to 0} \int_0^\epsilon \frac{dq_0^2}{q_0^2} \rho_A^L(q_0, \mathbf{q} = \mathbf{0}). \tag{66}$$

Physically, this means that the pion decay constant at finite temperature measures the strength of the coupling of the Goldstone boson to the longitudinal part of the retarded axial vector response function in the limit of zero momentum.

We shall study the pion's contribution to the spectral density only at temperatures small compared to f_π. We shall study both the nonlinear and the linear σ models. At low temperatures the σ meson's contribution as a material degree of freedom is frozen out and one might expect the same dynamics to be operative in both models; in other words, one may expect the result to be the same and so independent of λ. For temperatures approaching T_c, the problem is more difficult and is left for future investigation.

A. Nonlinear σ model

The nonlinear σ model was defined at the beginning of Sec. II B. One can make a nonlinear redefinition of the field without changing the physical content of the theory. Various redefinitions may be found in the literature. We will first list the most common ones, and then we will compute $f_\pi(T)$ for each of them, thereby illustrating that one always gets the same result. It is interesting to see how this comes about; it is also reassuring that it does.

A convenient way to express the σ and pion fields that explicitly contain the constraint is

$$\begin{aligned}\sigma &= f_\pi \cos(\phi / f_\pi), \\ \boldsymbol{\pi} &= f_\pi \hat{\boldsymbol{\phi}} \sin(\phi / f_\pi),\end{aligned} \tag{67}$$

where $\phi = |\boldsymbol{\phi}|$ and $\hat{\boldsymbol{\phi}} = \boldsymbol{\phi}/\phi$. The Lagrangian may then be expressed in terms of the fields of choice:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu \boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma \\ &= \frac{1}{2}\partial_\mu \boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} + \frac{1}{2}\frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi})}{f_\pi^2-\pi^2} \\ &= \frac{1}{2}\frac{f_\pi^2}{\phi^2}\sin^2\left(\frac{\phi}{f_\pi}\right)\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} \\ &\quad + \frac{1}{2}\left[1-\frac{f_\pi^2}{\phi^2}\sin^2\left(\frac{\phi}{f_\pi}\right)\right]\partial_\mu\phi\partial^\mu\phi. \end{aligned} \tag{68}$$

Another representation to consider is from Weinberg [6], who defines

$$\mathbf{p} = 2\frac{f_\pi^2}{\pi^2}\left(1-\sqrt{1-\frac{\pi^2}{f_\pi^2}}\right)\boldsymbol{\pi}, \tag{69}$$

or inversely,

$$\boldsymbol{\pi} = \frac{\mathbf{p}}{1+p^2/4f_\pi^2}. \tag{70}$$

In terms of Weinberg's field definition, the Lagrangian is very compact:

$$\mathcal{L} = \frac{1}{2}\frac{\partial_\mu\mathbf{p}\cdot\partial^\mu\mathbf{p}}{(1+p^2/4f_\pi^2)^2}. \tag{71}$$

The $(\sigma,\boldsymbol{\pi})$ representation is cumbersome because of the constraint, although it can be handled by the Lagrange multiplier method of Sec. II. However, it is inconvenient for exposing the physical particle content and for doing perturbation theory in terms of physical particles. Among the three physical representations we choose to work with here, it is interesting to note the range of allowed values of the fields. The magnitude of the $\mathbf{p}$ field can range from zero to infinity, the magnitude of the $\boldsymbol{\pi}$ field can range from 0 to f_π, and the magnitude of the $\boldsymbol{\phi}$ field can range from 0 to πf_π. This distinction is important when dealing with nonperturbative large amplitude motion; whether it makes any difference in low orders of perturbation theory is not known to us.

The first step in our quest to extract the temperature dependence of f_π from the theory is to obtain the form of the axial vector current in terms of the chosen fields. Starting from

$$\mathcal{A}_\mu = -\sigma\partial_\mu\boldsymbol{\pi} + \boldsymbol{\pi}\partial_\mu\sigma, \tag{72}$$

one directly computes

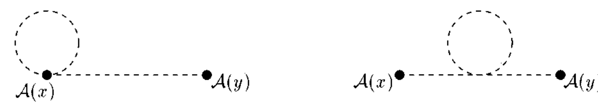

FIG. 4. Vertex and self-energy contributions to the axial vector correlation function in the nonlinear σ model.

$$\begin{aligned}\mathcal{A}_\mu &= -\sigma\left[\partial_\mu\boldsymbol{\pi} + \frac{\boldsymbol{\pi}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})}{f_\pi^2-\pi^2}\right] \\ &= -\frac{f_\pi^2}{2\phi}\sin\left(\frac{2\phi}{f_\pi}\right)\partial_\mu\boldsymbol{\phi} - f_\pi\hat{\boldsymbol{\phi}}\left[1-\frac{f_\pi}{2\phi}\sin\left(\frac{2\phi}{f_\pi}\right)\right]\hat{\boldsymbol{\phi}}\cdot\partial_\mu\boldsymbol{\phi} \\ &= -\frac{1}{f_\pi}\frac{1}{(1+p^2/4f_\pi^2)^2}\left[\left(f_\pi^2-\frac{1}{4}p^2\right)\partial_\mu\mathbf{p} + \frac{1}{2}\mathbf{p}(\mathbf{p}\cdot\partial_\mu\mathbf{p})\right]. \end{aligned} \tag{73}$$

Every form of the axial vector current is an odd function of the pion field.

Obviously, it is not possible to compute the axial vector correlation function exactly. We will restrict our attention to low temperature. Roughly speaking, a loop expansion of the correlation function is an expansion in powers of T^2/f_π^2 with each additional loop contributing one more such factor. To one-loop order we need the axial vector current to third order in the pion field:

$$\begin{aligned}\mathcal{A}_\mu &= -f_\pi\partial_\mu\boldsymbol{\pi} + \frac{\pi^2}{2f_\pi}\partial_\mu\boldsymbol{\pi} - \frac{1}{f_\pi}\boldsymbol{\pi}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}) \\ &= -f_\pi\partial_\mu\boldsymbol{\phi} + \frac{2\phi^2}{3f_\pi}\partial_\mu\boldsymbol{\phi} - \frac{2}{3f_\pi}\boldsymbol{\phi}(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi}) \\ &= -f_\pi\partial_\mu\mathbf{p} + \frac{3p^2}{4f_\pi}\partial_\mu\mathbf{p} - \frac{1}{2f_\pi}\mathbf{p}(\mathbf{p}\cdot\partial_\mu\mathbf{p}). \end{aligned} \tag{74}$$

We will also need the Lagrangian to fourth order in the pion field:

$$\begin{aligned}\mathcal{L}_4 &= \frac{1}{2f_\pi^2}(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}) \\ &= \frac{1}{6f_\pi^2}[(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}) - \phi^2\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}] \\ &= -\frac{1}{4f_\pi^2}p^2\partial_\mu\mathbf{p}\cdot\partial^\mu\mathbf{p}. \end{aligned} \tag{75}$$

The correlation function $\langle \mathcal{A}_\mu^i(x)\mathcal{A}_\nu^j(y)\rangle$ will have a zero-loop contribution from the π-π correlation function $\langle\partial_\mu\pi^i(x)\partial_\nu\pi^j(y)\rangle$, a one-loop self-energy correction to the same π-π correlation function, and a one-loop contribution from the correlation function $\langle\partial_\mu\pi^i(x)\pi^j(y)\pi^k(y)\partial_\nu\pi^l(y)\rangle$ involving four pions. These three contributions are illustrated in Fig. 4.

The contribution of the bare pion propagator D_0 to the longitudinal spectral density is easily found to be

$$\rho_A^L(q_0,\mathbf{q})=f_\pi^2 q^2\delta(q^2). \tag{76}$$

At zero temperature this is just the definition of the pion decay constant.

The one-loop pion self-energy may be computed by standard diagrammatic or functional integral techniques. The results are

$$\Pi_{\boldsymbol{\pi}}(q)=-\frac{T^2}{12f_\pi^2}q^2,$$

$$\Pi_{\mathbf{p}}(q)=(N-1)\frac{T^2}{24f_\pi^2}q^2,$$

$$\Pi_{\boldsymbol{\phi}}(q)=\tfrac{1}{3}\Pi_{\boldsymbol{\pi}}(q)+\tfrac{2}{3}\Pi_{\mathbf{p}}(q). \tag{77}$$

These are quite dependent on the definition of the pion field. Nevertheless, it is worth noting that the Goldstone theorem is satisfied on account of the fact that the self-energy is always proportional to q^2.

The final contribution comes from the correlation function of a pion at point x with three pions at point y. Again, standard diagrammatic or functional integral techniques may be used. To express the answers, we gather together the contributions from the bare propagator, from the one-loop self-energy, and from this correlation function, and quote the coefficient of the term $f_\pi^2 q^2\delta(q^2)$ in the longitudinal part of the axial vector spectral density:

$$\boldsymbol{\pi}:\quad \left[1-\frac{T^2}{12f_\pi^2}\right]-(N-3)\frac{T^2}{12f_\pi^2},$$

$$\mathbf{p}:\quad \left[1+(N-1)\frac{T^2}{24f_\pi^2}\right]-\left(N-\frac{5}{3}\right)\frac{T^2}{8f_\pi^2},$$

$$\boldsymbol{\phi}:\quad \left[1+(N-2)\frac{T^2}{36f_\pi^2}\right]-(N-2)\frac{T^2}{9f_\pi^2}. \tag{78}$$

In all three cases the results are the same and amount to a temperature dependence of

$$f_\pi^2(T)=f_\pi^2\left[1-\frac{N-2}{12}\frac{T^2}{f_\pi^2}\right]. \tag{79}$$

It agrees with Eq. (2) for the only case that they can be compared to: $N_f^2=N=4$. The calculation of Gasser and Leutwyler was verified by Eletsky and Kogan [42].

B. Linear σ model

It is now not surprising to discover that the linear σ model gives the same result for $f_\pi(T)$ at low temperature as the nonlinear σ model. This is because the σ meson is very heavy at low temperature and cannot contribute materially the way the pions do. However, the way in which it works out is very different.

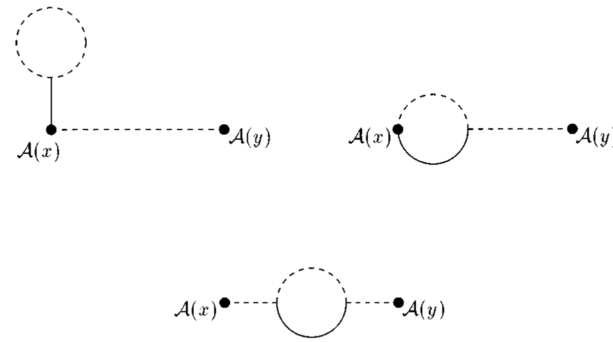

FIG. 5. Vertex and self-energy contributions to the axial vector correlation function in the linear σ model. See Fig. 1 for remarks.

Let us go back to the axial vector current *before* shifting the σ field:

$$\mathcal{A}_\mu=-\sigma\partial_\mu\boldsymbol{\pi}+\boldsymbol{\pi}\partial_\mu\sigma. \tag{80}$$

After making the shift $\sigma\to v+\sigma$, the current takes the form

$$\mathcal{A}_\mu=-v\,\partial_\mu\boldsymbol{\pi}-\sigma\partial_\mu\boldsymbol{\pi}+\boldsymbol{\pi}\partial_\mu\sigma. \tag{81}$$

By maximizing the pressure (minimizing the effective potential) with respect to v at each temperature, we effectively sum all tadpole diagrams, leaving only 1PI diagrams in any subsequent perturbative treatment. If this is done, one has the inclination to identify $v(T)$ with $f_\pi(T)$. This is wrong; $f_\pi(T)$ has additional contributions, as we shall now see.

The first contribution to $f_\pi^2(T)$ does come from $v^2(T)$ since it involves the cross term of $\partial_\mu\pi^a(x)$ with $\partial_\nu\pi^a(y)$. Following the analysis of Sec. II A, but at low temperature rather than high, we simply leave out the contribution of the heavy σ meson. This gives

$$P(T,v)=(N-1)\frac{\pi^2}{90}T^4+\frac{\lambda}{2}v^2\left[f_\pi^2-\frac{N-1}{12}T^2\right]-\frac{\lambda}{4}v^4. \tag{82}$$

Maximizing with respect to v gives

$$v^2(T)=f_\pi^2-\frac{N-1}{12}T^2. \tag{83}$$

The T^2/f_π^2 correction is identically the tadpole contribution to the vertex shown in Fig. 5.

There is another, nonlocal, contribution to the vertex shown in Fig. 5, corresponding to the emission and absorption of a virtual σ meson. One might think that it is suppressed by the large σ mass, $m_\sigma^2=2\lambda f_\pi^2$ but in fact this is compensated by the coupling constant λ in the extra vertex. Evaluation of this diagram gives a contribution of $T^2/6f_\pi^2$ to $f_\pi^2(T)$.

Finally, there is a contribution coming from the dressed pion propagator analogous to the nonlinear σ model. The full one-loop 1PI pion self-energy diagrams were already shown in Fig. 1. We know that the sum of the momentum-independent pieces is zero on account of Goldstone's theorem. We only need the contribution that is quadratic in the energy and momentum of the pion. This can only arise from the so-called exchange diagram involving two $\sigma\pi\pi$ vertices, also shown in Fig. 5. In imaginary time (Euclidean space) it is

$$\Pi_{\rm ex}(\omega_n,\mathbf{q})=-4\lambda^2 f_\pi^2 T\sum_l \int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_l^2+k^2} \times \frac{1}{(\omega_l+\omega_n)^2+(\mathbf{k}+\mathbf{q})^2+m_\sigma^2}. \quad (84)$$

Since $T\ll m_\sigma$, it is easy to extract the piece quadratic in the momentum. Analytically continuing to Minkowski space ($\omega_n\to iq_0$) it is $q^2T^2/12f_\pi^2$.

The residue of the pion pole in the axial vector correlation function can now be obtained by adding the vacuum contribution, the pion self-energy correction, and the tadpole and nonlocal vertex corrections as

$$\left[1-\frac{1}{12}\frac{T^2}{f_\pi^2}\right]-\frac{N-1}{12}\frac{T^2}{f_\pi^2}+\frac{1}{6}\frac{T^2}{f_\pi^2}.$$

The final result,

$$f_\pi^2(T)=f_\pi^2\left[1-\frac{N-2}{12}\frac{T^2}{f_\pi^2}\right], \quad (85)$$

is identical to that of the nonlinear σ model. We remark that this cannot be used to compute the critical temperature since it was obtained under the condition that $T\ll f_\pi$.

IV. SCALAR CONDENSATE AT LOW TEMPERATURE

The scalar condensate is defined as $|\langle\mathbf{\Phi}\rangle|$. Our convention has been to allow the last, Nth component of the field to condense, and to refer to this as either v (if the field is shifted) or $\langle\sigma\rangle$ (if the field is not shifted). In this section we use the latter convention.

It is interesting to ask what happens to this condensate as a function of temperature in the nonlinear model. The constraint as an operator equation is $f_\pi^2=\mathbf{\Phi}^2$ and as a thermal average is $f_\pi^2=\langle\mathbf{\Phi}^2\rangle$; it is not $f_\pi=|\langle\mathbf{\Phi}\rangle|$. The condensate indeed can change with temperature. In fact, we can quite easily compute it to two-loop order. Before doing so, we first discuss the connection of this condensate with the quark condensate $\langle\bar{q}q\rangle$.

In two-flavor QCD, one oftentimes associates the σ and pion fields with certain bilinears of the quark fields:

$$\bar{q}q\sim\sigma,$$

$$i\bar{q}\gamma_5\boldsymbol{\tau}q\sim\boldsymbol{\pi}.$$

This association is made because the quark bilinears transform in the same way under SU(2)×SU(2) as the corresponding meson fields. The dimensions do not match so there must be some dimensionful coefficient relating them; this coefficient could even be a function of the group-invariant $\sigma^2+\boldsymbol{\pi}^2\sim(\bar{q}q)^2-(\bar{q}\gamma_5\boldsymbol{\tau}q)^2$. Does this particular combination of four-quark condensates change with temperature? The temperature dependence of the four-quark condensates at low temperatures was first calculated in [43] with the help of the fluctuation-dissipation theorem. The contribution of pions alone was later discussed in [44] using soft pion techniques. From [43,44] one can read off the two condensates separately:

$$\langle(\bar{q}q)^2\rangle=\left[1-\frac{T^2}{4f_\pi^2}\right]\langle 0|(\bar{q}q)^2|0\rangle-\frac{T^2}{12f_\pi^2}\langle 0|(\bar{q}\gamma_5\boldsymbol{\tau}q)^2|0\rangle \quad (86)$$

and

$$\langle(\bar{q}\gamma_5\boldsymbol{\tau}q)^2\rangle=\left[1-\frac{T^2}{12f_\pi^2}\right]\langle 0|(\bar{q}\gamma_5\boldsymbol{\tau}q)^2|0\rangle -\frac{T^2}{4f_\pi^2}\langle 0|(\bar{q}q)^2|0\rangle. \quad (87)$$

Therefore, there is no correction to this group invariant to order T^2/f_π^2 inclusive:

$$\langle(\bar{q}q)^2-(\bar{q}\gamma_5\boldsymbol{\tau}q)^2\rangle=\langle 0|(\bar{q}q)^2-(\bar{q}\gamma_5\boldsymbol{\tau}q)^2|0\rangle. \quad (88)$$

This result is consistent with our analysis of the nonlinear σ model in Secs. II B and III A.

Now let us return to the business of computing the temperature dependence of the scalar condensate to one- and two-loop order. In terms of the three representations used in Sec. III A the σ field is

$$\begin{aligned}\sigma/f_\pi&=\sqrt{1-\frac{\boldsymbol{\pi}^2}{f_\pi^2}}=1-\frac{\boldsymbol{\pi}^2}{2f_\pi^2}-\frac{(\boldsymbol{\pi}^2)^2}{8f_\pi^4}+\cdots\\ &=\left[1-\frac{\mathbf{p}^2}{2f_\pi^2}+\frac{(\mathbf{p}^2)^2}{16f_\pi^2}\right]^{1/2}\left[1+\frac{\mathbf{p}^2}{4f_\pi^2}\right]^{-1}\\ &=1-\frac{\mathbf{p}^2}{2f_\pi^2}+\frac{(\mathbf{p}^2)^2}{8f_\pi^2}+\cdots\\ &=\cos(\phi/f_\pi)=1-\frac{\boldsymbol{\phi}^2}{2f_\pi^2}+\frac{(\boldsymbol{\phi}^2)^2}{24f_\pi^2}+\cdots.\end{aligned} \quad (89)$$

To second order in the pion field all three representations are the same. Using the free-field expression for the thermal average of the field squared we get

$$\langle\sigma\rangle/f_\pi=1-\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)+\cdots. \quad (90)$$

For $N=4$, the only value for which we can quantitatively compare with QCD, this agrees with the result of Gasser and Leutwyler as quoted in Eq. (3); it was also derived in an independent way by Eletsky [44].

The coefficient of the term that is fourth order in the pion field differs in sign and magnitude among the three representations. It would be a miracle if the thermal average of $\sqrt{1-\boldsymbol{\pi}^2/f_\pi^2}$, $\cos(\phi/f_\pi)$ and the Weinberg expression were all the same. But regarding the order $(T^2/12f_\pi^2)^2$ we must recognize that the term that is second order in the pion field gets modified because of a one-loop self-energy. This was computed for each representation in Sec. III A and the results listed in Eq. (77). The term fourth order in the pion field can be evaluated using free fields. The result is

$$\langle(\boldsymbol{\phi}^2)^2\rangle=(N^2-1)\left(\frac{T^2}{12}\right)^2, \tag{91}$$

and is obviously representation independent. The contributions for each representation are

$$\boldsymbol{\pi}:\ 1-\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)\left[1-\left(\frac{T^2}{12f_\pi^2}\right)\right]-\frac{N^2-1}{8}\left(\frac{T^2}{12f_\pi^2}\right)^2,$$

$$\mathbf{p}:\ 1-\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)\left[1+\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)\right]+\frac{N^2-1}{8}\left(\frac{T^2}{12f_\pi^2}\right)^2,$$

$$\boldsymbol{\phi}:\ 1-\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)\left[1+\frac{N-2}{3}\left(\frac{T^2}{12f_\pi^2}\right)\right]+\frac{N^2-1}{24}\left(\frac{T^2}{12f_\pi^2}\right)^2, \tag{92}$$

where the second term in each line comes from the square of the pion field and the last term comes from the pion field in fourth order. The sum of all terms is identical in all three representations.

$$\langle\sigma\rangle/f_\pi=1-(N-1)\left(\frac{T^2}{24f_\pi^2}\right)-\frac{(N-1)(N-3)}{2}\left(\frac{T^2}{24f_\pi^2}\right)^2+\cdots. \tag{93}$$

The miracle happens. It is a consequence of the fact that physical quantities must be independent of field redefinition. What is more, for $N=4$ it agrees with the result of Gasser and Leutwyler quoted in Eq. (3). However, we emphasize once more that this expression should not be used to infer a critical temperature because it was derived under the assumption that the temperature is small compared to f_π.

V. SUMMARY AND CONCLUSION

In this paper we have focused on the linear and nonlinear versions of the σ model based on the group O(N) at finite temperature. Models of this kind are prototypes for physical theories, such as QCD and electroweak theory. Our main goal was to understand, both conceptually and mathematically, whether the nonlinear model has a symmetry-restoring phase transition analogous to that of the linear model. We did show that the nonlinear model has a second-order phase transition by making use of the $1/N$ expansion. To leading and subleading orders the critical temperature is even the same as in the linear model. (This cannot be true in general; eventually there must be some dependence in the linear model on the value of the quartic coupling λ.) This expansion was facilitated by the introduction of a Lagrange multiplier field. In this way we could see that there is a condensate at low temperature; this condensate decreases in just the right way so as to conserve the constraint on the field vector. There is one particular temperature for which the thermally averaged constraint is satisfied with no condensate and with all excitations massless. This is the critical temperature. We had to make a mathematical approximation at the subleading order to get an analytical result. We referred to this as a "high energy approximation." It is directly analogous to what one does in the mean field approximation to the linear model. It would be interesting to relax this approximation; this is left as a future project.

Another goal was to carefully define and show how to compute the "pion decay constant" and the "scalar quark condensate" at finite temperature within the scope of these models. The definitions also apply to full QCD but, of course, the results will generally be different. Only at very low temperatures and for $N=4$ will the results be directly applicable to QCD for the reasons discussed in the introduction. Even within the context of the σ models, however, it would be interesting to compute the next order correction to $f_\pi^2(T)$ at low temperature. It would also be interesting to compute $f_\pi^2(T)$ near T_c. These computations are now underway.

In this paper we have not computed anything more complicated than a one-loop diagram. Even the calculation of the "scalar quark condensate" to order $(T^2/12f_\pi^2)^2$ only required knowledge of one-loop diagrams. What other interesting physical quantities can the reader compute in these models to one-loop order?

Natural extensions of these models to better approximate full QCD may be envisioned. Following the philosophy of chiral perturbation theory one may include higher derivative terms in the Lagrangian. One may also add other mesonic and baryonic fields, especially the vector mesons. However, no matter how many extra terms are added, one is still restricted from discussing the quark-gluon plasma.

If the Higgs particle turns out to have an exceptionally large mass, then a reasonable first approximation to the electroweak phase transition might begin with a nonlinear version of the Glashow-Weinberg-Salam model. Gauging the nonlinear σ model would be a step in this direction. This topic is also under investigation.

We hope to have stimulated the reader to make further progress on these very interesting topics at finite temperature.

ACKNOWLEDGMENTS

We are grateful to A. Kovner and S. Jeon for discussions. This work was supported by the U.S. Department of Energy under Grant No. DE-FG02-87ER40328.

[1] K. G. Wilson, Phys. Rev. D **7**, 2911 (1973).
[2] S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).
[3] D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena*, 2nd ed. (World Scientific, Singapore, 1984).
[4] M. Gell-Mann and M. Levy, Nuovo Cimento **16**, 705 (1960).
[5] J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model* (Cambridge University Press, Cambridge, England, 1992).
[6] S. Weinberg, Phys. Rev. **166**, 1568 (1968).
[7] S. Weinberg, Physica A **96A**, 327 (1979).
[8] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).
[9] F. Wilczek, Int. J. Mod. Phys. **7**, 3911 (1992).
[10] R. D. Pisarski, Phys. Lett. **110B**, 155 (1982).
[11] K. Rajagopal and F. Wilczek, Nucl. Phys. **B399**, 395 (1993); **B404**, 577 (1993).
[12] *Electroweak Physics and the Early Universe*, NATO ASI Series B: Physics Vol. **338**, edited by J. C. Romão and F. Freire (Plenum, New York, 1994).
[13] D. A. Kirzhnits and A. D. Linde, Phys. Lett. **42B**, 471 (1972).
[14] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
[15] S. Weinberg, Phys. Rev. D **9**, 3357 (1974).
[16] For updates on the latest lattice results at finite temperature see the proceedings of the lattice conferences, the most recent being *Lattice '93*, Proceedings of the International Symposium, Dallas, Texas, edited by T. Draper *et al.* [Nucl. Phys. B (Proc. Suppl.) **34** (1993)] and *Lattice '94*, Proceedings of the International Symposium, Bielefeld, Germany, edited by F. Karsch *et al.* [*ibid.* **42** (1995)], or the proceedings of the quark matter conferences, the most recent being *Quark Matter '93*, Proceedings of the Tenth International Conference on Ultrarelativistic Nucleus-Nucleus Collisions, Borlänge, Sweden, 1993, edited by H. A. Gustafsson [Nucl. Phys. **A566**, (1994)], and in *Quark Matter '95* [*ibid.* **A590**, (1995)].
[17] G. Boyd, S. Gupta, F. Karsch, and E. Laermann, Z. Phys. C **64**, 331 (1994).
[18] E. Shuryak, Comments Nucl. Part. Phys. **21**, 235 (1994).
[19] H. Leutwyler, in *QCD 20—Years Later*, Proceedings of the Workshop, Aachen, Germany, 1992, edited by P. M. Zerwas and H. A. Kastrup (World Scientific, Singapore, 1993), Vol. 2.
[20] F. Karsch, Phys. Rev. D **49**, 3791 (1994).
[21] A. Kocic and J. Kogut, Phys. Rev. Lett. **74**, 3109 (1995).
[22] M. Dine and W. Fischler, Phys. Lett. **105B**, 207 (1981).
[23] The linear and nonlinear σ model predictions for low energy properties of pions has been worked out. The result is that the linear model does not represent data as well as the nonlinear model, at least when the former is treated in perturbation theory with a σ mass of 500 to 1000 MeV; J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); J. F. Donoghue and B. R. Holstein, Phys. Rev. D **40**, 2378 (1989).
[24] J. Gasser and H. Leutwyler, Phys. Lett. B **184**, 83 (1987).
[25] P. Gerber and H. Leutwyler, Nucl. Phys. **B321**, 387 (1989).
[26] R. Dashen, S. Ma, and H. J. Bernstein, Phys. Rev. **187**, 345 (1969).
[27] J. I. Kapusta, in *Trends in Theoretical Physics*, Vol. 2, edited by P. J. Ellis and Y. C. Tang (Addison-Wesley, Reading, MA, 1991).
[28] G. Baym and G. Grinstein, Phys. Rev. D **15**, 2897 (1977).
[29] H. Meyer-Ortmanns, H. J. Pirner, and B.-J. Schaefer, Phys. Lett. B **311**, 213 (1993).
[30] J. I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
[31] V. F. Müller and W. Rühl, Ann. Phys. (N.Y.)**168**, 425 (1986).
[32] P. Dreher, Phys. Lett. B **281**, 127 (1992).
[33] W. Lin and B. D. Serot, Nucl. Phys. **A512**, 637 (1990).
[34] A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic, Chur, Switzerland, 1987).
[35] E. S. Fradkin, Proc. Lebedev Phys. Inst. Acad. Sci. USSR **29**, 6 (1965).
[36] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
[37] J. I. Kapusta and E. V. Shuryak, Phys. Rev. D **49**, 4694 (1994).
[38] J. Goldstone, Nuovo Cimento **19**, 154 (1961).
[39] J. Goity and H. Leutwyler, Phys. Lett. B **228**, 517 (1989).
[40] A. Schenk, Nucl. Phys. **A363**, 97 (1991); Phys. Rev. D **47**, 5138 (1993).
[41] C. Song, Phys. Rev. D **49**, 1556 (1994).
[42] V. L. Eletsky and I. I. Kogan, Phys. Rev. D **49**, R3083 (1994).
[43] A. Bochkarev and M. Shaposhnikov, Nucl. Phys. **B268**, 220 (1986).
[44] V. L. Eletsky, Phys. Lett. B **299**, 111 (1993).