New relativistic high-temperature Bose-Einstein condensation

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(Received 14 June 1996)

We discuss the properties of an ideal relativistic gas of events possessing Bose-Einstein statistics. We find that the mass spectrum of such a system is bounded by $\mu \leqslant m \leqslant 2M/\mu_K$, where μ is the usual chemical potential, M is an intrinsic dimensional scale parameter for the motion of an event in space time, and μ_K is an additional mass potential of the ensemble. For the system including both particles and antiparticles, with a nonzero chemical potential μ, the mass spectrum is shown to be bounded by $|\mu| \leqslant m \leqslant 2M/\mu_K$, and a special type of high-temperature Bose-Einstein condensation can occur. We study this Bose-Einstein condensation, and show that it corresponds to a phase transition from the sector of continuous relativistic mass distributions to a sector in which the boson mass distribution becomes sharp at a definite mass M/μ_K. This phenomenon provides a mechanism for the mass distribution of the particles to be sharp at some definite value. [S0556-2821(96)00416-X]

PACS number(s): 03.75.Fi, 05.20.Gg, 05.30.Ch, 98.20.−d

I. INTRODUCTION

There have been a number of papers in the past [1–4], which discuss the properties of an ideal relativistic Bose gas with a nonzero chemical potential μ. Particular attention has been given to the behavior of the Bose-Einstein condensation and the nature of the phase transition in d space dimensions [4,5]. The basic work was done many years ago by Jüttner [6], Glaser [7], and more recently by Landsberg and Dunning-Davies [8] and Nieto [9]. These works were all done in the framework of the usual on-shell relativistic statistical mechanics.

To describe an ideal Bose gas in the grand canonical ensemble, the usual expression for the number of bosons N in relativistic statistical mechanics is

$$N = V \sum_{\mathbf{k}} n_k = V \sum_{\mathbf{k}} \frac{1}{e^{(E_k-\mu)/T}-1}, \tag{1.1}$$

where V is the system's three volume, $E_k = \sqrt{\mathbf{k}^2+m^2}$, and T is the absolute temperature [we use the system of units in which $\hbar = c = k_B = 1$; we also use the metric $g^{\mu\nu} = (-,+,+,+)$], and one must require that $\mu \leqslant m$ in order to ensure a positive-definite value for n_k, the number of bosons with momentum $\mathbf{k}$. Here, N is assumed to be a conserved quantity, so that it makes sense to talk of a box of N bosons. This can no longer be true once $T \gtrsim m$ [10]; at such temperatures quantum field theory requires consideration of particle-antiparticle pair production. If $\bar{N}$ is the number of antiparticles, then N and $\bar{N}$ by themselves are not conserved but $N - \bar{N}$ is. Therefore, the high-temperature limit of Eq. (1.1) is not relevant in realistic physical systems.

The introduction of antiparticles into the theory in a systematic way was made by Haber and Weldon [10,11]. They considered an ideal Bose gas with a conserved quantum number (referred to as ''charge'') Q, which corresponds to a quantum mechanical particle number operator commuting with the Hamiltonian H.[1] All thermodynamic quantities may be then obtained from the grand partition function $\mathrm{Tr}\{\exp[-(H-\mu Q)/T]\}$ considered as a function of T, V, and μ [12]. The formula for the conserved net charge, which replaces Eq. (1.1), reads[2] [10]

$$Q = V \sum_{\mathbf{k}} \left[\frac{1}{e^{(E_k-\mu)/T}-1} - \frac{1}{e^{(E_k+\mu)/T}-1} \right]. \tag{1.2}$$

In such a formulation a boson-antiboson system is described by only one chemical potential μ; the sign of μ indicates whether particles outnumber antiparticles or vice versa. The requirement that both n_k and $\bar{n}_k$ be positive definite leads to the important relation

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[1] In the manifestly covariant theory which we shall use in our study, this charge is naturally associated with particles and antiparticles which are distinguished by the off-shell structure, as in quantum field theory [10].

[2] The standard recipe according to which all additive thermodynamic quantities are reversed for antiparticles is used.

$$|\mu| \leqslant m. \tag{1.3}$$

The sum over **k** in Eq. (1.2) can be replaced by an integral, so that the charge density $\rho \equiv Q/V$ becomes

$$\rho = \frac{1}{2\pi^2}\int_0^\infty k^2 dk\left[\frac{1}{e^{(E_k-\mu)/T}-1} - (\mu \to -\mu)\right], \tag{1.4}$$

which is an implicit formula for μ as a function of ρ and T, and in the region $T \gg m$ it reduces to

$$\rho \cong \frac{\mu T^2}{3}. \tag{1.5}$$

For T above some critical temperature T_c, one can always find a μ ($|\mu| \leqslant m$) such that Eq. (1.4) holds. Below T_c, no such μ can be found, and Eq. (1.4) should be interpreted as the charge density of the excited states: $\rho - \rho_0$ where ρ_0 is the charge density of the ground state [10] [with $\mathbf{k}=0$; clearly, this state is given with zero weight in the integral (1.4)]. The critical temperature T_c at which Bose-Einstein condensation occurs corresponds to $\mu = \pm m$ (depending on the sign of ρ). Thus, one sets $|\mu| = m$ in Eq. (1.4) and obtains, via Eq. (1.5) (provided that $|\rho| \gg m^3$),

$$T_c = \sqrt{\frac{3|\rho|}{m}}. \tag{1.6}$$

Below T_c, Eq. (1.4) is an equation for $\rho - \rho_0$, so that the charge density in the ground state is

$$\rho_0 = \rho[1-(T/T_c)^2]. \tag{1.7}$$

It follows from Eq. (1.6) that any ideal Bose gas will condense at a relativistic temperature ($T_c \gg m$), provided that $|\rho| \gg m^3$.

Recently, an analogous phenomenon has been studied in relativistic quantum field theory [11,13–15]. For relativistic fields, Bose-Einstein condensation occurs at high temperatures and can be interpreted in terms of a spontaneous symmetry breaking [11].

In this paper we shall use a manifestly covariant form of statistical mechanics which has more general structure than the standard forms of relativistic statistical mechanics, but which reduces to those theories in a certain limit, to be described precisely below. In fact, it is one of the principle aims of this work to provide a mechanism for which this limit can be realized on a statistical level. The results that we obtain are different from those of the standard theories at high temperatures. These theories, which are characterized classically by mass-shell constraints, and the use, in quantum field theory, of fields which are constructed on the basis of on-mass-shell free fields, are associated with the statistical treatment of *world lines* and hence, considerable coherence (in terms of the macroscopic structure of whole world lines as the elementary objects of the theory) is implied. In nonrelativistic statistical mechanics, the elementary objects of the theory are points. The relativistic analogue of this essentially structureless foundation for a statistical theory is the set of points in spacetime, i.e., the so-called *events*, not the world lines (Currie, Jordan, and Sudarshan [16] have discussed the difficulty of constructing a relativistic mechanics on the basis of world lines).

The mass of particles in a mechanical theory of events is necessarily a dynamical variable, since the classical phase space of the relativistic set of events consists of the spacetime and energy-momentum coordinates $\{\mathbf{q}_i, t_i; \mathbf{p}_i, E_i\}$, with no *a priori* constraint on the relation between the $\mathbf{p}_i$ and the E_i, and hence, such theories are ‘‘off shell.’’ It is well known from the work of Newton and Wigner [17] that on-shell relativistic quantum theories such as those governed by Klein-Gordon or Dirac-type equations do not provide local descriptions (the wave functions corresponding to localized particles are spread out); for such theories the notion of ensembles over local initial conditions is difficult to formulate. The off-shell theory that we shall use here is, however, precisely local in both its first and second quantized forms [18,19].

The phenomenologial predictions of on-shell theories, furthermore, provide equations of state which appear to be too rigid. Shuryak [20] has obtained equations of state which are more realistic by taking into account the spectrum of mass as seen in the resonance spectrum of strongly interacting matter. We have shown [21] that Shuryak's ‘‘realistic’’ equation of state follows in a natural way from the mass distribution functions of the off-shell theory.

We finally remark that the standard formulations of quantum relativistic statistical mechanics, and quantum field theory at finite temperature, lack manifest covariance on a fundamental level. As for nonrelativistic statistical mechanics, the partition function is described by the Hamiltonian, which is not an invariant object, and hence, thermodynamic mean values do not have tensor properties. [One could consider the invariant $p_\mu n^\mu$ in place of the Hamiltonian [22], where n^μ is a unit four-vector; this construction (supplemented by a spacelike vector othogonal to n^μ) implies an induced representation for spacetime. The quantity that takes the place of the parameter t is then $x_\mu n^\mu$. This construction is closely related to the problem pointed out by Currie, Jordan, and Sudarshan [16], for which different world lines are predicted dynamically by the change in the form of the effective Hamiltonian in different frames.] Since the form of such a theory is not constrained by covariance requirements, its dynamical structure and predictions may be different than those for a theory which satisifies these requirements. For example, the canonical distribution of Pauli [23] for the free Boltzmann gas has a high-temperature limit in which the energy is given by $3k_BT$, which does not correspond to any known equipartition rule, but for the corresponding distribution for the manifestly covariant theory, the limit is $2k_BT$, corresponding to $\frac{1}{2}k_BT$ for each of the four relativistic degrees of freedom. For the quantum field theories at finite temperature, the path integral formulation [24] replaces the Hamiltonian in the canonical exponent by the Lagrangian due to the infinite product of factors $\langle\phi|\pi\rangle$ (transition matrix element of the canonical field and its conjugate required to give a Weyl-ordered Hamiltonian its numerical value). However, it is the t variable which is analytically continued to construct the finite-temperature canonical ensemble, completely removing the covariance of the theoretical framework. One may argue that some frame has to be chosen for the statistical theory to be developed, and perhaps even for temperature to have a meaning, but as we have remarked above, the requirement of relativistic covariance has dynamical consequences (note that the model Lagrangians used in the noncovariant formulations are established with the criterion of relativistic covariance in mind), and we argue that the choice of a frame, if necessary for some physical reason, such as the definition and measurement of temperature, should be made in the framework of a manifestly covariant structure.

We consider, in this paper, a relativistic Bose gas within the framework of a manifestly covariant relativistic statistical mechanics [25–27]. We obtain the expressions for characteristic thermodynamic quantities and show that they coincide quantitatively, in the narrow mass-width approximation, with those of the relativistic on-shell theory, except for the value of the average energy (which differs by a factor 2/3, as remarked above). We introduce antiparticles and discuss the high-temperature Bose-Einstein condensation in such a particle-antiparticle system. We show that it corresponds to a phase transition to a high-temperature form of the usual on-shell relativistic kinetic theory. In the following, we briefly review the manifestly covariant mechanics and quantum mechanics which forms the basis of our study of relativistic statistical mechanics.

In the framework of a manifestly covariant relativistic statistical mechanics, the dynamical evolution of a system of N particles, for the classical case, is governed by equations of motion that are of the form of Hamilton equations for the motion of N *events* which generate the space time trajectories (particle world lines) as functions of a continuous Poincaré-invariant parameter τ, called the ''historical time'' [28,29]. These events are characterized by their positions $q^\mu=(t,\mathbf{q})$ and energy-momenta $p^\mu=(E,\mathbf{p})$ in an $8N$-dimensional phase space. For the quantum case, the system is characterized by the wave function $\psi_\tau(q_1,q_2,\ldots,q_N)\in L^2(R^{4N})$, with the measure $d^4q_1d^4q_2\cdots d^4q_N\equiv d^{4N}q$, $(q_i\equiv q_i^\mu;\mu=0,1,2,3;$ $i=1,2,\ldots,N)$, describing the distribution of events, which evolves with a generalized Schrödinger equation [29]. The collection of events (called ''concatenation'' [30]) along each world line corresponds to a *particle,* and hence, the evolution of the state of the N-event system describes, *a posteriori*, the history in space and time of an N-particle system.

For a system of N interacting identical events (and, hence, particles), one takes [29]

$$K=\sum_i \frac{p_i^\mu p_{i\mu}}{2M}+V(q_1,q_2,\ldots,q_N), \tag{1.8}$$

where M is a given fixed parameter (an intrinsic property of the particles), with the dimension of mass, taken to be the same for all the particles of the system. The Hamilton equations are

$$\frac{dq_i^\mu}{d\tau}=\frac{\partial K}{\partial p_{i\mu}}=\frac{p_i^\mu}{M},$$

$$\frac{dp_i^\mu}{d\tau}=-\frac{\partial K}{\partial q_{i\mu}}=-\frac{\partial V}{\partial q_{i\mu}}. \tag{1.9}$$

In the quantum theory, the generalized Schrödinger equation

$$i\frac{\partial}{\partial\tau}\psi_\tau(q_1,q_2,\ldots,q_N)=K\psi_\tau(q_1,q_2,\ldots,q_N) \tag{1.10}$$

describes the evolution of the N-body wave function $\psi_\tau(q_1,q_2,\ldots,q_N)$. To illustrate the meaning of this wave function, consider the case of a single free event. In this case (1.10) has the formal solution

$$\psi_\tau(q)=(e^{-iK_0\tau}\psi_0)(q) \tag{1.11}$$

for the evolution of the free wave packet. Let us represent $\psi_\tau(q)$ by its Fourier transform, in the energy-momentum space:

$$\psi_\tau(q)=\frac{1}{(2\pi)^2}\int d^4p\,e^{-i(p^2/2M)\tau}e^{ip\cdot q}\psi_0(p), \tag{1.12}$$

where $p^2\equiv p^\mu p_\mu$, $p\cdot q\equiv p^\mu q_\mu$, and $\psi_0(p)$ corresponds to the initial state. Applying the Ehrenfest arguments of stationary phase to obtain the principal contribution to $\psi_\tau(q)$ for a wave packet at p_c^μ, one finds [p_c^μ is the peak value in the distribution $\psi_0(p)$]

$$q_c^\mu\simeq\frac{p_c^\mu}{M}\tau, \tag{1.13}$$

consistent with the classical equations (1.9). Therefore, the central peak of the wave packet moves along the classical trajectory of an event, i.e., the classical world line.

In the case that $p_c^0=E_c<0$, we see, as in Stueckelberg's classical example [28], that

$$\frac{dt_c}{d\tau}\simeq\frac{E_c}{M}<0.$$

It has been shown [30] in the analysis of an evolution operator with minimal electromagnetic interaction, of the form

$$K=\frac{[p-eA(q)]^2}{2M},$$

that the CPT-conjugate wave function is given by

$$\psi_\tau^{CPT}(t,\mathbf{q})=\psi_\tau(-t,-\mathbf{q}), \tag{1.14}$$

with $e\to -e$. For the free wave packet, one has

$$\psi_\tau^{CPT}(q)=\frac{1}{(2\pi)^2}\int d^4p\,e^{-i(p^2/2M)\tau}e^{-ip\cdot q}\psi_0(p). \tag{1.15}$$

The Ehrenfest motion in this case is

$$q_c^\mu\simeq-\frac{p_c^\mu}{M}\tau;$$

if $E_c<0$, we see that the motion of the event in the CPT-conjugate state is in the positive direction of time, i.e.,

$$\frac{dt_c}{d\tau}\simeq-\frac{E_c}{M}=\frac{|E_c|}{M}, \tag{1.16}$$

and one obtains the representation of a positive energy generic event with the opposite sign of charge, i.e., the antiparticle.

It is clear from the form of Eq. (1.10) that one can construct relativistic transport theory in a form analogous to that of the nonrelativistic theory; a relativistic Boltzmann equation and its consequences, for example, was studied in Ref. [26].

As a simple example of the implications of the classical dynamical equations (1.9), consider the problem of a relativistic particle in a uniform external ''gravitational'' field, with evolution function

$$K=\frac{p_\mu p^\mu}{2M}+Mgz \tag{1.17}$$

(the external potential breaks the invariance of the evolution function, but that will not affect the illustrative value of the example) with initial conditions $t(0)=0,\ \dot t(0)=\alpha$, $z(0)=h,\ \dot z(0)=0$, resulting in the solution

$$z=-\frac{1}{2}g\tau^2+h,\quad t=\alpha\tau+t_0,$$

$$E=Mc^2\alpha,\quad p_z=-Mg\tau. \tag{1.18}$$

The invariant variable τ replaces t in describing the dynamical evolution of the system. The generator of the motion

$$K=\frac{p_z^2-E^2/c^2}{2M}+mgz=\frac{1}{2}Mc^2\alpha^2=\text{ const}, \tag{1.19}$$

as required. The total energy of the particle in this case, including *both* increase of momentum and decrease of dynamical mass, is constant also. The effective particle mass $\widetilde{m}$ is given by

$$\widetilde{m}=\frac{1}{c}\sqrt{(E/c)^2-p_z^2}=M\alpha\sqrt{1-\frac{g^2\tau^2}{c^2\alpha^2}}. \tag{1.20}$$

Expanding this out in the nonrelativistic limit $c\to\infty$, one obtains [with $\tau^2=2(h-z)/g$]

$$\widetilde{m}\cong M\alpha-\frac{Mg}{\alpha c^2}(h-z), \tag{1.21}$$

and we recognize $Mg(h-z)/c^2$ as the mass shift induced by the potential term. The factor α arises due to the choice of initial conditions, i.e., for $\tau=0, \widetilde{m}=M\alpha$, and not M (for τ sufficiently large, under this unbounded potential, the quantity in the square root could become negative, and the particle could become tachyonic). Note that it is the *mass* of the particle which carries dynamical information (the total energy is constant, but the mass is ''redshifted'' by the potential) and that has the correspondence with nonrelativistic energy, through the mass-energy equivalence, that we observe in the laboratory. This point is discussed in more detail in, for example, Refs. [31,32].

II. IDEAL RELATIVISTIC BOSE GAS WITHOUT ANTIPARTICLES

To describe an ideal gas of events obeying Bose-Einstein statistics in the grand canonical ensemble, we use the expression for the number of events found in [25],

$$N=V^{(4)}\sum_{k^\mu}n_{k^\mu}=V^{(4)}\sum_{k^\mu}\frac{1}{e^{(E-\mu-\mu_K m^2/2M)/T}-1}, \tag{2.1}$$

where $V^{(4)}$ is the system's four volume and $m^2\equiv-k^2=-k^\mu k_\mu$; μ_K is an additional mass potential [25], which arises in the grand canonical ensemble as the derivative of the free energy with respect to the value of the dynamical evolution function K, interpreted as the invariant mass of the system. In the kinetic theory [25], μ_K enters as a Lagrange multiplier for the equilibrium distribution for K, as μ is for N, and $1/T$ for E. We shall see, in the following, how μ_K plays a fundamental role in determining the structure of the mass distribution. In order to simplify subsequent considerations, we shall take it to be a fixed parameter.

To ensure a positive-definite value for n_{k^μ}, the number density of bosons with four-momentum k^μ, we require that

$$m-\mu-\mu_K\frac{m^2}{2M}\geqslant 0. \tag{2.2}$$

The discriminant for the left-hand side (lhs) of the inequality must be non-negative, i.e.,

$$\mu\leqslant\frac{M}{2\mu_K}. \tag{2.3}$$

For such μ, Eq. (2.2) has the solution

$$m_1\equiv\frac{M}{\mu_K}\left(1-\sqrt{1-\frac{2\mu\mu_K}{M}}\right)\leqslant m\leqslant\frac{M}{\mu_K}\left(1+\sqrt{1-\frac{2\mu\mu_K}{M}}\right)$$
$$\equiv m_2. \tag{2.4}$$

For small $\mu\mu_K/M$, the region (2.4) may be approximated by

$$\mu\leqslant m\leqslant\frac{2M}{\mu_K}. \tag{2.5}$$

One sees that μ_K determines an upper bound of the mass spectrum, in addition to the usual lower bound $m\geqslant\mu$. In fact, small μ_K admits a very large range of off-shell mass, and, hence, can be associated with the presence of strong interactions [33].

Replacing the sum over k^μ (2.1) by an integral, one obtains, for the density of events per unit spacetime volume $n\equiv N/V^{(4)}$ [34],

$$n=\frac{1}{4\pi^3}\int_{m_1}^{m_2}dm\int_{-\infty}^{\infty}d\beta\frac{m^3\sinh^2\beta}{e^{(m\cosh\beta-\mu-\mu_K m^2/2M)/T}-1}, \tag{2.6}$$

where m_1 and m_2 are defined in Eq. (2.4), and we have used the parametrization [26]

$$p^0=m\cosh\beta,$$

$$p^1=m\sinh\beta\sin\theta\cos\phi,$$

$$p^2=m\sinh\beta\sin\theta\sin\phi,$$

$$p^3=m\sinh\beta\cos\theta,$$

$$0\leqslant\theta<\pi,\quad 0\leqslant\phi<2\pi,\quad -\infty<\beta<\infty.$$

In this paper we shall restrict ourselves to the case of high temperature alone:

$$T\gg\frac{M}{\mu_K}. \tag{2.7}$$

It is then possible to use, for simplicity, the Maxwell-Boltzmann form for the integrand, and to rewrite Eq. (2.6) in the form

$$n=\frac{e^{\mu/T}}{4\pi^3}\int_{m_1}^{m_2}m^3dm\int_{-\infty}^{\infty}\sinh^2\beta d\beta e^{-m\cosh\beta/T}e^{\mu_K m^2/2MT}, \tag{2.8}$$

which reduces, upon integrating out β, to [27]

$$n=\frac{Te^{\mu/T}}{4\pi^3}\int_{m_1}^{m_2}dm m^2K_1\left(\frac{m}{T}\right)e^{\mu_K m^2/2MT}, \tag{2.9}$$

where $K_\nu(z)$ is the Bessel function of the third kind (imaginary argument). Since $\mu\leqslant m\leqslant m_2\leqslant 2M/\mu_K$,

$$\frac{\mu_K m^2}{2MT}\leqslant\frac{\mu_K(2M/\mu_K)^2}{2MT}=\frac{2M}{T\mu_K}\ll 1, \tag{2.10}$$

in view of Eq. (2.7), and also

$$\frac{\mu}{T}\leqslant\frac{m}{T}\leqslant\frac{2M}{T\mu_K}\ll 1. \tag{2.11}$$

Therefore, one can neglect the exponentials in Eq. (2.9), and for $K_1(m/T)$ use the asymptotic formula [35]

$$K_\nu(z)\sim\frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu},\quad z\ll 1. \tag{2.12}$$

Then, we obtain

$$n\cong\frac{T^2}{4\pi^3}\int_{m_1}^{m_2}dm m=\frac{T^2}{2\pi^3}\left(\frac{M}{\mu_K}\right)^2\sqrt{1-\frac{2\mu\mu_K}{M}}. \tag{2.13}$$

From this equation, one can identify the high-temperature mass distribution for the system we are studying, so that, now,

$$\langle m^\ell\rangle=\frac{\int_{m_1}^{m_2}dm m^{\ell+1}}{\int_{m_1}^{m_2}dm m}=\frac{2}{\ell+2}\,\frac{m_2^{\ell+2}-m_1^{\ell+2}}{m_2^2-m_1^2}. \tag{2.14}$$

In particular,

$$\langle m\rangle=\frac{4}{3}\frac{M}{\mu_K}\left(1-\frac{\mu\mu_K}{2M}\right), \tag{2.15}$$

$$\langle m^2\rangle=2\left(\frac{M}{\mu_K}\right)^2\left(1-\frac{\mu\mu_K}{M}\right). \tag{2.16}$$

Extracting the joint distribution for β and m from Eq. (2.8) in the same way, we also obtain the average values of the energy and the square of the energy for high T. The average energy is given by

$$\langle E\rangle\equiv\langle m\cosh\beta\rangle\cong\frac{\int_{m_1}^{m_2}m^4dm\sinh^2\beta\cosh\beta d\beta e^{-m\cosh\beta/T}}{\int_{m_1}^{m_2}m^3dm\sinh^2\beta d\beta e^{-m\cosh\beta/T}}. \tag{2.17}$$

Integrating out β, one finds

$$\langle E\rangle\cong\frac{1}{4T}\frac{\int_{m_1}^{m_2}dm m^4[K_3(m/T)-K_1(m/T)]}{\int_{m_1}^{m_2}dm m^2K_1(m/T)}. \tag{2.18}$$

It is seen, with the help of Eq. (2.12), that it is possible to neglect K_1 in comparison with K_3 in the numerator of Eq. (2.18) and obtain, via Eq. (2.12),

$$\langle E\rangle\cong\frac{1}{4T}\frac{\int_{m_1}^{m_2}dm m^4K_3(m/T)}{\int_{m_1}^{m_2}dm m^2K_1(m/T)}\simeq 2T, \tag{2.19}$$

in agreement with Refs. [25–27]. Similarly, one obtains

$$\langle E^2\rangle\equiv\langle m^2\cosh^2\beta\rangle\cong\frac{\int_{m_1}^{m_2}m^5dm\sinh^2\beta\cosh^2\beta d\beta e^{-m\cosh\beta/T}}{\int_{m_1}^{m_2}m^3dm\sinh^2\beta d\beta e^{-m\cosh\beta/T}}$$

$$=\frac{\int_{m_1}^{m_2}dm[m^4K_1(m/T)+3Tm^3K_2(m/T)]}{\int_{m_1}^{m_2}dm m^2K_1(m/T)}$$

$$\cong 3T\frac{\int_{m_1}^{m_2}dm m^3K_2(m/T)}{\int_{m_1}^{m_2}dm m^2K_1(m/T)}\simeq 6T^2. \tag{2.20}$$

Let us assume, as is generally done, that the average $\langle p^\mu p^\nu\rangle$ has the form

$$\langle p^\mu p^\nu\rangle = a u^\mu u^\nu + b g^{\mu\nu}, \tag{2.21}$$

where $u^\mu=(1,\mathbf{0})$ in the local rest frame. The values of a and b can then be calculated as follows: for $\mu=\nu=0$ one has $\langle (p^0)^2\rangle = a-b$, while contraction of Eq. (2.21) with $g^{\mu\nu}$ gives $-g^{\mu\nu}\langle p_\mu p_\nu\rangle = a-4b$. The use of the expressions (2.20) for $\langle (p^0)^2\rangle \equiv \langle E^2\rangle$, and Eq. (2.16) for $-g^{\mu\nu}\langle p_\mu p_\nu\rangle \equiv \langle m^2\rangle$ yields

$$\begin{cases} a-b=6T^2, \\ a-4b=2\left(\dfrac{M}{\mu_K}\right)^2 (1-\mu\mu_K/M), \end{cases}$$

so that

$$a = 8T^2 - \frac{2}{3}\left(\frac{M}{\mu_K}\right)^2\left(1-\frac{\mu\mu_K}{M}\right), \tag{2.22}$$

$$b = 2T^2 - \frac{2}{3}\left(\frac{M}{\mu_K}\right)^2\left(1-\frac{\mu\mu_K}{M}\right). \tag{2.23}$$

For $T \gg M/\mu_K$, it is possible to take $a\cong 8T^2$, $b\cong 2T^2$, and obtain, therefore,

$$\langle p^\mu p^\nu\rangle \cong 8T^2 u^\mu u^\nu + 2T^2 g^{\mu\nu}. \tag{2.24}$$

To find the expressions for the pressure and energy density in our ensemble, we study the particle energy-momentum tensor defined by the relation [26]

$$T^{\mu\nu}(q) = \sum_i \int d\tau \frac{p_i^\mu p_i^\nu}{M/\mu_K} \delta^4(q-q_i(\tau)), \tag{2.25}$$

in which M/μ_K is the value around which the mass of the bosons making up the ensemble is distributed, i.e., it corresponds to the limiting mass-shell value when the inequality (2.3) becomes equality. Upon integrating over a small space time volume $\triangle V$ and taking the ensemble average, Eq. (2.25) reduces to [26]

$$\langle T^{\mu\nu}\rangle = \frac{T_{\triangle V}}{M/\mu_K} n \langle p^\mu p^\nu\rangle. \tag{2.26}$$

In this formula, $T_{\triangle V}$ is the average passage interval in τ for the events which pass through the small (typical) four volume $\triangle V$ in the neighborhood of the R^4 point. The four volume $\triangle V$ is the smallest that can be considered a macrovolume in representing the ensemble. Using the standard expression

$$\langle T^{\mu\nu}\rangle = (p+\rho)u^\mu u^\nu + p g^{\mu\nu}, \tag{2.27}$$

where p and ρ are the particle pressure and energy density, respectively, we obtain

$$p \equiv p(\mu) = \frac{T_{\triangle V}}{\pi^3}\frac{M}{\mu_K}\sqrt{1-\frac{2\mu\mu_K}{M}}T^4, \quad \rho = 3p. \tag{2.28}$$

To interpret these results we calculate the particle number density per unit three-volume. The particle four current is given by the formula [26]

$$J^\mu(q) = \sum_i \int d\tau \frac{p_i^\mu}{M/\mu_K}\delta^4(q-q_i(\tau)), \tag{2.29}$$

which, upon integrating over a small space time volume and taking the average, reduces to

$$\langle J^\mu\rangle = \frac{T_{\triangle V}}{M/\mu_K} n\langle p^\mu\rangle; \tag{2.30}$$

then

$$N_0 \equiv \langle J^0\rangle = \frac{T_{\triangle V}}{M/\mu_K} n\langle E\rangle, \tag{2.31}$$

so that

$$N_0 \equiv N_0(\mu) = \frac{T_{\triangle V}}{\pi^3}\frac{M}{\mu_K}\sqrt{1-\frac{2\mu\mu_K}{M}}T^3, \tag{2.32}$$

and we recover the ideal gas law

$$p = N_0 T. \tag{2.33}$$

Since, in view of Eq. (2.4),

$$\frac{2M}{\mu_K}\sqrt{1-\frac{2\mu\mu_K}{M}} = \triangle m$$

is the width of the mass distribution around the value M/μ_K, Eqs. (2.28) and (2.32) can be rewritten as

$$p = \frac{T_{\triangle V}\triangle m}{2\pi^3}T^4, \quad \rho = 3p,$$

$$N_0 = \frac{T_{\triangle V}\triangle m}{2\pi^3}T^3. \tag{2.34}$$

In Ref. [36] we obtained the formulas for thermodynamic variables, under the assumption of narrow mass width, which depend on $T_{\triangle V}\triangle m$ as well; the requirement that these results coincide with those of the usual on-shell theories implies the relation[3]

$$T_{\triangle V}\triangle m = 2\pi. \tag{2.35}$$

One can understand this relation, up to a numerical factor, in terms of the uncertainty principle [rigorous in the $L^2(R^4)$ quantum theory] $\triangle E\cdot\triangle t \gtrsim 1/2$. Since the time interval for the particle to pass the volume $\triangle V$ (this smallest macroscopic volume is bounded from below by the size of the wave packets), $\triangle t \cong E/M\triangle\tau$, and the dispersion of E due to the mass distribution is $\triangle E \sim m\triangle m/E$, one obtains a lower bound for $T_{\triangle V}\triangle m$ of order unity.

Thus, with Eq. (2.35) holding, the formulas (2.34) reduce to

[3] In cgs units, this relation has a factor $\hbar/c^2$ on the right-hand side.

$$p=\frac{T^4}{\pi^2}, \quad \rho = 3p, \tag{2.36}$$

$$N_0=\frac{T^3}{\pi^2}, \tag{2.37}$$

which are the standard expressions for high temperature [37]. The formulas for characteristic thermodynamic quantities and the equation of state for a relativistic gas of off-shell events have the same form as those of the relativistic gas of on-shell particles. They coincide with them [under the condition (2.35)] in the narrow mass-shell limit, except for the expression for the average energy which takes the value $2T$ in the relativistic gas of events, in contrast to $3T$, as for the high-temperature limit of the usual theory [23]. Experimental measurement of average energy at high temperature can, therefore, affirm (or negate) the validity of the off-shell theory. There seems to be no empirical evidence which distinguishes between these results at the present time. The quantity $\sigma = M_0c^2/k_BT$, a parameter which distinguishes the relativistic regime from the nonrelativistic regime (see, e.g. [25],) is very large for M_0 of the order of the pion mass, at ordinary temperatures; the ultrarelativistic limit corresponding to σ small becomes a reasonable approximation for $T \gtrsim 10^{12}$ K.

III. ANTIPARTICLES AND CONDENSATION

The introduction of antiparticles into the theory as the CPT conjugate of negative energy events leads, by application of the arguments of Haber and Weldon [10], or Actor [38], to a change in sign of μ in the distribution function for antiparticles. We, therefore, write down the following relation which represents the analogue of the formula (1.2):[4]

$$N=V^{(4)}\sum_{k^\mu}\left[\frac{1}{e^{(E-\mu-\mu_K m^2/2M)/T}-1} - \frac{1}{e^{(E+\mu-\mu_K m^2/2M)/T}-1}\right]. \tag{3.1}$$

With respect to the determination of the sign of the second term, let us consider a space time picture in which we have many world lines, generated by events moving monotonically in the positive t direction. The addition of a particle-antiparticle pair which annihilates corresponds to the addition of a world line which is generated by an event initially moving in the positive direction of time to some upper bound t_0, where annihilation takes place, and returning in the negative direction of time. At times later than t_0, the total particle number is unaffected. At times earlier than t_0, a particle and an antiparticle are added to the total particle number. Since, as also assumed by Haber and Weldon [10], the total particle number is a conserved quantity, the antiparticle trajectory must be counted with a sign opposite to that of the particle trajectory. The second term in Eq. (3.1), counting antiparticles must therefore carry a negative sign. We require that both $n_{k\mu}$ terms in Eq. (3.1) be positive definite. In this way, we obtain the two quadratic inequalities,

$$m-\mu-\mu_K\frac{m^2}{2M}\geqslant 0,$$

$$m+\mu-\mu_K\frac{m^2}{2M}\geqslant 0, \tag{3.2}$$

which give the following relation representing the nonnegativeness of the corresponding discriminants:

$$-\frac{M}{2\mu_K}\leqslant\mu\leqslant\frac{M}{2\mu_K}. \tag{3.3}$$

It then follows that we must consider the intersection of the ranges of validity of the two inequalities (3.2). Indeed, if each inequality is treated separately, there would be some values of m for which one and not another would be physically acceptable. One finds the bounds of this intersection region by solving these inequalities, and obtains[5]

$$\frac{M}{\mu_K}\left(1-\sqrt{1-\frac{2|\mu|\mu_K}{M}}\right)\leqslant m\leqslant\frac{M}{\mu_K}\left(1+\sqrt{1-\frac{2|\mu|\mu_K}{M}}\right), \tag{3.4}$$

which for small $|\mu|\mu_K/M$ reduces, as in the no-antiparticle case (2.5), to

$$|\mu|\leqslant m\leqslant\frac{2M}{\mu_K}. \tag{3.5}$$

Replacing the summation in Eq. (3.1) by integration, we obtain a formula for the number density:

$$n=\frac{1}{4\pi^3}\int_{m_1}^{m_2}m^3dm\int_{-\infty}^{\infty}\sinh^2\beta d\beta \times\left[\frac{1}{e^{(m\cosh\beta-\mu-\mu_K m^2/2M)/T}-1} - \frac{1}{e^{(m\cosh\beta+\mu-\mu_K m^2/2M)/T}-1}\right], \tag{3.6}$$

where m_1 and m_2 are defined in Eq. (3.4), which for large T reduces, as above, to

$$n=\frac{e^{\mu/T}-e^{-\mu/T}}{4\pi^3}T\int_{m_1}^{m_2}dm\, m^2K_1\left(\frac{m}{T}\right)e^{\mu_K m^2/2MT}.$$

[4]As for the nonrelativistic theory, the ‘‘free’’ distribution functions describe quasiparticles in a form which takes interactions into account entering through the chemical potential. By definition, good quasiparticles are not frequently emitted or absorbed; we, therefore, consider the (quasi)particles and antiparticles as two species. Since the particle number is determined by the derivative of the free energy with respect to the chemical potential, μ must change sign for the antiparticles [10]. Similarly, the average mass (squared) is obtained by the derivative with respect to μ_K [25]; since the mass (squared) of the antiparticle is positive, μ_K does not change sign.

[5]This is actually the solution of one of the inequalities (3.2) (the most restrictive), depending on the sign of μ.

Now, using the estimates (2.10) and (2.11), and $\sinh(\mu/T)\cong\mu/T$ for $\mu/T\ll 1$, we obtain [in place of Eq. (2.13)] the net event charge

$$n=\frac{1}{\pi^3}\left(\frac{M}{\mu_K}\right)^2\sqrt{1-\frac{2|\mu|\mu_K}{M}}\mu T. \tag{3.7}$$

The pressure and energy density are obtained by the sum of particle and antiparticle contributions [proportional to $\exp(\pm\mu/T)$], with the number density (3.7). To second order in $(\mu/T)^2$, one finds

$$p=2p(|\mu|),\quad \rho=2\rho(|\mu|),$$

where $p(\mu)$ and $\rho(\mu)$ are given by Eq. (2.28) with μ replaced by $|\mu|$. On the other hand, from Eqs. (2.31) and (3.7), one finds

$$N_0=2\frac{T_{\triangle V}}{\pi^3}\frac{M}{\mu_K}\sqrt{1-\frac{2|\mu|\mu_K}{M}}\mu T^2, \tag{3.8}$$

where the factor of $2\mu/T$, as compared to Eq. (2.32), arises from the *difference* between the factors $\exp(\pm\mu/T)$ (the sign of μ indicates whether particles or antiparticles predominate). One then obtains the following expressions for the Bose gas including both particles and antiparticles[6] (here, $\triangle m$ is not necessarily small):

$$p=\frac{T_{\triangle V}\triangle m}{2\pi}\frac{2T^4}{\pi^2},\quad \rho=3p, \tag{3.9}$$

$$N_0=\frac{T_{\triangle V}\triangle m}{2\pi}\frac{2T^2}{\pi^2}\mu. \tag{3.10}$$

We now wish to show that the dynamical properties of the current, which follow from the relativistic canonical equations of motion, are consistent with the thermodynamic relation

$$N_0=\frac{N}{V}, \tag{3.11}$$

where N is the number of bosons in a three-dimensional box of volume V. Since the event-number density n is, by definition,

$$n=\frac{N}{V^{(4)}}=\frac{N}{V\triangle t},$$

where $\triangle t$ is the (average) extent of the ensemble along the q^0 axis [as in our discussion after Eq. (2.35)], one has

$$N_0=n\triangle t. \tag{3.12}$$

The equation of motion (1.9) for q^0 [with M/μ_K, the central value of the mass distribution, instead of M, which corresponds to a change of scale parameter in the expression (1.8) for the generalized Hamiltonian K],

$$\frac{dq_i^0}{d\tau}=\frac{p_i^0}{M/\mu_K},$$

upon averaging over the whole ensemble, reduces to

$$\frac{\triangle t}{T_{\triangle V}}=\frac{\langle E\rangle}{M/\mu_K}, \tag{3.13}$$

where $T_{\triangle V}$ is the average passage interval in τ used in the previous consideration. Then, in view of Eqs. (3.12) and (3.13), one obtains the Eq. (2.31).

Since in the particle-antiparticle case, $N_{\rm rel}\equiv N-\overline{N}$, where N and $\overline{N}$ are the numbers of particles and antiparticles, respectively, is a conserved quantity, according to the arguments of Haber and Weldon [10] pointed out in Sec. I, and our discussion above, $N_0=N_{\rm rel}/V$ is also a conserved quantity, so that it makes sense to talk of $|N_{\rm rel}|$ bosons in a spatial box of the volume V. Therefore, in Eq. (3.10), N_0 is a conserved quantity, so that, the dependence of μ on temperature is defined by (we assume that N_0 is continuous at the phase transition)

$$\mu=\frac{2\pi}{T_{\triangle V}\triangle m}\frac{\pi^2N_0}{2T^2}. \tag{3.14}$$

For T above some critical temperature, one can always find μ satisfying Eq. (3.3) such that the relation (3.14) holds; no such μ can be found for T below the critical temperature. The value of the critical temperature is defined by putting $|\mu|=M/2\mu_K$ in Eq. (3.14). In the narrow mass-shell limit, inserting Eq. (2.35), one obtains

$$T_c=\pi\sqrt{\frac{|N_0|}{M/\mu_K}}. \tag{3.15}$$

For $|\mu|=M/2\mu_K$, the width of the mass distribution is zero, in view of Eq. (3.4), and hence the ensemble approaches a distribution sharply peaked at the mass-shell value M/μ_K. The fluctuations $\delta m=\sqrt{\langle m^2\rangle-\langle m\rangle^2}$ also vanish. Indeed, as follows from Eqs. (2.15) and (2.16) with μ replaced by $|\mu|$, and Eqs. (3.14) and (3.15),

$$\delta m=\frac{M}{3\mu_K}\sqrt{2-\left(\frac{T_c}{T}\right)^2-\left(\frac{T_c}{T}\right)^4}, \tag{3.16}$$

so that, at $T=T_c$, $\delta m=0$. It follows from Eq. (3.16) that for T in the vicinity of T_c ($T\geqslant T_c$),

[6]If we did not neglect indistinguishability of bosons at high temperature, we would obtain, instead of Eq. (2.37) [36], $N_0=(T^3/\pi^2){\rm Li}_3(e^{\mu/T})$, where ${\rm Li}_\nu(z)\equiv\Sigma_{s=1}^\infty z^s/s^\nu$ is the polylogarithm [39], so that, for the system including both particles and antiparticles, $N_0=(T^3/\pi^2)[{\rm Li}_3(e^{\mu/T})-{\rm Li}_3(e^{-\mu/T})]$. It then follows from the properties of the polylogarithms [39] that, for $x\equiv|\mu|/T\ll 1$, ${\rm Li}_3(e^x)-{\rm Li}_3(e^{-x})\cong(\pi^2/3)x$, so that, we would obtain, instead of Eq. (3.10), $N_0=\mu T^2/3$, which coincides with Haber and Weldon's equation (1.5).

$$\delta m \simeq \frac{M}{3\mu_K}\sqrt{\frac{6}{T_c}}\sqrt{T-T_c}, \tag{3.17}$$

as for a second order phase transition, for which fluctuations go to zero smoothly.

We note that Eq. (2.36) and Eq. (2.37) do not contain explicit dependence on the chemical potential, and hence no phase transition is induced. In fact, at lower temperature (or small μ_K), one or the other of the particle or antiparticle distribution dominates, and one returns to the case of the high temperature strongly interacting gas [40]. The remaining phase transition is the usual low-temperature Bose-Einstein condensation discussed in the textbooks.

One sees, with the help of Eq. (3.4), that the expression for n (3.7) can be rewritten as

$$n = \frac{1}{2\pi^3}\frac{M}{\mu_K}\triangle m \mu T; \tag{3.18}$$

since at $T=T_c$, $\triangle m=0$, it follows that $n=0$ at all temperatures below T_c. Therefore, the behavior of an ultrarelativistic Bose gas including both particles and antiparticles, which is governed by the relation (3.14), can be thought of as a special type of Bose-Einstein condensation to a ground state with $p^\mu p_\mu = -(M/\mu_K)^2$ [this ground state occurs with zero weight in the integral (3.6)]. In such a formulation, every state with temperature $T>T_c$, given by Eq. (3.6), should be considered as an *off-shell* excitation of the on-shell ground state. At $T=T_c$, all such excitations freeze out and the distribution becomes strongly peaked at a definite mass, i.e., the system undergoes a phase transition to the on-shell sector. Note that, for $n=0$, Eq. (3.12) gives $\triangle t=\infty$. Then, since $\langle E\rangle \sim T$, one obtains from Eq. (3.13) that $T_{\triangle V}=\infty$ [this relation can be also obtained from Eq. (2.35) for $\triangle m=0$], which means that in the mean, all the events become particles.

As the distribution function enters the on-shell phase at $T=T_c$, the underlying off-shell theory describes fluctuations around the sharp mean mass. This phenomenon provides a mechanism, based on equilibrium statistical mechanics, for understanding how the general off-shell theory is constrained to the neighborhood of a sharp universal mass shell for each particle type. At temperatures below T_c, the results of the theory for the main thermodynamic quantities coincide with those of the usual on-shell theories.

In order that our considerations be valid, the relation $T_c \gg M/\mu_K$ must hold; this relation reduces, with Eq. (3.15), to

$$|N_0| \gg \frac{1}{\pi^2}\left(\frac{M}{\mu_K}\right)^3. \tag{3.19}$$

For $M/\mu_K \sim m_\pi \simeq 140$ MeV, this inequality yields $N_0 \gg 3\times 10^5$ MeV^3. Taking $N_0 \sim 5\times 10^6$ MeV^3, which corresponds to temperature ~ 350 MeV, in view of Eq. (2.37), one gets $T_c \sim 550$ MeV$\simeq 4m_\pi$.

If μ_K is very small, it is difficult to satisfy Eq. (3.19) and the possibility of such a phase transition may disappear. This case corresponds, as noted above, to that of strong interactions and is discussed in a succeeding paper [40].

IV. CONCLUDING REMARKS

We have considered the ideal relativistic Bose gas within the framework of a manifestly covariant relativistic statistical mechanics, taking account antiparticles. We have shown that in such a particle-antiparticle system, at some critical temperature T_c, a special type of relativistic Bose-Einstein condensation sets in, which corresponds to a phase transition from the sector of relativistic mass distributions to a sector in which the boson mass distribution peaks at a definite mass. The results which can be computed from the latter coincide with those obtained in a high-temperature limit of the usual on-shell relativistic theory.

The relativistic Bose-Einstein condensation in particle-antiparticle system considered in the present paper can represent (as for the Galilean limit $c\rightarrow\infty$ [36]) a possible mechanism of acquiring a given sharp mass distribution by the particles of the system, as a phase transition between the corresponding sectors of the theory. Since this phase transition can occur at an ultrarelativistic temperature, it might be relevant to cosmological models. The relativistic Bose-Einstein condensation considered in the present paper may also have properties which could be useful in the study of relativistic boson stars [41]. These and other aspects of the theory are now under further investigation.

The extension and generalization of Bose-Einstein condensation to curved spacetimes and spacetimes with boundaries, for which the work reported here may have constructive application, have also been the subject of much study. The nonrelativistic Bose gas in the Einstein-static universe was treated in Ref. [1]. The generalization to relativistic scalar fields was given in Refs. [42,43]. The extension to higher-dimensional spheres was given in Ref. [44]. Bose-Einstein condensation on hyperbolic manifolds [45], and in the Taub universe [46], has also been considered. More recently, by calculating the high-temperature expansion of the thermodynamic potential when boundaries are present, Kirsten [47] examined Bose-Einstein condensation in certain cases. Later work of Toms [48] showed how to interpret Bose-Einstein condensation in terms of symmetry breaking, in the manner of flat space time calculations [11,13]. The most recent study by Lee *et al.* [49] showed how interacting scalar fields can be treated. Bose-Einstein condensation for self-interacting complex scalar fields was considered in Ref. [50]. It is to be hoped that the techniques developed here can contribute to the development of this subject as well.

ACKNOWLEDGMENTS

We wish to thank the Los Alamos National Laboratory and the Ilya Prigogine Center for Studies in Statistical Mechanics and Complex Systems at Austin, Texas for their hospitality in enabling us to collaborate closely in the final stages of this work.

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