Bosonic description of spinning strings in 2+1 dimensions

B. Harms and A. Stern
Department of Physics and Astronomy, University of Alabama, Tuscaloosa, Alabama 35487
(Received 17 January 1996)

We write down a general action principle for spinning strings in (2+1)-dimensional space-time *without introducing Grassmann variables*. The action is written solely in terms of coordinates taking values in the 2+1 Poincaré group, and it has the usual string symmetries; i.e., it is invariant under (a) diffeomorphisms of the world sheet and (b) Poincaré transformations. [S0556-2821(96)00518-8]

PACS number(s): 11.25.Sq, 11.10.Kk, 11.30.Cp

It is well known that the classical spin of relativistic particles can be described by using either classical or pseudoclassical variables [1]. Although the description in terms of pseudoclassical variables is commonly given, the description in terms of classical (or bosonic) variables has the advantage that from it one can obtain all irreducible representations of spinning particles upon quantization.

For the case of spinning strings, pseudoclassical descriptions are well known, but a general description solely in terms of classical variables is not. We rectify the situation in this paper. It is hoped that, in analogy with the particle case, the quantization of the system presented here will lead to all representations of spinning strings.

For reasons of simplicity, we shall examine strings in 2+1 space-time only. (The string action has a particularly elegant form in 2+1 dimensions due to the existence of a nondegenerate scalar product on the Poincaré algebra $\underline{\mathrm{ISO}(2,1)}$ [2].) Our system contains the general description of (spinless) strings due to Balachandran, Lizzi, and Sparano [3] as a special case.[1] Furthermore, it can be generalized to an arbitrary number of space-time dimensions, and also to spinning membranes and p-branes. We shall discuss such generalizations in a later article [4].

Analogous to the bosonic formulation of a spinning particle in 2+1 dimensions (cf. [5]), the spinning string action can be written on the (2+1)-dimensional Poincaré group manifold ISO(2,1). That is, the string variables are maps from the two-dimensional world sheet to ISO(2,1). We shall express the action for strings in (2+1)-dimensional Minkowski space in terms of these variables and their derivatives. It will be seen to be invariant under (a) diffeomorphisms of the world sheet and (b) global Poincaré transformations. The strings can be classified in terms of ISO(2,1) orbits, and we shall find that certain orbits correspond to strings with a nonvanishing spin current. Lastly, we shall show how to embed such strings in curved space-time, the resulting action being invariant under (a), and now (b) local Poincaré transformations.

We begin with some mathematical preliminaries. We denote the string variables by g. g can be decomposed into an SO(2,1) matrix $\Lambda=\{\Lambda^i{}_j,\ i,j=0,1,2\}$ and an SO(2,1) vector $x=\{x^i,\ i=0,1,2\}$. The latter will serve as the Minkowski coordinates of the string. Under the left action of the Poincaré group, $g=(\Lambda,x)$ transforms according to the usual semidirect product rule:

$$g\to h\circ g=(\theta,y)\circ(\Lambda,x)=(\theta\Lambda,\theta x+y). \tag{1}$$

We let t_i and u_i, $i=0,1,2$ denote a basis for the Lie algebra $\underline{\mathrm{ISO}(2,1)}$. For their commutation relations we can take

$$[t_i,t_j]=\epsilon_{ijk}t^k,\quad [t_i,u_j]=\epsilon_{ijk}u^k,\quad [u_i,u_j]=0, \tag{2}$$

where we raise and lower indices using the Minkowski metric $[\eta_{ij}]=\mathrm{diag}(-1,1,1)$, and we define the totally antisymmetric tensor ϵ_{ijk} such that $\epsilon^{012}=1$.

A left invariant Maurer-Cartan form can be expanded in this basis as follows:

$$g^{-1}dg=\frac{1}{2}\,\epsilon^{ijk}(\Lambda^{-1}d\Lambda)_{ij}t_k+(\Lambda^{-1}dx)^i u_i. \tag{3}$$

It is easy to check that $g^{-1}dg$ is unchanged under the left action of the Poincaré group (1). Under the right action of the Poincaré group,

$$g\to g\circ h^{-1}=(\Lambda,x)\circ(\theta^{-1},-\theta^{-1}y)=(\Lambda\theta^{-1},x-\Lambda\theta^{-1}y), \tag{4}$$

and, consequently, the Maurer-Cartan form transforms as follows:

$$g^{-1}dg\to{}^h[g^{-1}dg]=\frac{1}{2}\,\epsilon^{ijk}(\Lambda^{-1}d\Lambda)_{ij}\,{}^h[t_k]+(\Lambda^{-1}dx)^i\,{}^h[u_i], \tag{5}$$

where

$${}^h[t_i]=\theta^j{}_i t_j+\epsilon^{jkl}\theta_{ki}y_j u_l,\quad {}^h[u_i]=\theta^{\ j}{}_i u_j. \tag{6}$$

Here ${}^h[t_i]$ and ${}^h[u_i]$ denote basis vectors which are transformed under the adjoint action by $h\in\mathrm{ISO}(2,1)$. They are given explicitly for $h=(\theta,y)$. These equations can be utilized to define the adjoint action $v\to{}^h[v]$ by $h\in\mathrm{ISO}(2,1)$ on any vector $v=\alpha^i t_i+\beta^i u_i$ in $\underline{\mathrm{ISO}(2,1)}$.

[1] Spinning strings were also considered in [3] using a Wess-Zumino term. Here we shall show that there are more possibilities for including spin.

Two scalar products exist on $\underline{\mathrm{ISO}(2,1)}$ which are invariant under the above defined adjoint action. We denote them with angular brackets and parentheses. The former satisfies

$$\langle t_i, u_j\rangle = \eta_{ij}, \quad \langle t_i, t_j\rangle = \langle u_i, u_j\rangle = 0, \tag{7}$$

and is nondegenerate. The latter satisfies

$$(t_i, t_j) = \eta_{ij}, \quad (u_i, t_j) = (u_i, u_j) = 0, \tag{8}$$

and is degenerate. Thus for any two vectors v and v' in $\underline{\mathrm{ISO}(2,1)}$, the invariance property implies that $\langle {}^h[v], {}^h[v']\rangle = \langle v, v'\rangle$ and $({}^h[v], {}^h[v']) = (v, v')$.

The nondegenerate scalar product $\langle\ \rangle$ was utilized previously in writing down the action for a relativistic spinning (or spinless) particle, which we now review [5]. The expression for the action is linear in the Maurer-Cartan form and it is therefore invariant under (left) Poincaré transformations, as well as diffeomorphisms of the particle world line. The action is just

$$S_{\text{particle}} = \int \langle K, g^{-1}dg\rangle, \tag{9}$$

where here g is a function of the world line and K is a constant vector in $\underline{\mathrm{ISO}(2,1)}$. The direction of K in $\underline{\mathrm{ISO}(2,1)}$ determines whether the particle is massive, massless, or tachyonic, and whether it is spinning or spinless. Actually, for this purpose, it is sufficient to specify the ISO(2,1) orbit on which K lies. This is because both $K = K_0$ and $K = {}^h[K_0]$, $h \in \mathrm{ISO}(2,1)$, lead to the same classical equations of motion. This follows from the invariance property of the scalar product $\langle\ \rangle$:

$$\langle K, g^{-1}dg\rangle = \langle {}^h[K], {}^h[g^{-1}dg]\rangle = \langle {}^h[K], g'^{-1}dg'\rangle, \tag{10}$$

where $g' = g \circ h^{-1}$. Thus the action is invariant under $K \to {}^h[K]$ and the change of coordinates $g \to g'$. Now to specify an ISO(2,1) orbit we can use the two invariants $\langle K, K\rangle$ and (K, K). Spin is associated with the former invariant, and we shall find an analogous result for strings as well.

For the case $K = mt_0 - \kappa u_0$, we end up with the known [5] bosonic description of a massive spinning particle. That choice corresponds to both invariants being nonvanishing: $\langle K, K\rangle = m\kappa$ and $(K, K) = -m^2$. For this case the integrand in the action can be expressed according to

$$\langle K, g^{-1}dg\rangle = m\Lambda^i_0 dx_i + \kappa(\Lambda^{-1}d\Lambda)_{12}. \tag{11}$$

The equations of motion for the particle are easy to obtain. Transforming g according to: $g \to (\mathbb{1} + \epsilon) \circ g$, where ϵ is an infinitesimal element of ISO(2,1), induces the following variation of the Maurer-Cartan form:

$$\delta(g^{-1}dg) = {}^{g^{-1}}[d\epsilon]. \tag{12}$$

Then

$$\delta S_{\text{particle}} = \int \langle K, {}^{g^{-1}}[d\epsilon]\rangle = \int \langle {}^g[K], d\epsilon\rangle = -\int \langle d({}^g[K]), \epsilon\rangle, \tag{13}$$

where we have used the invariance property of the scalar product. The equations of motion thus state that ${}^g[K] = p^i t_i - j^i u_i$ are constants of the motion. Upon once again choosing $K = mt_0 - \kappa u_0$, we get the following expressions for these constants:

$$p^i = m\Lambda^i{}_0, \quad j^i = m\epsilon^{ijk}x_j\Lambda_{k0} + \kappa\Lambda^i{}_0. \tag{14}$$

The former can be identified with the momenta of the particle, while the latter can be identified with the angular momenta, the first term being the orbital angular momenta and the second being the spin. The spin is thus proportional to κ which is nonvanishing when $\langle K, K\rangle$ is.

We now apply an analogous procedure to the description of spinning strings. Once again we shall express the action in terms of the Maurer-Cartan form, and consequently, it will be invariant under (left) Poincaré transformations. The action should be quadratic in $g^{-1}dg$ in order for it to also be invariant under diffeomorphisms of the world sheet. We then take the tensor product of two such Maurer-Cartan forms and write

$$S_{\text{string}} = \int \langle \mathcal{K}, g^{-1}dg \otimes g^{-1}dg\rangle. \tag{15}$$

Now $\mathcal{K}$ is a constant antisymmetric tensor with values in $\underline{\mathrm{ISO}(2,1)} \otimes \underline{\mathrm{ISO}(2,1)}$. Analogous to the particle case, it is sufficient to specify the ISO(2,1) orbit on which $\mathcal{K}$ lies. This is because both $\mathcal{K} = \mathcal{K}_0$ and $\mathcal{K} = {}^h[\mathcal{K}_0]$, $h \in \mathrm{ISO}(2,1)$ lead to the same classical equations of motion, which is once again due to the invariance property of the scalar product $\langle\ \rangle$:

$$\langle \mathcal{K}, g^{-1}dg \otimes g^{-1}dg\rangle = \langle {}^h[\mathcal{K}], {}^h[g^{-1}dg] \otimes {}^h[g^{-1}dg]\rangle = \langle {}^h[\mathcal{K}], g'^{-1}dg' \otimes g'^{-1}dg'\rangle, \tag{16}$$

where $g' = g \circ h^{-1}$. Thus the action is invariant under $\mathcal{K} \to {}^h[\mathcal{K}]$ and the change of coordinates $g \to g'$. Now to specify the ISO(2,1) orbit we can use its invariants, among which are $\langle \mathcal{K}, \mathcal{K}\rangle$ and $(\mathcal{K}, \mathcal{K})$.

The string action (15) has already been studied [3] for the case $\mathcal{K} = \epsilon^{ijk} n_k t_i \otimes t_j$. For that choice

$$\langle \mathcal{K}, g^{-1}dg \otimes g^{-1}dg\rangle = \epsilon_{ijk}(\Lambda n)^i dx^j dx^k. \tag{17}$$

(The wedge product between the one forms dx^j and dx^k is understood.) It remains to specify the constant vector n_i. For n_i spacelike, lightlike, or timelike one recovers the Nambu string, the null string [3,6] or the tachyonic string, respectively. Thus to get the Nambu string we may write $n_i = 1/(4\pi\alpha')\delta_{i2}$. To see how this works we may extremize the action with repect to the variations of Λ: $\delta\Lambda_{ij} = \epsilon_{ikl}\Lambda^k{}_j\zeta^l$, ζ^l being infinitesimal. This leads to the equations of motion

$$\epsilon^{ijk}\Lambda_{j2}V_k = 0, \quad V_k = \epsilon_{ijk}dx^i dx^j. \tag{18}$$

Then V_i is parallel to Λ_{i2}. Upon fixing the normalization, we get $\Lambda_{i2} = V_i/\sqrt{V_j V^j}$. Substituting this result back into the integrand (17) yields the usual form for the Nambu action:

$$S_{\text{Nambu}}=\frac{1}{4\pi\alpha'}\int\sqrt{V_jV^j}$$
$$=\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{(\partial_0\vec{x})^2(\partial_1\vec{x})^2-(\partial_0\vec{x}\cdot\partial_1\vec{x})^2},\tag{19}$$

where $\sigma=(\sigma_0,\sigma_1)$ parametrize the world sheet and $\partial_0=\partial/\partial\sigma_0$ and $\partial_1=\partial/\partial\sigma_1$.

We now return to the general form for the string action (15). The equations of motion are obtained in identical fashion as was done for the particle. Transforming g according to $g\rightarrow(\mathbb{1}+\epsilon)\circ g$, where ϵ is an infinitesimal element of $\underline{\text{ISO}(2,1)}$, once again induces the variation (12) of the Maurer-Cartan form. Then

$$\delta S_{\text{string}}=2\int\langle\mathcal{K},{}^{g^{-1}}[d\epsilon]\otimes g^{-1}dg\rangle$$
$$=2\int\langle{}^{g}[\mathcal{K}],d\epsilon\otimes dgg^{-1}\rangle,\tag{20}$$

where we have used the invariance property of the scalar product and the identity

$${}^{g^{-1}}[dgg^{-1}]=g^{-1}dg.\tag{21}$$

Upon integrating by parts we arrive at the equations of motion

$$d\langle{}^{g}[\mathcal{K}],T_A\otimes dgg^{-1}\rangle=0,\tag{22}$$

where the T_A's denote the generators of ISO(2,1). These equations state that there are six conserved currents. For $T_A=u_i$, we have

$$\partial_\alpha P_i^\alpha=0,\quad P_i^\alpha=\epsilon^{\alpha\beta}\langle{}^{g}[\mathcal{K}],u_i\otimes\partial_\beta gg^{-1}\rangle,\tag{23}$$

which we identify with the momentum current conservation. (Here $\alpha,\beta,...$ denote world sheet indices.) For $T_A=t_i$, we have

$$\partial_\alpha J_i^\alpha=0,\quad J_i^\alpha=\epsilon^{\alpha\beta}\langle{}^{g}[\mathcal{K}],t_i\otimes\partial_\beta gg^{-1}\rangle,\tag{24}$$

which we identify with the angular momentum current conservation.

We now examine the conserved currents P_i^α and J_i^α for several choices of $\mathcal{K}$.

Case 1. $\mathcal{K}=\mathcal{K}_1=\epsilon^{ijk}n_k\ t_i\otimes t_j$. This is the case we considered earlier which contains the Nambu string, as well as the null and tachyonic strings. It corresponds to the ISO(2,1) orbit with $\langle\mathcal{K},\mathcal{K}\rangle=0$ and $(\mathcal{K},\mathcal{K})=-2n_in^i$. Here we get that

$${}^{g}[\mathcal{K}_1]=\epsilon^{ijk}(\Lambda n)_k\ t_i\otimes t_j-\epsilon^{ijk}(\Lambda n)_lx^lx_k\ u_i\otimes u_j$$
$$+(\Lambda n)_lx^l(t_i\otimes u^i-u_i\otimes t^i)$$
$$+(\Lambda n)^ix^j(u_i\otimes t_j-t_j\otimes u_i).\tag{25}$$

Using this and the expression for the right invariant Maurer-Cartan form

$$dgg^{-1}=\frac{1}{2}\epsilon^{ijk}(d\Lambda\Lambda^{-1})_{ij}t_k+[dx^i-(d\Lambda\Lambda^{-1}x)^i]u_i,\tag{26}$$

we compute the currents

$$P_i^\alpha=P_{(1)i}^\alpha=\epsilon^{\alpha\beta}\bigg(\frac{1}{2}\epsilon_{mjk}(\partial_\beta\Lambda\Lambda^{-1})^{jk}[\delta_i^m(\Lambda n)_lx^l-(\Lambda n)^mx_i]$$
$$+\epsilon_{ijk}[\partial_\beta x^j-(\partial_\beta\Lambda\Lambda^{-1}x)^j](\Lambda n)^k\bigg),\tag{27}$$

$$J_i^\alpha=J_{(1)i}^\alpha=-\epsilon^{\alpha\beta}((\partial_\beta\Lambda\Lambda^{-1})_i{}^j(\Lambda n)_lx^lx_j$$
$$+[\partial_\beta x^j-(\partial_\beta\Lambda\Lambda^{-1}x)_j][\delta_i^j(\Lambda n)_lx^l-(\Lambda n)_ix^j]).\tag{28}$$

These two currents can be shown to be related by

$$J_{(1)i}^\alpha-\epsilon_{ijk}x^jP_{(1)}^{k\alpha}=0.\tag{29}$$

We therefore argue that for such strings the angular momentum current consists only of an orbital term, and that no spin is present. This is a well known result for the Nambu string.

Case 2. $\mathcal{K}=\mathcal{K}_2=1/2\ \chi^{ij}(u_i\otimes t_j-t_j\otimes u_i)$. Since $\mathcal{K}$ is antisymmetric with respect to the exchange of the two vector spaces, this is the most general ansatz which is linear in u_i and in t_j. It corresponds to the ISO(2,1) orbit with $\langle\mathcal{K},\mathcal{K}\rangle=\chi_{ij}\chi^{ij}$ and $(\mathcal{K},\mathcal{K})=0$. From it we get

$${}^{g}[\mathcal{K}_2]=\tfrac{1}{2}(\Lambda\chi\Lambda^{-1})^{ij}(u_i\otimes t_j-t_j\otimes u_i)$$
$$+\tfrac{1}{2}[\text{tr}\chi\ x_k-(\Lambda\chi\Lambda^{-1})^l{}_kx_l]\epsilon^{ijk}u_i\otimes u_j.\tag{30}$$

We now obtain the currents

$$P_i^\alpha=P_{(2)i}^\alpha=-\frac{1}{4}\epsilon^{\alpha\beta}\epsilon_{jkl}(\Lambda^{-1}\partial_\beta\Lambda)^{jk}(\chi\Lambda^{-1})^l{}_i,\tag{31}$$

$$J_i^\alpha=J_{(2)i}^\alpha=\frac{1}{2}\epsilon^{\alpha\beta}([\Lambda\chi\partial_\beta(\Lambda^{-1}x)]_i$$
$$-[\text{tr}\chi\ (\partial_\beta\Lambda\Lambda^{-1})_{ji}+(\Lambda\chi\partial_\beta\Lambda^{-1})_{ji}]x^j).\tag{32}$$

Now the analogue of Eq. (29) is no longer true: i.e.,

$$S_i^\alpha=J_{(2)i}^\alpha-\epsilon_{ijk}x^jP_{(2)}^{k\alpha}\neq0.\tag{33}$$

We then conclude that a spin current is present in this case.

Case 3. $\mathcal{K}=\mathcal{K}_3=\epsilon^{ijk}\nu_k\ u_i\otimes u_j$. Here both invariants vanish, $\langle\mathcal{K},\mathcal{K}\rangle=0$ and $(\mathcal{K},\mathcal{K})=0$. The currents P_i^α and J_i^α are trivially conserved in this case. This is because the integrand in Eq. (15) can be expressed as an exact two form on $\underline{\text{ISO (2,1)}}$:

$$\langle\mathcal{K},g^{-1}dg\otimes g^{-1}dg\rangle=\frac{1}{2}\epsilon_{ijk}(\Lambda^{-1}d\Lambda)^{ij}(\Lambda^{-1}d\Lambda\nu)^k$$
$$=-d\epsilon_{ijk}\nu^k(\Lambda^{-1}d\Lambda)^{ij}.\tag{34}$$

Although this term does not contribute to the classical equations of motion, it can affect the quantum dynamics. Furthermore, it is known to be associated with the θ vacua of string theory [3,7].

Case 4. $\mathcal{K}=\mathcal{K}_1+\mathcal{K}_2$. This defines the most general classical system with the string action given by Eq. (15). It therefore contains the case of the Nambu string. Now both invariants can be nonzero, $\langle\mathcal{K},\mathcal{K}\rangle=\chi_{ij}\chi^{ij}$ and $(\mathcal{K},\mathcal{K})=-2n_i n^i$. The conserved currents are now given by

$$P_i^\alpha=P_{(1)i}^\alpha+P_{(2)i}^\alpha, \tag{35}$$

$$J_i^\alpha=J_{(1)i}^\alpha+J_{(2)i}^\alpha. \tag{36}$$

For such strings, we can identify both an orbital and a spin angular momentum current, i.e., $J_i^{\ \alpha}=L_i^{\ \alpha}+S_i^{\ \alpha}$. The spin current $S_i^{\ \alpha}$ is defined in Eq. (33), while the orbital angular momentum $L_i^{\ \alpha}$ is given by

$$L_i^\alpha=J_{(1)i}^\alpha+\epsilon_{ijk}x^j P_{(2)}^{k\alpha}. \tag{37}$$

From the above discussion we conclude that a spin current is present for the case of ISO(2,1) orbits with $\langle\mathcal{K},\mathcal{K}\rangle\neq0$. If we include a Wess-Zumino term as is done in Ref. [3] an additional term of the form $\epsilon_{ijk}\epsilon^{\alpha\beta}(\Lambda^{-1}\partial_\beta\Lambda)^{jk}$ contributes to the angular momentum current.

It is easy to embed our spinning strings in curved space-time. For this the action should be invariant under local Poincaré transformations:

$$g\rightarrow h_L\circ g, \tag{38}$$

where like g,h_L are functions on the two-dimensional world sheet, taking values in ISO(2,1). We recall that the action (15) was instead invariant under global Poincaré transformations. To elevate it to a local invariance, we replace $g^{-1}dg$ by

$$g^{-1}Dg=g^{-1}dg+{}^{g^{-1}}[A], \tag{39}$$

where $A=\omega^i t_i+e^i u_i$ are the connection one-forms for ISO(2,1) evaluated on the string world sheet. Under Poincaré gauge transformations (38),

$$A\rightarrow{}^{h_L}[A]-dh_L h_L^{\ -1}, \tag{40}$$

and as a result Dgg^{-1} is invariant. Then

$$\mathcal{S}_{\rm string}=\int\langle\mathcal{K},g^{-1}Dg\otimes g^{-1}Dg\rangle \tag{41}$$

is gauge invariant, and hence gives the string action in curved space-time.

The equations of motion obtained by varying g now state that the momentum and angular momentum currents are covariantly conserved. To see this we can again use Eq. (12) along with $\delta({}^{g^{-1}}[A])={}^{g^{-1}}[A,\epsilon]$. Then

$$\delta\mathcal{S}_{\rm string}=2\int\langle\mathcal{K},{}^{g^{-1}}[D\epsilon]\otimes g^{-1}Dg\rangle$$

$$=2\int\langle{}^g[\mathcal{K}],D\epsilon\otimes Dgg^{-1}\rangle, \tag{42}$$

where we have used the invariance property of the scalar product, $D\epsilon=d\epsilon+[A,\epsilon]$ and the identity ${}^{g^{-1}}[Dgg^{-1}]=g^{-1}Dg$. Upon integrating by parts we now arrive at the equations of motion

$$d\langle{}^g[\mathcal{K}],T_A\otimes Dgg^{-1}\rangle-\langle{}^g[\mathcal{K}],[A\otimes\mathbb{1},T_A\otimes Dgg^{-1}]\rangle=0, \tag{43}$$

where the T_A's once again denote the generators of ISO(2,1). We then get the following generalizations of Eqs. (23) and (24):

$$\partial_\alpha\mathcal{P}_i^\alpha+\epsilon_{ijk}\omega_\alpha^j\mathcal{P}^{\alpha k}=0,\quad \mathcal{P}_i^\alpha=\epsilon^{\alpha\beta}\langle{}^g[\mathcal{K}],u_i\otimes\mathcal{D}_\beta gg^{-1}\rangle, \tag{44}$$

$$\partial_\alpha\mathcal{J}_i^\alpha+\epsilon_{ijk}(\omega_\alpha^j\mathcal{J}^{\alpha k}+e_{\ \alpha}^j\mathcal{P}^{\alpha k})=0,$$

$$\mathcal{J}_i^\alpha=\epsilon^{\alpha\beta}\langle{}^g[\mathcal{K}],t_i\otimes\mathcal{D}_\beta gg^{-1}\rangle, \tag{45}$$

ω_α^j, $e_{\ \alpha}^j$ and $\mathcal{D}_\beta gg^{-1}$ denoting the world sheet components of the one forms ω^j, e^j, and Dgg^{-1}, respectively. The currents play the role of sources for the ISO(2,1) curvature. For this we can take the Chern-Simons action [2] for the fields. Then $\mathcal{P}_i^\alpha$ is a source for the SO(2,1) curvature, while $\mathcal{J}_i^\alpha$-is a source for the torsion.

We are grateful for discussions with R. Casadio. We were supported in part by the U.S. Department of Energy under Contract No. DE-FG05-84EP-40141.

[1] A. P. Balachandran, G. Marmo, B.-S. Skagerstam, and A. Stern, Phys. Lett. **89B**, 199 (1980); *Gauge Symmetries and Fibre Bundles, Applications to Particle Dynamics*, Lecture Notes in Physics Vol. 188 (Springer-Verlag, Berlin, 1982).

[2] E. Witten, Nucl. Phys. **B311**, 46 (1988).

[3] A. P. Balachandran, F. Lizzi, and G. Sparano, Nucl. Phys. **B263**, 608 (1986); **B277**, 359 (1986); A. P. Balachandran, F. Lizzi, D. Sorkin, and G. Sparano, *ibid.* **B287**, 508 (1987).

[4] A. Stern, ''Bosonization and current algebra of spinning strings,'' Alabama Report No. UAHEP968 (unpublished).

[5] B.-S. Skagerstam and A. Stern, Int. J. Mod. Phys. A **5**, 1575 (1990).

[6] A. Schild, Phys. Rev. D **16**, 1722 (1977); F. Lizzi and G. Sparano, Phys. Lett. B **232**, 311 (1989); I. A. Bandos, D. P. Sorokin, M. Tonon, and D. V. Volkov, *ibid.* **319**, 445 (1993).

[7] P. O. Mazur and V. P. Nair, Nucl. Phys. **B284**, 146 (1987); V. G. J. Rodgers, Mod. Phys. Lett. A **7**, 1001 (1992); J. Pawelczyk, Phys. Lett. B **311**, 98 (1993); R. Efraty, *ibid.* **322**, 84 (1994).