Formation and evaporation of primordial black holes in scalar-tensor gravity theories

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We discuss how the constraints on models of the early Universe derived from considering the formation and evaporation of primordial black holes (PBH's) are modified if the gravitational ''constant'' varies with time. The modifications depend crucially on whether G has the same value everywhere (so that it maintains the evolving background value) or whether the local value within the black hole is preserved (corresponding to what is termed ''gravitational memory''). The simplest varying-G scenario is Brans-Dicke theory, in which one has a scalar field ϕ (with $G\sim\phi^{-1}$) and a coupling constant ω_0. In this case, solar system observations imply that ω_0 is very large and the modifications to the PBH constraints are negligible whether or not there is gravitational memory. However, in more general scalar-tensor theories, the coupling ''constant'' $\omega(\phi)$ varies, so ω may be large today but small at early times. In this case, the value of G and the dynamics of the early Universe could be strongly modified during the period when PBH's form. We present a class of scalar-tensor models which exhibit this feature and discuss how the PBH constraints are modified according to whether or not one has gravitational memory. [S0556-2821(96)01218-0]

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I. INTRODUCTION

Primordial black holes (PBH's) may have formed in the early Universe as a result of initial inhomogeneities [1,2], inflation [3,4], phase transitions [5], bubble collisions [6,7] or the decay of cosmic loops [8]. Such black holes are of special interest because they are the only ones which could be small enough to have evaporated by the present epoch as a result of quantum emission [9]. There are severe constraints on the fraction of the Universe which can have gone into PBH's at early times—even for those which do not evaporate—and this in turn places important restrictions on models of the early Universe (see [10] for a recent review).

Recently, Barrow [11] has pointed out that these constraints would be modified if the gravitational ''constant'' were to vary with time. This happens in various cosmological scenarios, in particular, with the scalar-tensor theory, in which one has a scalar field ϕ (with G varying as ϕ^{-1}), a coupling function $\omega(\phi)$ and a cosmological ''constant'' $\lambda(\phi)$. This reduces to the Brans-Dicke theory [12] when $\lambda=0$ and ω is constant ($\omega=\omega_0$) and, in this case, solar system observations imply $\omega_0>500$ [13]. However, there are also more complicated theories in which ω_0 exceeds 500 today but is small at early epochs. Thus G could deviate considerably from its present value at the time when PBH's formed. As Barrow emphasized, there are then two possibilities: in what he terms ''scenario A,'' G has the same value everywhere at a given time, so that PBH evaporation is always determined by its current value; in ''scenario B,'' the local value of G within the black hole is always preserved, so that the evaporation is determined by the value of G when the PBH formed (i.e., there is ''gravitational memory'').

The purpose of the present paper is to discuss how the various PBH constraints are modified in these two situations and to examine how PBH limits can constrain scalar-tensor theories. During most of the period in which PBH's form and evaporate, the Universe is radiation dominated and this simplifies the situation considerably. This is because, in any scalar-tensor gravity theory, the Friedmann radiation universe of general relativity is a particular exact solution of the field equations with ϕ constant and, in general, this solution is a late-time attractor in radiation-dominated models. However, there are at least two situations in which the early Universe will not be radiation dominated and this is why the standard PBH constraints are modified. First, the Universe becomes pressureless after the radiation density falls below the matter density, around 10^4 yr after the big bang, and this may also apply for some earlier period if the density was ever dominated by nonrelativistic particles [5,14]. Secondly, the Universe could have been vacuum dominated at sufficiently early times. This certainly happens in the inflationary scenario but it is also a generic feature of scalar-tensor theories since the scalar field itself induces vacuum domination for some initial period. In some theories (including variants of Brans-Dicke itself) the scalar field may even cause the Universe to bounce (so that it avoids an initial singularity), in which case PBH formation will be suppressed at sufficiently early times.

The plan of this paper is as follows. In Sec. II we review how the formation and evaporation of PBH's proceeds in the standard cosmological scenario (with no variation of G) and summarize the constraints on the fraction of the Universe going into them. In Sec. III we examine some of the cosmological consequences of scalar-tensor theories, focusing first on Brans-Dicke itself and then on more general scalar-tensor theories. In either case we show that, to a good approximation, the history of the Universe can be modeled by a Brans-Dicke vacuum phase followed by a standard Friedmann radiation phase. In Sec. IV we consider the formation and evaporation of PBH's in Brans-Dicke and more general scalar-tensor theories, covering both scenario A and scenario B. In Sec. V we infer how the standard constraints on the fraction of the Universe going into PBH's are modified.

II. PBH EVAPORATION IN THE STANDARD SCENARIO

Let us first recall how the formation and evaporation of PBH's proceeds in the standard cosmological scenario with G constant. Providing the equation of state is hard ($p=\gamma\rho$ with $0<\gamma<1$), PBH's which form from inhomogeneities at time t must have an initial mass of order the particle horizon mass:

$$M_i \approx M_H(t) \approx G^{-1}t \approx 10^5 (t/\mathrm{s}) M_\odot, \tag{2.1}$$

where we choose units with $c=\hbar=1$. This is because they must be bigger than the Jeans mass $\sim\gamma^{3/2}M_H$ in order to collapse against the pressure but smaller than M_H itself in order not to be a separate Universe [15]. PBH's forming at the Planck time ($t_{\mathrm{Pl}}\sim G^{1/2}\sim 10^{-43}$ s) would, therefore, have the Planck mass ($M_{\mathrm{Pl}}\sim G^{-1/2}\sim 10^{-5}$ g), whereas those forming at 10^{-23} s would have the mass ($M\sim 10^{15}$ g) required for PBH's which evaporate at the present epoch. The probability of PBH formation on any mass scale depends on the amplitude of the density fluctuations on that scale when it enters the particle horizon and to form PBH's over an extended mass range this amplitude must be scale invariant [2]. PBH's forming via phase transitions or bubble collisions or the collapse of cosmic loops might have a somewhat smaller mass than is indicated by Eq. (2.1) but they would only form over a limited period and hence over a limited mass range.

In the standard picture a black hole emits particles like a blackbody with temperature [9]

$$T(M)=(8\pi GM)^{-1} \approx 10 (M/10^{15}\ \mathrm{g})^{-1}\ \ \mathrm{MeV}. \tag{2.2}$$

It therefore loses mass at a rate

$$-dM/dt = 4\pi R_g^2 f(M) a T^4 = \alpha f(M) M^{-2}, \tag{2.3}$$

where $R_g(M)=2GM$ is the radius of the black hole, the factor $f(M)$ measures the number of particle species which can be emitted [i.e., the number of species with rest mass below $T(M)$] and

$$\alpha \approx M_{\mathrm{Pl}}^3/t_{\mathrm{Pl}} \approx G^{-2} \approx 10^{-70} M_\odot^3\ \ \mathrm{s}^{-1} \tag{2.4}$$

$f(M)$ only has a weak dependence on M and, if we give it the value $f(M_i)$ associated with the initial mass M_i, Eq. (2.3) can be integrated to give

$$M^3 = M_i^3 + 3\alpha f(M_i)(t_i - t), \tag{2.5}$$

where t_i is the formation time of the PBH. The black hole therefore evaporates completely ($M=0$) at a time

$$\tau = [3\alpha f(M_i)]^{-1} M_i^3 + t_i. \tag{2.6}$$

Note that Eqs. (2.1) and (2.4) imply that the t_i term is negligible for $M_i \gg M_{\mathrm{Pl}}$. The PBH's evaporating at the present epoch ($t_0 \approx 10^{17}$ s) have a mass

$$M_{\mathrm{crit}} = (3\alpha t_0 f_{\mathrm{crit}})^{1/3} \approx (t_0 f_{\mathrm{crit}}/t_{\mathrm{Pl}})^{1/3} M_{\mathrm{Pl}} \approx 10^{15}\ \mathrm{g}, \tag{2.7}$$

where the factor $f_{\mathrm{crit}}=f(M_{\mathrm{crit}})$ has been taken to be about 3, corresponding to the emission of zero-rest-mass particles (i.e., photons, relativistic neutrinos, and gravitons).

We now review the various constraints that can be placed on the number of PBH's [10,16,17]. If the current density parameter associated with PBH's forming at time t is $\Omega_B(t)$, then the fraction of the Universe's mass going into such PBH's at that time is [2]

$$\beta(t) = [\Omega_B(t)/\Omega_R](1+z)^{-1} \approx 10^{-5} \Omega_B(t/s)^{1/2} \quad (t<t_e), \tag{2.8}$$

where z is the redshift associated with time t and $\Omega_R \sim 10^{-4}$ is the microwave background density. Equation (2.8) assumes that the PBH's form before the time of matter-radiation equality (t_e) and also neglects dependencies on the Hubble parameter H_0. The mass of the PBH's forming at time t is given by Eq. (2.1), so the fraction of the Universe going into PBH's of mass M is

$$\beta(M) \approx 10^{-8} \Omega_B(M)(M/M_\odot)^{1/2}. \tag{2.9}$$

Any constraint on $\Omega_B(M)$ therefore imposes a constraint on $\beta(M)$.

Observations of the cosmological deceleration parameter imply $\Omega_B(M)<1$ over all mass ranges for which PBH's have not yet evaporated, so Eqs. (2.7) and (2.9) imply

$$\beta(M) < 10^{-17}(M/M_{\mathrm{crit}})^{1/2} \quad (M>M_{\mathrm{crit}}). \tag{2.10}$$

Considerably stronger limits apply for $M\sim M_{\mathrm{crit}}$ since such PBH's generate a γ-ray background, most of the energy appearing at around 100 MeV [18]. If the fraction of the emitted energy which goes into photons is ε_γ, the density of the radiation at this energy is expected to be $\Omega_\gamma = \varepsilon_\gamma \Omega_B(M_{\mathrm{crit}})$. Since $\varepsilon_\gamma \sim 0.1$ and the observed γ-ray background density around 100 MeV is $\Omega_\gamma \sim 10^{-9}$, we infer $\Omega_B < 10^{-8}$ and Eq. (2.9) then implies

$$\beta(M_{\mathrm{crit}}) < 10^{-25}. \tag{2.11}$$

(See [17] for a more precise limit.) Constraints on $\beta(M)$ associated with the evaporation of PBH's smaller than M_{crit} derive from entropy production [19]

$$\beta(M) < 10^{-8}(M/10^{11}\ \mathrm{g})^{-1} \quad (M<10^{11}\ \mathrm{g}), \tag{2.12}$$

distortion of the microwave background spectrum [20]

$$\beta(M) < 10^{-18}(M/10^{11}\ \mathrm{g})^{-1} \quad (10^{11}\ \mathrm{g}<M<10^{13}\ \mathrm{g}), \tag{2.13}$$

and cosmological nucleosynthesis effects

$$\beta(M) < \begin{cases} 10^{-15}(M/10^9\ \mathrm{g})^{-1} & (10^9 g<M<10^{13}\ \mathrm{g}), \quad (2.14\mathrm{a})\\ 10^{-21}(M/10^{10}\ \mathrm{g})^{1/2} & (M>10^{10}\ \mathrm{g}), \quad (2.14\mathrm{b})\\ 10^{-16}(M/10^9\ \mathrm{g})^{-1/2} & (10^9 g<M<10^{10}\ \mathrm{g}). \quad (2.14\mathrm{c}) \end{cases}$$

The latter constraints are associated with (a) the increase of the photon-to-baryon ratio by PBH photons emitted after nucleosynthesis [21]; (b) the photodissociation of deuterium by such photons [22]; and (c) the modification of the neutron-to-proton ratio by PBH nucleons emitted before nucleosyn-

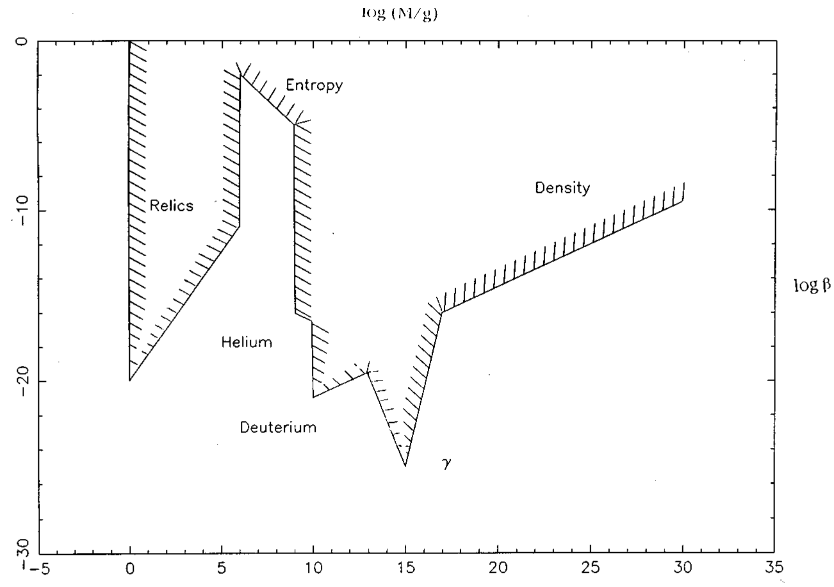

FIG. 1. The constraints on the fraction of the Universe going into PBH's of mass M in the standard radiation-dominated model with constant G. These derive from upper limits on the cosmological density parameter, the γ-ray background intensity, primordial entropy production, modifications to primordial helium and deuterium production, and the density of stable Planck mass relics.

thesis [23]. If evaporating black holes leave stable Planck mass relicts, as some people have argued [4,24], then one also gets a limit

$$\beta(M) < 10^{-27}(M/10^{-5}\ \mathrm{g})^{3/2} \quad (M < 10^{11} M_{\mathrm{Pl}} \sim 10^{6}\ \mathrm{g}). \tag{2.15}$$

These limits on $\beta(M)$ are summarized in Fig. 1. If the equation of state in the early Universe is ever soft, then the limits are modified (as discussed in [5]) but we do not consider this case here.

III. MODELS WITH VARIABLE G

Most studies of cosmological models with varying gravitational coupling have focused upon the simplest case of Brans-Dicke (BD) theory. This particular scalar-tensor theory was created in the early 1960's in response to claims that the deflection of light by the Sun was observed to lie outside the range predicted by general relativity (GR). Subsequently, those observations were found to be affected by significant errors in the determination of the solar diameter, so there was no further need to consider deviations from GR. However, recently scalar-tensor gravity theories have reemerged in the context of early universe studies. They provide scalar field sources for inflation [7,25–29] and also serve as models for the low-energy behavior of string cosmologies [30,31] or the dimensional reduction of higher-dimensional cosmologies [32].

The early studies of BD theories have therefore been extended to a more general class of scalar-tensor theories with a coupling function $\omega(\phi)$ governing the interactions with the scalar field ϕ. This was first introduced by Bergmann [33], Wagoner [34], and Nordvedt [35]. We will assume that the effective gravitational ''constant'' still varies as ϕ^{-1} in such theories. Nordvedt [35] and Will [13] argue that one should add a first order correction factor $[4+2\omega(\phi)]/[3+\omega(\phi)]$ after ϕ^{-1}, based on the study of light travel in the weak field limit. However, this factor should not appear in the cosmological context because, for any choice of $\omega(\phi)$, one replaces the term $8\pi G\rho$ in the Friedmann equation with $8\pi\phi^{-1}\rho$, so the effective ''cosmological'' G is just ϕ^{-1}.

Scalar-tensor gravity theories have been formulated in two different ways. Steinhardt and Accetta [29] express the Lagrangian of the theory in the form

$$L = -f(\Phi)R + \tfrac{1}{2}\partial_a\Phi\,\partial^a\Phi + 16\pi L_m, \tag{3.1}$$

where Φ is a scalar field, $f(\Phi)$ is the coupling to the four-curvature, and L_m is the Lagrangian of the remaining matter fields. If we define a new scalar field $\phi = f(\Phi)$ and a coupling function

$$\omega(\phi) = \tfrac{1}{2} f (df/d\Phi)^{-2}, \tag{3.2}$$

then Eq. (3.1) becomes

$$L = -\phi R + \phi^{-1}\omega(\phi)\,\partial_a\phi\,\partial^a\phi + 16\pi L_m. \tag{3.3}$$

The theory proposed by Brans and Dicke [12] arises in the special case with ω constant and $f(\Phi) \propto \Phi^2$. The relative merits of adopting Eq. (3.1), as do La and Steinhardt [7], or Eq. (3.3), as do Barrow and Maeda [27], have been discussed by Liddle and Wands [36]. Here we adopt Eq. (3.3).

By varying the action associated with Eq. (3.3) with respect to the space-time metric and the scalar field ϕ, respectively, we obtain the generalized Einstein equations and the wave equation for ϕ:

$$R_{ab} - \tfrac{1}{2} g_{ab} R = -8\pi\phi^{-1}T_{ab} - \omega(\phi)\phi^{-2}\{\phi_a\phi_b - \tfrac{1}{2} g_{ab}\phi_i\phi^i\} - \phi^{-1}\{\phi_{a;b} - g_{ab}\Box\phi\}, \tag{3.4a}$$

$$\{3 + 2\omega(\phi)\}\Box\phi = 8\pi T - (d\omega/d\phi)\phi_i\phi^i, \tag{3.4b}$$

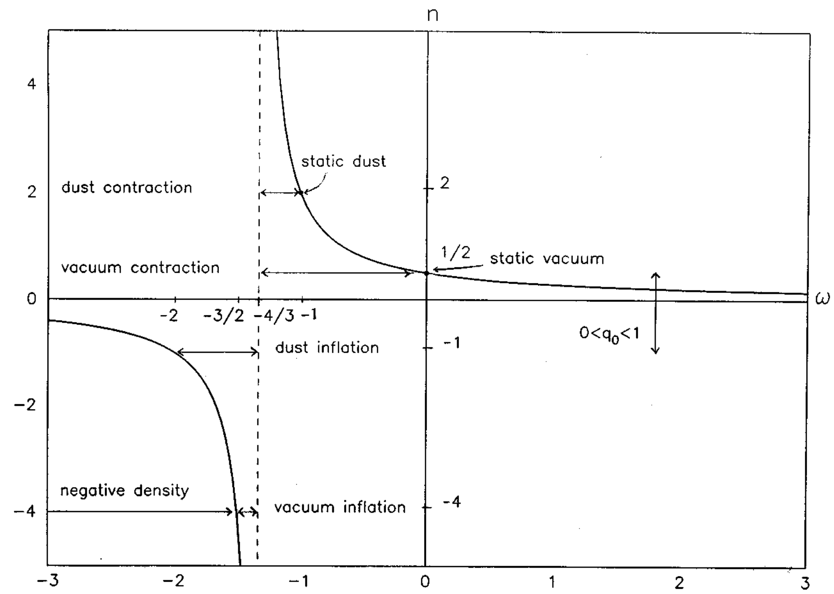

FIG. 2. The relation between the exponent n in the time dependence of G and the Brans-Dicke parameter ω. In Brans-Dicke theory itself, $\omega > 500$ but generalized scalar-tensor theories may be approximated by Brans-Dicke models with ω down to $-3/2$ for some period in the early Universe. Models with $\omega < -3/2$ are probably unrealistic since the scalar field has negative energy density. The consequences of various values of n and ω for the cosmic evolution are indicated.

where $\phi_a \equiv \partial_a \phi$, $\Box \equiv g^{ab}\partial_a \partial_b$, T^{ab} is the energy momentum tensor of the matter content of the theory and T is its trace. Clearly if T vanishes (which only applies for a radiation fluid) and if ϕ is constant, then Eq. (3.4b) is satisfied identically and Eq. (3.4a) reduces to the standard Einstein equations with a gravitational constant $G = \phi^{-1}$. Hence any exact solution of Einstein's equations with a trace-free matter source (e.g., radiation) will also be a particular exact solution of the scalar-tensor theory with ϕ, and hence $\omega(\phi)$, constant. For an isotropic and homogeneous universe containing a perfect fluid with equation of state $p = \gamma\rho$, Eqs. (3.4) and (3.5) give

$$H^2 + H\dot{\phi}/\phi - \omega\dot{\phi}^2/(6\phi^2) + k/a^2 = 8\pi\rho/(3\phi), \tag{3.5a}$$

$$\ddot{\phi} + [3H + \dot{\omega}/(2\omega+3)]\dot{\phi} = 8\pi\rho(1-3\gamma)/(2\omega+3), \tag{3.5b}$$

$$\dot{H} + H^2 + \omega\dot{\phi}^2/(3\phi^2) - H\dot{\phi}/\phi = -[8\pi\rho/(3\phi)]\{[(3\gamma+1)\omega+3]/(2\omega+3)\} + \dot{\omega}\dot{\phi}/[2\phi(2\omega+3)], \tag{3.5c}$$

$$\dot{\rho} + 3(\gamma+1)H\rho = 0, \tag{3.5d}$$

where $a(t)$ is the scale factor, $H = \dot{a}/a$ is the Hubble parameter, k is the curvature constant, an overdot denotes differentiation with respect to time t, and $\omega = \omega(\phi(t))$ is regarded as a function of t.

A. Brans-Dicke theory

In BD theory, the isotropic homogeneous radiation model is not identical with the GR model: it is vacuum dominated at early times and only approaches the GR radiation solution ($a \propto t^{1/2}$) at late times. [Although there exist power-law solutions (termed ''Machian'' by some authors) which are *always* matter dominated [37], the general solution approaches these at late times (if they expand forever) but not early times.] The vacuum solutions were found by O'Hanlon and Tupper [38] (see also [39]). The full $k=0$ solution for a perfect fluid is known [40] and is given, together with the $k \neq 0$ vacuum and radiation solutions, by Barrow [26]. Mimoso and Wands [41] have also studied the stiff and radiation solutions for general k. The full radiation solution can be well approximated by joining the $k=0$ vacuum solution to the $k=0$ GR solution at some time t_1 which may be regarded as a free parameter of the theory. The exact $k=0$ vacuum solution has

$$G \propto \phi^{-1} \propto t^{-d/(1+d)}, \quad a \propto t^{1/[3(1+d)]},$$

$$d \equiv \omega^{-1}[1 + \sqrt{1 + 2\omega/3}] \tag{3.6}$$

so this applies from t_{Pl} until t_1. For $t > t_1$, ϕ is constant and $a \propto t^{1/2}$, as in the usual radiation-dominated model. However, G will vary again after the time $t_e \sim 10^{11}$ s when the matter density goes above the radiation density. During this era, one has a dust equation of state and obtains

$$G \propto \phi^{-1} \propto t^{-n}, \quad a \propto t^{(2-n)/3}, \quad n \equiv 2/(4+3\omega). \tag{3.7a}$$

In terms of the parameter n, in some ways the most useful characterization of the model, the evolution of $G(t)$ and $a(t)$ during the vacuum-dominated phase [given by Eq. (3.6)] becomes

$$G \propto t^{-(n+\sqrt{4n+n^2})/2}, \quad a \propto t^{(2-n-\sqrt{4n+n^2})/6}. \tag{3.7b}$$

The relationship between n and ω is shown in Fig. 2, al-

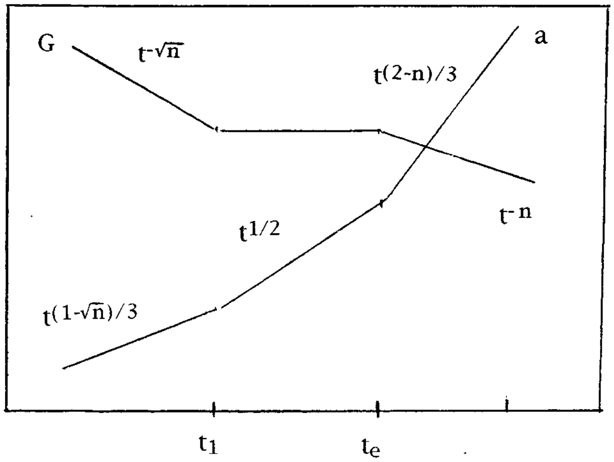

FIG. 3. The evolution of the scale factor and G in standard Brans-Dicke theory. These both evolve in the standard way in the radiation-dominated era but they deviate from the standard model in the late (matter-dominated) and early (vacuum-dominated) eras. The form of the exponents for $t<t_1$ assumes that n is small, as required in Brans-Dicke theory itself.

though not all the values indicated may be physically reasonable.

In BD theory itself, observations require that ω be large (viz. $\omega>500$), so that n is small (viz. $n<0.001$). In this case, the expressions for G and a during the vacuum phase can be simplified, Eqs. (3.7a) and (3.7b) being approximated by

$$a \propto \begin{cases} t^{(1-\sqrt{n})/3} & (t<t_1) \\ t^{1/2} & (t_1<t<t_e) \\ t^{(2-n)/3} & (t>t_e) \end{cases} \tag{3.8}$$

and

$$G(t)=\begin{cases} G_0(t_1/t)^{\sqrt{n}}(t_0/t_e)^n & (t<t_1) \\ G_0(t_0 t_e)^n & (t_1<t<t_e) \\ G_0(t_0/t)^n & (t>t_e), \end{cases} \tag{3.9}$$

where G_0 is the present value of G. This behavior is indicated in Fig. 3. The factor (t_0/t_e) in Eq. (3.9) can be approximated as 10^6. For if the Hubble parameter is $H_0=100$ h and the total density parameter is Ω_0, we have [42]

$$t_0=3\times10^{17}\ \mathrm{h}^{-1}f(\Omega_0)\ \mathrm{s},\quad t_e=4\times10^{10}\ \mathrm{h}^{-4}\Omega_0{}^{-2}\ \mathrm{s}, \tag{3.10a}$$

where $f(\Omega_0)$ goes from 2/3 for $\Omega_0=1$ to 1 for $\Omega_0=0$, so

$$t_0/t_e=8\times10^6\ \mathrm{h}^3\Omega_0^2 f(\Omega_0). \tag{3.10b}$$

For $\Omega_0=1$ (as implied by the inflationary scenario), the ratio has the value of 10^6 for $H_0=60$; for $\Omega_0=0.3$ (the smallest value consistent with dynamical constraints), one needs $H_0=100$. Both values of H_0 are reasonable, so 10^6 is always a good approximation.

As discussed below, ω may vary in scenarios other than BD. However, even in this case, the cosmological evolution may still be approximated by a BD model for a limited period. We therefore need to consider a wider range of values for n and ω than indicated above and this gives rise to more exotic possibilities. Figure 2 shows that the function $n(\omega)$ has both positive and negative branches, with the latter and some of the former encompassing negative values of ω. Although n diverges at $\omega=-4/3$, which is presumably unphysical, any other value of ω might in principle be permitted. However, a number of important restrictions on the values of n and ω can be imposed from cosmological considerations.

(i) Equations (3.7a) and (3.7b) imply that values of ω just below $-4/3$ have the interesting consequence that G and a increase with time very rapidly (since n is very negative). Indeed one has power-law inflation (with a growing faster than t) during any dust era with $n<-1$ $(-4/3>\omega>-2)$ or any vacuum era with $n<-4$ $(-4/3>\omega>-3/2)$. Negative values of n and ω are not necessarily precluded. However, the scalar field effectively has a negative energy density unless $\omega>-3/2$, as can be seen by writing the Friedmann equation (3.5a) in the form

$$[H+\dot{\phi}/(2\phi)]^2+k/a^2=\dot{\phi}^2(2\omega+3)/(12\phi^2)+8\pi\rho/(3\phi). \tag{3.11}$$

This precludes values of n in the range -4 to 0. Note that the exponent of t in the expression for a during the vacuum phase [given by Eq. (3.7b)] would also be complex for $0>n>-4$ and this is another indication that such solutions could not be cosmologically realistic.

(ii) In order that the Universe be expanding (with the exponent of t in the expression for a being positive), we require $n<1/2$ ($\omega>0$ or $\omega<-4/3$) during the vacuum era and $n<2$ ($\omega>-1$ or $\omega<-4/3$) during the dust era. Models with $1/2<n<2$ $(-1<\omega<0)$ may therefore expand during their dust phase but bounce during their vacuum phase, thereby avoiding the initial singularity [43]. Such models are not necessarily precluded but they have the important consequence that no PBH's could form before the bounce. Note that models with $n=2$ $(\omega=-1)$ contract during the vacuum phase and are static in the dust era, while models with $n=1/2$ $(\omega=0)$ are static during the vacuum phase and expand as $t^{1/2}$ in the dust era.

(iii) In order that the deceleration parameter, $q_0=(n+1)/(2-n)$ for $k=0$, lie between 0 and 1 at the present epoch (as indicated by observations), Eq. (3.8) requires $-1<n<1/2$ ($\omega>0$ or $\omega<-2$). However, this condition need not be imposed at early times.

These restrictions are summarized in Fig. 2. We see that only two bands of values for n (or ω) are permitted by cosmological considerations: $2>n\geqslant0$ $(\omega>-1)$ and $n<-4$ $(-3/2>\omega>-4/3)$. The latter gives power-law inflation in both the dust and vacuum phases (so no PBH's can form) but there is no inflationary phase for the former. In BD itself, the experimental limit $\omega>500$ would restrict one to just a small part of the first band. Note that the limit $n=2$ $(\omega=-1)$ has an interesting physical significance since the low-energy effective action for bosonic string theory can be written in a form that gives a description of its scalar content corresponding to $n=2$. In this context the scalar field is referred to as the dilation field. However, the two theories do not have the same coupling of the scalar field to the other forms of matter [30].

B. More general scalar-tensor theories

BD theory is the simplest scalar-tensor theory which permits time variation of G, but in general any $\omega(\phi)$ is permitted. These more general scalar-tensor theories will approach GR in the weak-field limit only if $\omega\to\infty$ and $\omega^{-3}(d\omega/d\phi)\to 0$ simultaneously [13,34,35]. However, such theories make a strong time variation of G much more acceptable because $\omega(\phi)$ can vary significantly early on but asymptote to a large constant value at late times, so that GR behaviour is achieved today. A number of exact cosmological solutions and a general solution-generating procedure for such $\omega(\phi)$ theories have recently been found [26,44] and their asymptotic behaviours are understood for a wide class of $\omega(\phi)$. As can be seen from Eqs. (3.4) and (3.5), they have the property that, in the radiation case, any GR solution is also a particular solution of any $\omega(\phi)$ theory with ϕ (and hence ω) constant. [However, the converse need not be true: there exist exact vacuum solutions of BD theories, such as those given by Eq. (3.6), which are not vacuum solutions of GR.] This particular solution is generally a late-time attractor of radiation-dominated cosmological models if the theory approaches GR in the weak-field limit.

Observations of the cosmic light element abundances imply that the standard big bang picture, in which G does not change with time, is an excellent description of the large-scale evolution of the Universe after $t\sim 1$ s. This is because the success of the primordial nucleosynthesis scenario implies that the current value of G equals that at $t\sim 1$ s to within 10–20% [45], so significant variation of $G(t)$ can only have occurred prior to this. Acceptable evolution will therefore be well described by a model which begins with an initial period of vacuum-dominated expansion ($t<t_v$ with $t_v<1$ s), during which significant time variation of $\phi(t)$ can occur, followed by a phase ($t_v<t<t_e$) described by the radiation-dominated Friedmann model with ϕ constant. During the radiation-dominated period, the evolution will be driven ever closer to the GR radiation solution for a very general class of $\omega(\phi)$. In the subsequent dust-dominated phase ($t>t_e$), the initial state is so close to that of the GR solution that there will be little evolution of $G(t)$ away from the value $G(t_v)$. If the Universe is open, so that there is a final period of curvature-dominated evolution, then the solution will be driven ever closer to the GR Milne solution with $a\propto t$ and G constant during the curvature-dominated era.

We will be interested in assessing the observational implications of PBH formation occurring during the vacuum-dominated era ($t<t_v$), when $G(t)$ differs from its current value. Its variation will be determined by the specification of $\omega(\phi)$. It is expedient to use a model in which $\phi(t)$ exhibits simple behavior and a wide class of theories is described by the choice

$$2\omega(\phi)+3=2\beta|1-\phi/\phi_c|^{-\alpha} \tag{3.12}$$

where α, β, and ϕ_c are constants (with $\alpha>0$ and $\beta>0$). This representation has been introduced by Garcia-Bellido and Quiros [46] and investigated by Barrow [26]. The case $\alpha=0$ corresponds to BD theory with $\beta=\omega+(3/2)$. For $\alpha>0$, one expects ω to increase with t, as required, but the discussion of such theories by Barrow [26] shows that only a range of values of α permits both approach to GR at late times and the observed behavior of light-bending and perihelion precession in the weak-field limit. The latter condition requires $\omega^{-3}(d\omega/d\phi)\to 0$ as $\omega\to\infty$ and hence $\alpha>1/2$; the former condition requires $\alpha<2$, with ϕ tending to ϕ_c from below (see later).

For simplicity, we focus on the $\alpha=1$ solution since this can be solved analytically for both a radiation-dominated and dust-dominated homogeneous, isotropic model. [Note that the theory of Barker [47] is obtained when $\alpha=1$ and $\beta=-1/2$; in this case, the value of G defined by geodesics, $G=\phi^{-1}(4+2\omega)/(3+\omega)$, is constant to first order.] In the radiation-dominated case, the solution is expressed most simply in terms of the conformal time η, defined by $dt=a\,d\eta$:

$$\phi(\eta)=4\phi_c\eta^{\lambda}(\eta+2\eta_0)^{\lambda}/[(\eta+2\eta_0)^{\lambda}+\eta^{\lambda}]^2, \tag{3.13}$$

$$a^2(\eta)=\Gamma\eta(\eta+\eta_0)[(\eta+2\eta_0)^{\lambda}+\eta^{\lambda}]^2/[4\phi_c\eta^{\lambda}(\eta+2\eta_0)^{\lambda}], \tag{3.14}$$

where Γ and η_0 are constants and

$$\lambda=\sqrt{3/(2\beta)}. \tag{3.15}$$

Thus $\phi\to\phi_c$ and $a(\eta)\propto\eta\propto t^{1/2}$ as $\eta\to\infty$, so the radiation-dominated Friedmann universe of GR is approached asymptotically. This is expected since as $t\to\infty$:

$$2\omega+3\propto|1-\phi/\phi_c|^{-1}\propto\eta^2\propto t\to\infty, \tag{3.16}$$

$$\omega^{-3}(d\omega/d\phi)\propto|1-\phi/\phi_c|\propto\eta^{-2}\propto t^{-1}\to 0. \tag{3.17}$$

Although the solutions of Eq. (3.12) for other values of α in the range 1/2 to 2 do not admit convenient closed form, they display the following asymptotic behavior as $t\to\infty$:

$$a(\eta)\propto\eta[1+C\eta^{-2/(2-\alpha)}]\to\eta, \tag{3.18}$$

$$\phi(\eta)\propto\phi_c[1-D\eta^{-2/(2-\alpha)}]\to\phi_c, \tag{3.19}$$

$$2\omega+3\propto\eta^{2\alpha/(2-\alpha)}\propto t^{\alpha/(2-\alpha)}\to\infty, \tag{3.20}$$

$$\omega^{-3}(d\omega/d\phi)\propto(1-\phi/\phi_c)^{2\alpha-1}\propto\eta^{2(1-2\alpha)/(2-\alpha)}\propto t^{(1-2\alpha)/(2-\alpha)}\to 0, \tag{3.21}$$

where C and D are constants. This shows that such theories always tend to GR at late times for $1/2<\alpha<2$ but not otherwise. In the limiting case with $\alpha=2$, $\phi\to\phi_c/2$, and $\omega(\phi)\to 4\beta-3/2$ at late times.

At early times ($\eta\to 0$) the $\alpha=1$ solution given by Eqs. (3.13) and (3.14) with $\eta_0>0$ has the asymptotic form

$$a\propto\eta^{(1-\lambda)/2}\propto t^{(1-\lambda)/(3-\lambda)}, \tag{3.22}$$

$$\phi\propto\eta^{\lambda}\propto t^{2\lambda/(3-\lambda)}. \tag{3.23}$$

Hence there is a singularity in the radiation density ($\rho\propto a^{-4}$) at $\eta=0$ (where $a=0$) if $\lambda<1$ or $\lambda>3$. This asymptotic behavior also describes the early phase of models with $1/2<\alpha<2$ [44]. This provides a good model for the early vacuum-dominated phase ($t<t_v$) which must be completed

prior to $t \sim 1$ s. Since the transition to the radiation solution occurs at $\eta \sim \eta_0$, t_v is itself determined by η_0. By requiring that the temperature evolution ($T \propto a^{-1}$) match smoothly onto that of the GR radiation universe at t_v, we find that the temperature-time relation for $t < t_v$ must take the form

$$T \approx 10^{10}(t_v/\mathrm{s})^{-1/2}(t_v/t)^{(1-\lambda)/(3-\lambda)}\ \mathrm{K} \quad (t < t_v). \tag{3.24}$$

The behavior of a and ϕ is indicated in Fig. 4 for different ranges of values for λ. Note that models with $1 < \lambda < 3$ reach a maximum compression and bounce rather than encountering a singularity. Equation (3.22) and (3.23) are equivalent to Eq. (3.6) with $\lambda = 3d/(2+3d)$, so the vacuum phase of this particular model resembles the vacuum phase of the BD model with an ''effective'' coupling constant

$$\omega = 3(1-\lambda^2)/(2\lambda^2) = \beta - 3/2, \tag{3.25}$$

where we have used Eq. (3.15). The last equality also follows from Eq. (3.12) since $\phi \rightarrow 0$ as $t \rightarrow 0$ for $0 < \lambda < 3$.

The $k=0$ dust solution for $\alpha = 1$ can be found using the generating-function method of Barrow and Mimosa [28] and is conveniently expressed in terms of a time parameter θ, defined by $dt = \sqrt{2\omega + 3}\, d\theta$. The solution is given by

$$\phi(\theta) = \phi_c(1 - A\theta^{-\mu}), \tag{3.26}$$

$$a(\theta)^3 = B\phi_c^{\ -1}\theta^{2+\mu}, \tag{3.27}$$

$$2\omega + 3 = [B(2+\mu)\theta^{\mu} + \tfrac{1}{2} - 2\mu^{-1}]^2/\{3[B\theta^{\mu} + \tfrac{1}{4} - \mu^{-1}]\}, \tag{3.28}$$

where A, B, and μ are constants, related by $\mu AB = 1$, and θ is related to t by

$$B\theta^{2+\mu} + (1 - 4/\mu)\theta^2 \propto t^2. \tag{3.29}$$

We take $\mu > 0$, so that $\phi \rightarrow \phi_c$ as $\theta \rightarrow \infty$. At late times ($\theta \rightarrow \infty$), one has $\theta \propto t^{2/(2+\mu)}$ and so

$$2\omega + 3 = B(2+\mu)^2\theta^{\mu}/3 \propto t^{2\mu/(2+\mu)}, \tag{3.30}$$

$$a \propto \theta^{(2+\mu)/3} \propto t^{2/3}. \tag{3.31}$$

This shows that the solution tends asymptotically to the standard GR dust form. From Eqs. (3.12), (3.26), and (3.30), μ is related to the parameter β by

$$\beta = (2+\mu)^2(6\mu)^{-1}. \tag{3.32}$$

At early times ($\theta \rightarrow 0$), the dust solution would have $a \propto t^{(2+\mu)/3}$ and $\phi \propto t^{-\mu}$, so Eq. (3.7a) shows it would be BD-like with $\omega < -4/3$. However, one would not expect this solution to apply after decoupling.

To summarize, in the $\alpha = 1$ scalar-tensor theory the Universe behaves like a BD model during the vacuum era but like the standard GR solution at all other times. Although these properties may not apply for more general scalar-tensor theories, one might speculate that any theory can be approximated by a BD model with an effective value of ω providing one restricts attention to a sufficiently short time interval. In the present context we are only interested in the value of ω during the period when the PBH's form. From this perspective, it is reasonable to consider all the values of ω permitted

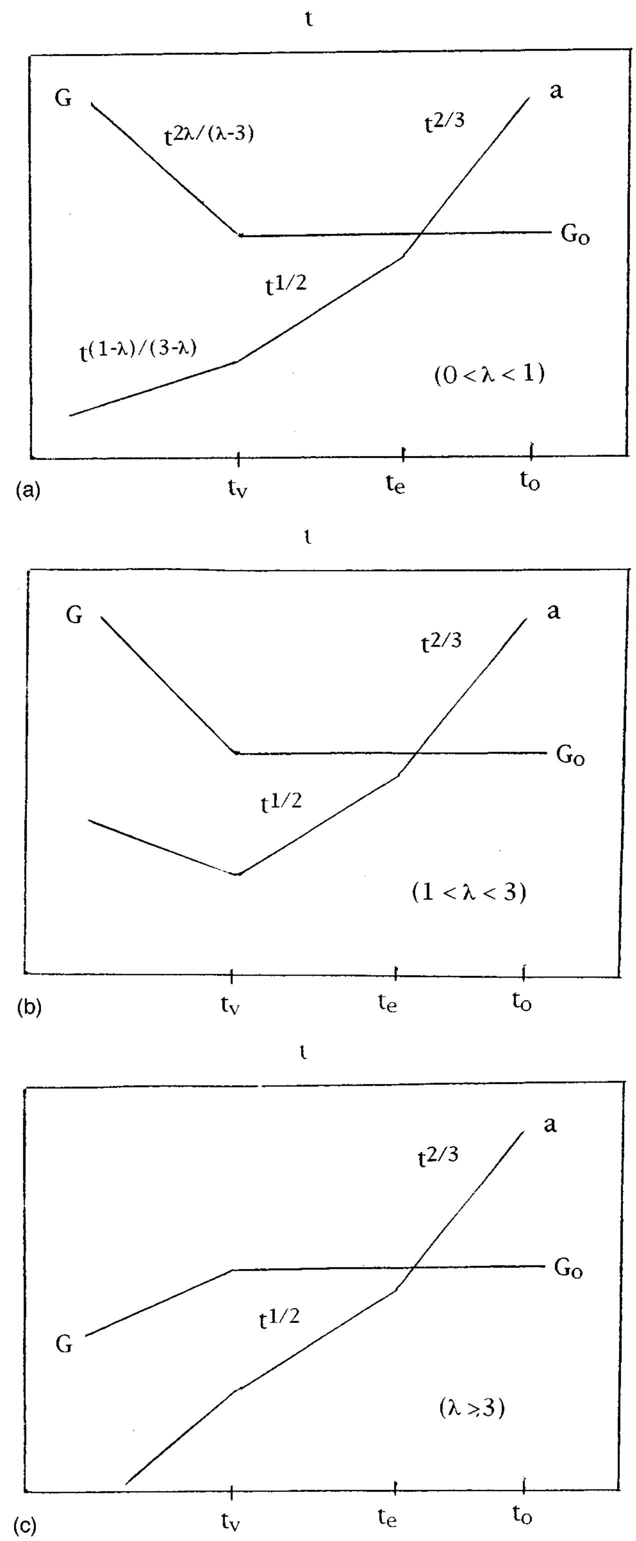

FIG. 4. The evolution of the scale factor and G in the ''$\alpha = 1$'' scalar-tensor theory. These deviate from the standard model only in the vacuum period before t_v and this necessarily precedes 1 s. (a) shows models with $0 < \lambda < 1$; these expand continuously from an initial singularity. (b) shows models with $1 < \lambda < 3$; these bounce during the vacuum era. (c) shows models with $\lambda > 3$; these undergo power-law inflation and have G increasing before t_v.

in our earlier discussion of BD theory. In this context, note that the exponent of t in Eq. (3.22) is negative for $3>\lambda>1$ (corresponding to $0>\omega>-4/3$) and such models should bounce at $\eta_{\rm min}\sim(1-\lambda)^{1/\lambda}$. Presumably no PBH's can form before this epoch. It may also be impossible for PBH's to form in models with $\lambda>3$ (corresponding to $-3/2<\omega<-4/3$) since such models always undergo power-law inflation during the vacuum and dust eras. Henceforth we therefore assume $0<\lambda<1$ (corresponding to $\omega>0$).

IV. PBH EVAPORATIONS IN SCALAR-TENSOR THEORIES

In this section we will discuss how PBH formation and evaporation is modified in scalar-tensor theories, considering separately scenario A (in which G evolves in the same way everywhere) and scenario B (in which black holes locally preserve the value of G at their formation epoch, while G continues to change in the background Universe). We will also consider the two variants of the scalar-tensor theories discussed in Sec. III: BD itself (in which G varies in both the vacuum and dust eras but only as a weak power of time) and the ''α=1'' model (in which G can vary as a high power of time early on but very little thereafter). Since the most interesting PBH constraints are associated with evaporations which occur at late times ($t>1$ s), they are essentially unaffected in the ''α=1'' model if one adopts scenario A. We will therefore only discuss scenario A in the context of exact BD theory, whereas we discuss scenario B in the context of both the BD and ''α=1'' models.

A. Scenario A

In this scenario G has the same value everywhere (both at the black hole horizon and in the background Universe) at a given epoch and it is this global value which determines the evaporation rate of a black hole. Equations (2.2) and (2.3) give $dM/dt\propto R_g^2T^4\propto G^{-2}$, so one obtains an extra time dependence whenever G varies. As explained above, we only consider this scenario in the context of BD theory itself. Although the equations below assume that n is small (since observations require $n<0.001$), we also show how to extend them to the more general case.

It is convenient to consider the evolution of the PBH's in the period before and after t_e separately. Before t_e Eqs. (2.3) and (3.9) give

$$-dM/dt=\begin{cases}\alpha(t_e/t_0)^{2n}M^{-2} & (t_1<t<t_e)\\ \alpha(t_e/t_0)^{2n}(t/t_1)^{2\sqrt{n}}M^{-2} & (t<t_1),\end{cases}\qquad(4.1)$$

where α is still given by Eq. (2.4), except that it has now absorbed the factor f. (If n were not small, the factor $2\sqrt{n}$ would be replaced by $n+\sqrt{4n+n^2}$ throughout the ensuing analysis.) Integrating Eq. (4.1) we obtain

$$M^3=\begin{cases}M_i^3+3\alpha(t_e/t_0)^{2n}(\min[1,(1+2\sqrt{n})^{-1}(t_i/t_1)^{2\sqrt{n}}]t_i-t) & (t_1<t<t_e)\\ M_i^3+3\alpha(1+2\sqrt{n})^{-1}(t_e/t_0)^{2n}(t_i^{1+2\sqrt{n}}t_1^{-2\sqrt{n}}-t^{1+2\sqrt{n}}t_1^{-2\sqrt{n}}) & (t<t_1),\end{cases}\qquad(4.2)$$

where M_i is the mass of the black hole at its formation time t_i and the first expression covers the cases in which the PBH forms before and after t_1. The black hole therefore evaporates completely ($M=0$) at a time

$$\tau=\begin{cases}(3\alpha)^{-1}(t_0/t_e)^{2n}M_i^3+\min[1,(1+2\sqrt{n})^{-1}(t_i/t_1)^{2\sqrt{n}}]t_i & (t_1<\tau<t_e)\\ [(3\alpha)^{-1}(1+2\sqrt{n})(t_0/t_e)^{2n}M_i^3t_1^{2\sqrt{n}}+t_i^{1+2\sqrt{n}}]^{1/(1+2\sqrt{n})} & (\tau<t_1).\end{cases}\qquad(4.3)$$

From Eq. (2.1), the mass of a PBH which forms at a time t_i before t_e (always the case for evaporating PBH's) is

$$M_i\approx c^3G(t_i)^{-1}t_i\approx\begin{cases}G_0^{-1}(t_e/t_0)^nt_i & (t_1<t_i<t_e)\\ G_0^{-1}(t_e/t_0)^n(t_i/t_1)^{\sqrt{n}}t_i & (t_i<t_1).\end{cases}\qquad(4.4)$$

Since the Planck time in this model is given implicitly by

$$t_{\rm Pl}=G^{1/2}=G_0^{1/2}(t_0/t_e)^{n/2}(t_1/t_{\rm P1})^{\sqrt{n}/2}\qquad(4.5)$$

corresponding to

$$t_{\rm Pl}=[G_0(t_0/t_e)^nt_1{}^{\sqrt{n}}]^{1/(2+\sqrt{n})},\qquad(4.6)$$

Eq. (4.4) implies that the t_i terms in Eq. (4.3) are negligible for PBH's which form after the Planck epoch. This is necessarily the case, so we henceforth drop the t_i terms. Equation (4.3) can then be approximated as

$$\tau\approx\begin{cases}(3\alpha)^{-1}(t_0/t_e)^{2n}M_i{}^3\approx10^{17+12n}(M_i/10^{15}\text{ g})^3\text{ s}\\ (t_1<\tau<t_e)\\ \{(3\alpha)^{-1}(1+2\sqrt{n})(t_0/t_e)^{2n}M_i^3t_1{}^{2\sqrt{n}}\}^{1/(1+2\sqrt{n})}\\ (\tau<t_1).\end{cases}\qquad(4.7)$$

Note that PBH's evaporate before t_e (as assumed) for masses below

$$M_e=(t_e/t_0)^{2n/3}(3\alpha t_e)^{1/3}\approx10^{13-4n}\text{ g}.\qquad(4.8)$$

For $n=0$, all these equations reduce to the form given in Sec. II.

We now consider the PBH evolution after t_e. During this era G is again time dependent and Eq. (2.3) gives

$$-dM/dt=\alpha t_0^{-2n}t^{2n}M^{-2}\quad(t>t_e).\qquad(4.9)$$

This can be integrated to give

$$M^3 = M(t_e)^3 + 3\alpha(2n+1)^{-1}t_0^{-2n}(t_e^{2n+1} - t^{2n+1})$$
$$\approx M_i^3 - 6\alpha n(2n+1)^{-1}(t_e/t_0)^{2n}t_e$$
$$-3\alpha(2n+1)^{-1}t^{2n+1}t_0^{-2n}, \qquad (4.10)$$

where $M(t_e)$ is the mass of the black hole at t_e, which is itself determined in terms of M_i by Eq. (4.2), and we have assumed that such PBH's form after t_1 (although it makes little difference if they form before t_1). We now have three cases, depending on the sign of the exponent of t in this equation. Since n must be small and positive, only the first case is applicable in BD itself, but we will consider all three cases for completeness.

$n > -1/2$. The exponent of t is positive, so the black hole evaporates completely in a time

$$\tau \approx [(2n+1)(3\alpha)^{-1}t_0^{2n}M_i^3 - 2nt_e^{2n+1}]^{1/(2n+1)} \quad (t > t_e). \qquad (4.11)$$

The second term on the right-hand side is negligible providing M_i greatly exceeds the value given by Eq. (4.8), i.e., providing the black hole evaporates well after t_e, and in this case we have

$$\tau \propto M_i^{3/(2n+1)}. \qquad (4.12)$$

The situation is therefore qualitatively similar to the standard one in that τ increases as a power of M_i, albeit with a different exponent. The black holes evaporating at the *present* epoch form with an initial mass of

$$M_* = (2n+1)^{-1/3}(3\alpha t_0)^{1/3} \approx 10^{15}(2n+1)^{-1/3} \text{ g}. \qquad (4.13)$$

Apart from the $(2n+1)^{-1/3}$ factor, this is the same as the mass $M_{\rm crit}$ of the PBH's currently evaporating in the standard scenario, given by Eq. (2.7), as expected since the current value of G is unaltered. However, M_* becomes very large as n tends to $-1/2$.

$n < -1/2$. The exponent of t in Eq. (4.10) is now negative, so this term is dominated by the preceding one at late times. As $t \to \infty$, we have

$$M \to [M_i^3 - 6n(2n+1)^{-1}\alpha t_0^{-2n}t_e^{2n+1}]^{1/3} \qquad (4.14)$$

so complete evaporation only ever occurs for

$$M_i < M_\infty \equiv [2n/(2n+1)]^{1/3}M_e, \qquad (4.15)$$

where M_e is defined by Eq. (4.8); this upper limit is only slightly larger than M_e itself unless $n \approx -1/2$. For M_i between M_e and M_∞, the time for evaporation is

$$\tau = t_e[-2n + (2n+1)(M_i/M_e)^3]^{1/(2n+1)} \quad (\tau > t_e) \qquad (4.16)$$

and this goes from t_e to infinity as M_i goes from M_e to M_∞. Contrary to the remark made by Barrow [11], τ always increases with M_i but the situation is very different from the usual one in that PBH's only slightly larger than M_e *never* evaporate in this picture. Indeed the PBH's evaporating at the present epoch have an initial mass

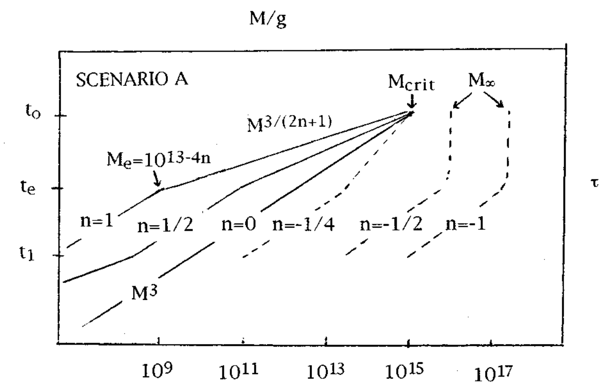

FIG. 5. The dependence of the PBH evaporation time on initial PBH mass in Brans-Dicke theory for scenario A, in which the value of G evolves in the same way everywhere. In practice, only small positive values of n may be allowed and negative values (shown dotted) would probably not permit PBH formation before t_1.

$$M_* = [1 + (2n)^{-1}(t_0/t_e)^{2n+1}]^{1/3}M_\infty$$
$$\approx 10^{13-4n}[2n/(2n+1)]^{1/3} \text{ g}, \qquad (4.17)$$

which is only slightly less than M_∞. Note that M_e, M_*, and M_∞ are all approximately the same and always exceed 10^{15} g for $n < -1/2$. The cosmological constraints discussed in Sec. III suggest that models with negative n are probably only realistic for $n < -4$, in which case the critical mass scale exceeds 10^{29} g. However, it is not clear that PBH's can form with $n < -4$ since there is then power-law inflation.

$n = -1/2$. This is a special case in which Eq. (4.10) breaks down and must be replaced by

$$M^3 = M(t_e)^3 - 3\alpha t_0 \ln(t/t_e)$$
$$\approx M_i^3 - 3\alpha t_0[1 + \ln(t/t_e)]. \qquad (4.18)$$

Evaporation, therefore, occurs at a time

$$\tau = t_e \exp[(M_i^3/3\alpha t_0)^{-1}] = t_e \exp[(M_i/M_{\rm crit})^3 - 1] \qquad (4.19)$$

and the PBH's evaporating at the present have an initial mass

$$M_* = (3\alpha t_0)^{1/3}[1 + \ln(t_0/t_e)] = 2 \times 10^{16} \text{ g}. \qquad (4.20)$$

However, as discussed above, this case is probably inapplicable.

The dependence of τ upon M_i is these different situations is indicated in Fig. 5. When n is not small, the form of the curves for $\tau < t_1$ comes from replacing $2\sqrt{n}$ by $n + \sqrt{4n + n^2}$ in the above equations. Curves with negative n (shown dotted) may be inapplicable since PBH's can probably not form in this situation.

B. Scenario B

In this scenario, we assume that the value of G associated with the black hole reflects the conditions when it first formed rather than the background cosmological value. In BD theory Eqs. (2.3) and (3.9) then imply

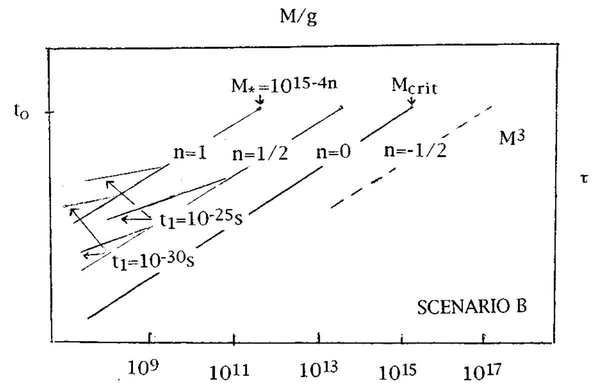

FIG. 6. The dependence of the PBH evaporation time on initial PBH mass in Brans-Dicke theory for scenario B, in which the PBH preserves the value of G at its formation epoch. In this case, one must specify both the value of n and the value of t_1 (which specifies when the vacuum-dominated era ends).

$$-dM/dt=\begin{cases}\alpha(t_e/t_0)^{2n}M^{-2} & (t_i>t_1)\\ \alpha(t_e/t_0)^{2n}(t_i/t_1)^{2\sqrt{n}}M^{-2} & (t_i<t_1).\end{cases} \tag{4.21}$$

The lifetime is, therefore,

$$\tau\approx\begin{cases}(3\alpha)^{-1}(t_0/t_e)^{2n}M_i^{\,3}\approx 10^{17+12n}(M_i/10^{15}\text{ g})^3\text{ s}\\ (t_i>t_1)\\ (3\alpha)^{-1}(t_0/t_e)^{2n}(t_1/t_i)^{2\sqrt{n}}M_i^{\,3}\\ (t_i<t_1).\end{cases} \tag{4.22}$$

The mass of the PBH's evaporating at the present epoch is

$$M_*=(t_e/t_0)^{2n/3}M_{\text{crit}}\approx 10^{15-4n}\text{ g} \tag{4.23}$$

[where M_{crit} is given by Eq. (2.7)] providing such PBH's form after t_1. For PBH's which form before t_1, Eqs. (4.4) and (4.22) imply

$$\tau\approx G_0^{\,2/(1+\sqrt{n})}(t_0/t_e)^{2n/(1+\sqrt{n})}t_1^{\,2\sqrt{n}/(1+\sqrt{n})}M_i^{\,(3+\sqrt{n})/(1+\sqrt{n})} \tag{4.24}$$

and, in this case, the mass evaporating now can be expressed as

$$M_*=(t_e/t_0)^{2n/(3+\sqrt{n})}(t_0/t_1^2)^{\sqrt{n}/(3+\sqrt{n})}M_{\text{crit}}^{\,3/(3+\sqrt{n})}. \tag{4.25}$$

Equation (4.25) only applies for $0<n\ll 1$ but can be extended to the more general case with the prescription used in scenario A. The dependence of τ upon M_i is shown in Fig. 6.

In the "$\alpha=1$" scalar-tensor model, G can be regarded as constant in the period $t>t_v$ but it varies as a power of t for $t<t_v$. We can therefore adapt Eqs. (4.21) to (4.25) by dropping $(t_e/t_0)^n$ terms and replacing $\sqrt{n}$ by $2\lambda/(3-\lambda)$. Equation (4.21) then becomes

$$-dM/dt=\begin{cases}\alpha M^{-2} & (t_i>t_v)\\ \alpha(t_i/t_v)^{4\lambda/(3-\lambda)}M^{-2} & (t_i<t_v),\end{cases} \tag{4.26}$$

so the lifetime is

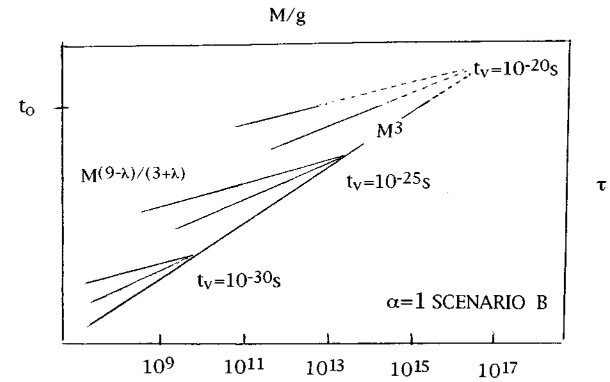

FIG. 7. The dependence of the PBH evaporation time on initial PBH mass in the $\alpha=1$ scalar-tensor theory. This assumes that scenario B applies, since otherwise the standard picture is unchanged. In this case, one must specify both the value of λ and the value of t_v. The latter may or may not exceed the time (10^{-23} s) at which PBH's evaporating at the present epoch form.

$$\tau\approx\begin{cases}(3\alpha)^{-1}M_i^{\,3}\approx 10^{17}(M_i/10^{15}\text{ g})^3\text{ s} & (t_i>t_v)\\ (3\alpha)^{-1}(t_v/t_i)^{4\lambda/(3-\lambda)}M_i^3 & (t_i<t_v).\end{cases} \tag{4.27}$$

Since t_i is itself related to M_i by Eq. (4.4), we can express τ in the $t_i<t_v$ case as

$$\tau\approx G_0^{2(3-\lambda)/(3+\lambda)}t_v^{4\lambda/(3+\lambda)}M_i^{(9-\lambda)/(3+\lambda)}\quad (t_i<t_v). \tag{4.28}$$

The mass of the PBH's evaporating at the present epoch is therefore M_{crit} if such PBH's form after t_1 but

$$M_*=M_{\text{crit}}^{3(3-\lambda)/(9-\lambda)}t_0^{2\lambda/(9-\lambda)}t_v^{-4\lambda/(9-\lambda)} \tag{4.29}$$

if they form before t_1. The dependence of τ upon M_i in this situation is indicated in Fig. 7. Solutions with $\lambda>1$ are probably inapplicable since such models either contract in the vacuum phase or undergo power-law inflation.

V. CONSTRAINTS ON PBH's

Figure 1 summarizes the constraints on the fraction of the Universe going into PBH's of mass M in the standard big bang model. In this section we discuss how the "density" and "γ-ray" constraints are modified in the various scenarios considered in Sec. III. We will assume $0<n<1/2$ in the context of BD theory and $0<\lambda<1$ in the context of "$\alpha=1$" scalar-tensor theory.

First we note that the final expression in Eq. (2.8) no longer applies since the relationship between z and t is changed. In BD theory Eq. (3.8) implies that the fraction of the Universe's mass going into PBH's at a time t between t_1 and t_e is

$$\beta(t)=[\Omega_B(t)/\Omega_R](t_e/t_0)^{(2-n)/3}(t/t_e)^{1/2}$$
$$\approx 10^{2n-5}\Omega_B(t/\text{s})^{1/2}, \tag{5.1}$$

where [from the discussion after Eq. (3.10)] we have assumed $t_0/t_e=10^6$. From Eq. (4.4) PBH's with initial mass M form at a time

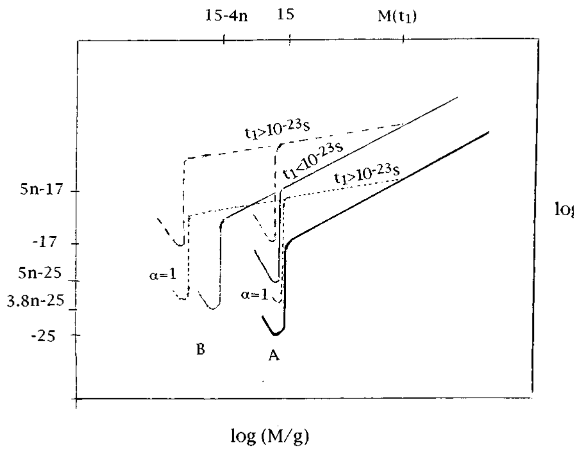

FIG. 8. The γ-ray background and cosmological density constraints on the fraction of the Universe going into PBH's of mass M for Brans-Dicke theory with scenarios A and B. The continuous and broken curves show the situations with $t_1<10^{-23}$ s and $t_1>10^{-23}$ s respectively. The dotted lines show the constraints in the ''$\alpha=1$'' scalar-tensor theory for scenarios A and B with $t_v>10^{-23}$ s. The heavy line corresponds to the standard situation with G constant.

$$t\approx G_0M(t_0/t_e)^n\approx 10^{6n-5}(M/M_\odot)\ \ \text{s} \tag{5.2}$$

so Eq. (2.9) for the fraction of the Universe going into PBH's of mass M is replaced by

$$\beta(M)\approx 10^{5n-8}\Omega_B(M)(M/M_\odot)^{1/2}. \tag{5.3}$$

For PBH's which form before t_1, Eqs. (5.1)–(5.3) become

$$\beta(t)\approx 10^{2n-5}\Omega_B(t_1/\text{s})^{1/2}(t/t_1)^{(1-\sqrt{n})/3}, \tag{5.4}$$

$$t\approx[G_0M(t_0/t_e)^n t_1^{\sqrt{n}}]^{1/(1+\sqrt{n})}, \tag{5.5}$$

$$\beta(M)\approx 10^{4n/(1+\sqrt{n})-5}\Omega_B(t_1/\text{s})^{1/2}(G_0M/t_1)^{(1-\sqrt{n})/3(1+\sqrt{n})}, \tag{5.6}$$

where we have assumed $n\ll 1$. In the ''$\alpha=1$'' scalar-tensor scenario

$$\beta(t)\approx 10^{-5}\Omega_B(t_v/\text{s})^{1/2}(t/t_v)^{(1-\lambda)/(3-\lambda)}, \tag{5.7}$$

$$t\approx(G_0M)^{(3-\lambda)/(3+\lambda)}t_v{}^{2\lambda/(3+\lambda)}, \tag{5.8}$$

$$\beta(M)\approx 10^{-5}\Omega_B(G_0M/t_v)^{(1-\lambda)/(3+\lambda)}(t_v/\text{s})^{1/2}. \tag{5.9}$$

All these equations apply independently of whether one adopts scenario A or B.

We require $\Omega_B(M)<1$ for nonevaporating PBH's and so in BD theory, for PBH's forming after t_1, we require

$$\beta(M)<10^{5n-8}(M/M_\odot)^{1/2}\quad (M>M_*), \tag{5.10}$$

where M_* is the mass of the PBH's evaporating today. M_* is given by Eq. (4.13) in scenario A and Eq. (4.23) in scenario B. For PBH's which form before t_1, Eq. (5.6) implies

$$\beta(M)<[10^{(12n-10\sqrt{n}-20)}(t_1/\text{s})^{(1+5\sqrt{n})}\times(M/M_\odot)^{(1-\sqrt{n})}]^{1/3(1+\sqrt{n})}\quad (M>M_*), \tag{5.11}$$

where M_* is given by Eq. (4.13) in scenario A and Eq. (4.25) in scenario B. In $\alpha=1$ scalar-tensor theory, Eq. (5.9) implies

$$\beta(M)<[10^{-20}(M/M_\odot)^{(1-\lambda)}\times(t_v/\text{s})^{(1+3\lambda)/2}]^{1/(3+\lambda)}\quad (M>M_*), \tag{5.12}$$

where M_* is given by Eq. (4.13) in scenario A and Eq. (4.29) in scenario B; this assumes that the PBH's form before t_v, else there is no change from the usual constraint.

For evaporating PBH's the most important constraint comes from the γ-ray background limit [17,18]. The strongest limit is always associated with the PBH's evaporating at the present epoch but the strength of the limit depends on M_* since this determines the energy at which the predicted background peaks. In BD theory with scenario A, the PBH's evaporating at the present epoch have nearly the standard mass $M_{\text{crit}}\approx 10^{15}$ g, so the γ-ray background still peaks at around 100 MeV and we again infer $\Omega_B(M_*)<10^{-8}$. A more precise comparison in the $\Omega=1$ case, allowing for the emission of quark and gluon jets, gives [17]

$$\Omega_B(M_*)<8\times 10^{-9}\ \text{h}^{-1.95}. \tag{5.13}$$

In BD theory with scenario B, assuming the PBH's form after the time t_1, the background radiation from the PBH's evaporating at the present epoch peaks at a current energy

$$E\approx T\approx G(t_e)^{-1}M_*^{-1}\approx G_0^{-1}M_*^{-1}(t_e/t_0)^n\approx 10^{2-2n}\ \ \text{MeV}, \tag{5.14}$$

which is below the usual γ-ray peak of 100 MeV for $n>0$. Since the observed γ-ray background density scales as $E^{-0.4}$ over the energy band 35–170 MeV, the associated limit on the PBH density is

$$\Omega_B(M_*)\approx 10^{0.8n-8}\quad (0<n<0.2), \tag{5.15}$$

where the upper limit on n comes from putting E equal to 35 MeV in Eq. (5.14). Equation (5.10), therefore, implies

$$\beta(M_*)<\begin{cases}10^{5n-25} & (\text{scenario A})\\ 10^{3.8n-25} & (\text{scenario B}).\end{cases} \tag{5.16}$$

If the PBH's form before t_1, Eq. (5.11) implies

$$\beta(M_*) < [10^{(12n-16\sqrt{n}-62)} \times (t_1/\mathrm{s})^{1+5\sqrt{n}}]^{1/3(1+\sqrt{n})} \quad \text{(scenario A)}. \tag{5.17}$$

In $\alpha=1$ scalar-tensor theory, the γ-ray limit is unaffected if the PBH forms after t_v but, if they form before t_v, Eq. (5.12) gives

$$\beta(M_*) < [10^{10\lambda-62}(t_v/\mathrm{s})^{(1+3\lambda)/2}]^{1/(3+\lambda)} \quad \text{(scenario A)}. \tag{5.18}$$

Equations (5.17) and (5.18) are modified in scenario B (since M_* is modified) but we do not show the equations explicitly in this case since they are complicated. Figure 8 summarizes the constraints on $\beta(M_*)$ in these various situations.

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