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Entropy of a quantum field in rotating black holes

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By using the brick wall method we calculate a free energy and the entropy of the scalar field in rotating black holes. As one approaches the stationary limit surface rather than the event horizon in a comoving frame, these become divergent. Only when the field is comoving with the black hole (i.e., $\Omega_0=\Omega_H$) do the free energy and entropy become divergent at the event horizon. In the Hartle-Hawking state the leading terms of the entropy are $A(1/h)+B\ln(h)+$ finite, where h is the cutoff in the radial coordinate near the horizon. In terms of the proper distance cutoff ϵ it is written as $S=NA_H/\epsilon^2$. The origin of the divergence is that the density of states on the stationary surface and beyond it diverges. [S0556-2821(96)01816-4]

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I. INTRODUCTION

By comparing black hole physics with thermodynamics and from the discovery of black hole evaporation by Hawking, it was shown that the black hole entropy is proportional to the horizon area [1,2]:

$$S_{\mathrm{BH}}=\frac{A_H}{4} \tag{1}$$

in units $\hbar=c=G=1$. In the Euclidean path integration approach it was shown that the tree level contribution of the gravitational action gives the black hole entropy [3]. However, the exact statistical origin of the Bekenstein-Hawking black hole entropy is unclear.

Recently many efforts have been concentrated on understanding the statistical origin of black hole thermodynamics, especially black hole entropy, by various methods (for review see [4]): 't Hooft calculated the entropy of a quantum field propagating outside the black hole. After regularization, he obtained $S=\frac{1}{4}A_H$ (the brick wall method) [5–8]. Another approach is to identify the black hole entropy with the entanglement entropy S_{ent}. The entanglement entropy arises from ignoring the degree of freedom of a proper region of space: $S=-\mathrm{Tr}\rho\ln\rho$. It is found that the entropy is proportional to the area of the boundary [9]. In fact, the entanglement entropy and the entropy in the brick wall method are equivalent. Frolov and Novikov argued that black hole entropy can be obtained by identifying the dynamical degrees of freedom with the states of all fields which are located inside the black hole [10]. The leading term of the entropy obtained by these methods is proportional to the surface area of the horizon. However the proportional coefficient diverges as the cutoff goes to zero. The conical approach also gives results similar to others [11]. The divergence is because of an infinite number of states near the horizon, which can be explained by the equivalence principle [12]. An alternative approach by Frolov is to identify the black hole entropy with the thermodynamic one. In this approach the entropy is finite [13]. However they all treat only the spherical symmetrical black hole.

If the black hole has a rotation, what is changed? It is well known that in a rotating black hole spacetime a particle with zero angular momentum dropped from infinity is dragged just by the influence of gravity so that it acquires an angular velocity in the same direction in which the black hole rotates. The dragging becomes more and more extreme the nearer one approaches the horizon of the black hole. This effect is called the dragging of inertial frames [14].

Thus the field at equilibrium with the rotating black hole must also be rotating. The rotation is not rigid but locally is different. So the velocity of the radiation does not exceed the velocity of light. However we do not know how to treat the equilibrium state with a locally different angular velocity. More precisely there are no global static coordinates. So we assume that the radiation has a rigid rotation Ω_0 smaller than or equal to the extremum value of the local rotation. In a rotating black hole the extremum value of it is Ω_H, which is the angular velocity of the event horizon.

Recently we considered the black hole entropy by the brick wall method in the charged Kerr black hole in [15] and showed the entropy is proportional to the event horizon in Hartle-Hawking states. In this paper to more deeply understand the black hole entropy we shall investigate the black hole entropy by the brick wall method in various stationary black holes: the Kaluza-Klein black hole [16] which is the solution of the four-dimensional effective theory reduced from the five-dimensional Kaluza-Klein theory, and the Sen black hole [17] which is the solution of the Einstein-Maxwell dilaton-antisymmetric tensor gauge field theory came from the heteroitic string theory, and the Kerr-Newman black hole [18] which is the solution of the Einstein-Maxwell theory.

In order to understand the equilibrium state of the radiation (the field) in the rotating black hole spacetime in Sec. II we will first consider the rotating heat bath in the flat spacetime. In Sec. III we will consider the radiation in the equilibrium state in Rindler spacetime with rotation, which is the most simple spacetime having the event horizon and a rotation. In Sec. IV we will investigate the entropy of the quantum field in the stationary black hole background. We find the condition to give the finite value to the free energy and

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the entropy. In Sec. V we calculate the entropy in the Hartle-Hawking state for the rotating black holes. The final section is devoted to the summary.

II. A ROTATING HEAT BATH

Let us consider a massless scalar field with a constant angular velocity Ω_0 about the z axis at thermal equilibrium with a temperature $T=1/\beta$ in Minkowski spacetime, for which the line element in cylindrical coordinates is given by

$$ds^2=-dt^2+r^2d\phi^2+dr^2+dz^2. \tag{2}$$

In this spacetime the positive frequency field mode can be written as $\Phi_{q,m}(x)=f_{qm}(r,z)e^{-i\omega t+im\phi}$, where q denotes a quantum number and m is the azimuthal quantum number.

For such an equilibrium ensemble of the states of the scalar field the partition function is given by

$$Z=\sum_{n_q,m} e^{-n_q(\omega_q-m\Omega_0)\beta} \tag{3}$$

and the free energy is given by

$$\beta F=\sum_m \int_0^\infty d\omega g(\omega,m)\ln(1-e^{-\beta(\omega-m\Omega_0)}), \tag{4}$$

where $g(\omega,m)$ is the density of state for a fixed ω and m.

Following 't Hooft we assume that all possible modes of a scalar field vanish at $r=r_1$ (r_1 is very small) and at $r=L$. In the WKB approximation with $\Phi=e^{iS(r)-i\omega t+im\phi+ikz}$ the radial wave number $K(x,\omega,m)=\partial_r S$ is given by

$$K^2(x,\omega,m)=\omega^2-\frac{m^2}{r^2}-k^2. \tag{5}$$

This expression denotes the ellipsoid in momentum phase space at a fixed frequency ω. The total number of modes with energy less than ω and a fixed m is obtained by integrating over the volume of phase space, which is determined by Eq. (5):

$$\begin{aligned}\Gamma(\omega,m)&=\sum_m \int d\phi dz\int_{r_1}^L dr\frac{1}{\pi}\int dkK(x,\omega,m)\\&=\frac{1}{\pi}\sum_m \int d\phi dz\int_{r_1}^L dr\int dk\left(\omega^2-\frac{m^2}{r^2}-k^2\right)^{1/2}.\end{aligned} \tag{6}$$

The integration over k must be carried out over the phase space that satisfies $K^2\geqslant 0$. $\Gamma(\omega,m)$ can be obtained by investigating the shape of expression (5) in momentum phase space. Thus the free energy, after the integration by parts, becomes

$$\begin{aligned}\beta F&=-\beta\sum_m \int_0^\infty d\omega\Gamma(\omega,m)\frac{1}{e^{\beta(\omega-m\Omega_0)}-1}\\&=-\frac{\beta}{2}\int_0^\infty d\omega\int_{-r\omega}^{r\omega}dm\left(\omega^2-\frac{m^2}{r^2}\right)\frac{1}{e^{\beta(\omega-m\Omega_0)}-1},\end{aligned} \tag{7}$$

where we assume that the azimuthal quantum number m is a continuous parameter. By making the change of variable $m=r\omega u$ we obtain the free energy

$$\beta F=-\frac{N}{\beta^3}\int d\phi dz\int_{r_1}^L\frac{r}{(1-v^2)^2}dr, \tag{8}$$

where N is a constant and $v=r\Omega_0$. Note that as L goes to $1/\Omega_0$ this partition function diverges as γ^4, where $\gamma=(1-v^2)^{-1/2}$.

From expression (8) it is easy to obtain expressions for the energy E, angular momentum J, and entropy S of radiation:

$$J=\langle m\rangle_{\rm av}=\frac{1}{\beta}\frac{\partial}{\partial\Omega_0}(\beta F)=4N\frac{1}{\beta^4}\Omega_0\int r^2\gamma^6 r dr d\phi dz, \tag{9}$$

$$E=\langle\omega\rangle_{\rm av}=\Omega_0 J-\frac{\partial}{\partial\beta}(\beta F)=N\frac{1}{\beta^4}\int (3+v^2)\gamma^4 r dr d\phi dz, \tag{10}$$

$$S=\beta^2\frac{\partial}{\partial\beta}F=4N\frac{1}{\beta^3}\int \gamma^4 r dr d\phi dz. \tag{11}$$

These coincide with those in [19]. Similarly to the free energy F these expressions J,E, and S diverge as $L\to 1/\Omega_0$. The divergence is related to the rigid rotation. In a rigid rotating system the velocity of the comoving observer grows as one moves from the origin to infinity. So beyond some point the velocity exceeds the velocity of the light. This is unphysical. Thus a rotating system cannot have the size greater than $1/\Omega_0$. Therefore to obtain a finite value for J,E, and S, we must take $L<1/\Omega_0$. In such a finite system $\omega>m\Omega_0$.

Now let us consider the above problem in the comoving coordinate that is rotating with angular velocity Ω_0. The line element in the comoving frame is given by

$$ds^2=-(1-\Omega_0^2r^2)dt^2+2\Omega_0 r d\phi' dt+dr^2+dz^2, \tag{12}$$

where we have used $\phi'=\phi-\Omega_0 t$. In this coordinate the positive frequency field mode is written as $\Phi_{qm}(x)=\overline{f_{qm}}(r,z)e^{-i\omega' t+im\phi'}$.

Because in the comoving frame the field has no rotation the free energy is given by

$$\beta F=\int_0^\infty d\omega' g'(\omega')\ln(1-e^{-\beta\omega'}), \tag{13}$$

where $g'(\omega')$ is the density of state for a fixed ω'. In the WKB approximation the Klein-Gordon equation $\Box\Phi=0$ yields the constraint [20]

$$g^{ab}k_ak_b=0 \tag{14}$$

or

$$-(\omega'-\Omega_0 m)^2+\left(\frac{1}{r^2}m^2+k^2+p^2\right)=0, \tag{15}$$

where $p=\partial S/\partial r$. In the region where $\Omega_0 r<1$, for a fixed ω', this expression represents the ellipsoid in momentum space. Therefore the total number of modes with energy less than ω' is given by

$$\Gamma'(\omega')=\frac{1}{\pi}\sum_m \, d\phi dz\int dr\int dk\left((\omega'-m\Omega_0)^2-\frac{m^2}{r^2}-k^2\right)^{1/2} \tag{16}$$

$$=\frac{4}{3}\int d\phi dz\int_{r_1}^{L} dr\frac{r}{(1-\Omega_0^2r^2)^2}\,\omega'^3, \tag{17}$$

which is the volume of the ellipsoid. Expression (16) is just the same form as Eq. (6) when $\omega\to\omega-m\Omega_0$. The phase volume (17) diverges as $L\to 1/\Omega_0$. Inserting expression (17) into Eq. (13) and integrating we get

$$\beta F=-\frac{N}{\beta^3}\int d\phi dz\int_{r_1}^{L} dr\frac{r}{(1-\Omega_0^2r^2)^2}. \tag{18}$$

This expression is the same with Eq. (8). From this we get the energy E' and the entropy S:

$$E'=\langle\omega'\rangle_{\rm av}=-\frac{\partial}{\partial\beta}(\beta F)=3\frac{N}{\beta^4}A\int_{r_1}^{L} dr\frac{r}{(1-\Omega_0^2r^2)^2}, \tag{19}$$

$$S=\beta^2\frac{\partial}{\partial\beta}(\beta F)=4\frac{N}{\beta^3}A\int_{r_1}^{L} dr\frac{r}{(1-\Omega_0^2r^2)^2}, \tag{20}$$

where $A=\int d\phi dz$. It is noted that the entropy S is the same with Eq. (11) and the energy E' is satisfied with $E'=E-\Omega_0 J$. This fact shows that the coordinate transformation to the comoving frame only changes the energy and does not change the entropy in the WKB approximation. Thus in the case of calculating the entropy or the free energy it is convenient to choose the comoving frame. It is noted that in the comoving frame the divergence is related to the time component g_{tt} of the metric (12).

III. A THERMAL BATH IN RINDLER SPACETIME WITH A ROTATION

In this section we will consider the thermal equilibrium state of the scalar field with the mass μ and a uniform rotation about the z axis in Rindler spacetime. The line element of the Rindler spacetime in cylindrical coordinates is given by

$$ds^2=-\xi^2 d\eta^2+d\xi^2+r^2d\phi^2+dr^2. \tag{21}$$

In this spacetime the event horizon is at $\xi=0$, and ξ=const represent the trajectory of the uniform acceleration [21]. The importance of Rindler spacetime is that in the large black hole mass limit the metric of the black spacetime reduces to that of Rindler spacetime [7].

As in Sec. II, the WKB approximation with $\Phi(x)=e^{-i\omega t+im\phi+iS(\xi,r)}$ yields

$$K^2(\xi,r,\omega,m)=\frac{\omega^2}{\xi^2}-\frac{1}{r^2}m^2-p_r^2-\mu^2, \tag{22}$$

where $K=\partial_\xi S$ and $p_r=\partial_r S$. In this section we will calculate the free energy by using the slightly different method with that in Sec. II.

It is important to note that in the WKB approximation the density of state $g(\omega,m)$ is determined by the constraint (22), and that the free energy is singular at $\omega=m\Omega_0$. In particular if $\omega-m\Omega_0<0$ the free energy becomes an imaginary number. However in the WKB approximation we can easily see $\overline{\omega}=\omega-m\Omega_0>0$ in the region such that $\xi-\Omega_0 r>0$. But in the region such that $\xi-\Omega_0 r<0$ it is possible that $\omega-m\Omega_0<0$. (More details are in Sec. IV.) Therefore to obtain the finite value for the free energy we must require the system to be in the region such that $\xi-\Omega_0 r>0$. Then the free energy is written as

$$\begin{aligned}\beta F&=\sum_m \int_{m\Omega_0}^{\infty} d\omega g(\omega,m)\ln(1-e^{-\beta(\omega-m\Omega_0)})\\&=\int_0^\infty d\omega\sum_m g(\omega+m\Omega_0,m)\ln(1-e^{-\beta\omega})\\&=-\beta\int_0^\infty d\omega\frac{1}{e^{\beta\omega}-1}\int dm\Gamma(\omega+m\Omega_0,m),\end{aligned} \tag{23}$$

where we have integrated by parts and assumed that the quantum number m is a continuous variable. The total number of modes with energy less than ω is obtained by integrating over the volume of phase space

$$\begin{aligned}\Gamma(\overline{\omega})&=\int dm\Gamma(\omega+m\Omega_0,m)\\&=\int dm\int d\phi dr\int_{r_1}^{L} d\xi\frac{1}{\pi}\int dp_r K(\xi,r,\omega+m\Omega_0,m)\\&=\frac{1}{\pi}\int dm\int d\phi dr\int_{r_1}^{L} d\xi\int dp_r\left(\frac{\omega^2}{\xi^2}+\frac{2}{\xi^2}m\Omega_0\omega+\frac{m^2\Omega_0^2}{\xi^2}-\frac{1}{r^2}m^2-p_r^2-\mu^2\right)^{1/2}.\end{aligned} \tag{24}$$

The integrations over m and p_r must be carried out over the phase space that satisfies $K^2(\omega+m\Omega_0,m)\geqslant 0$. After the integration we obtain the number of states with energy less than ω, which is given by

$$\Gamma(\omega)=\frac{4}{3}\int d^3x\frac{\xi r}{\sqrt{(\xi^2-\Omega_0^2r^2)}}\left(\frac{\omega^2}{\xi^2-\Omega_0^2r^2}-\mu^2\right)^{3/2}. \tag{25}$$

Thus the free energy becomes

$$\beta F=-\frac{4}{3}\beta\int d^3x\int_{\mu\sqrt{\xi^2-\Omega_0^2r^2}}^{\infty}d\omega\frac{1}{e^{\beta\omega}-1}\times\frac{\xi r}{\sqrt{(\xi^2-\Omega_0^2r^2)}}\left(\frac{\omega^2}{\xi^2-\Omega_0^2r^2}-\mu^2\right)^{3/2}. \quad (26)$$

For a massless scalar field ($\mu=0$) the free energy becomes

$$\beta F=-\frac{N}{\beta^3}\int d\phi dr\int_{\xi_1}^{L}d\xi\frac{\xi r}{(\xi^2-\Omega^2r^2)^2}. \quad (27)$$

From this we get the energy E, the angular momentum J, and the entropy S of the field

$$J=\langle m\rangle_{\rm av}=4\frac{N}{\beta^4}\Omega_0\int\frac{r^2}{(\xi^2-\Omega_0^2r^2)^3}\xi rd\xi drdz, \quad (28)$$

$$E=\langle E\rangle_{\rm av}=\frac{N}{\beta^4}\int\frac{3\xi^2+\Omega_0^2r^2}{(\xi^2-\Omega_0^2r^2)^3}\xi rd\xi drdz, \quad (29)$$

$$S=4\frac{N}{\beta^3}\int\frac{1}{(\xi^2-\Omega_0^2r^2)^2}\xi rd\xi drdz. \quad (30)$$

It is noted that the thermodynamic quantities F,E, and S are divergent as $\xi\to\Omega_0 r$ rather than the event horizon. Only in the $\Omega_0=0$ case the divergence occurs at the horizon $\xi=0$. Such a fact can be easily understand in the comoving frame, of which the line element is given by

$$\begin{aligned}ds^2&=-\xi^2d\eta^2+r^2(d\phi'+\Omega_0d\eta)^2+d\xi^2+dr^2\\&=-(\xi^2-\Omega_0^2r^2)d\eta^2+2\Omega_0r^2d\eta d\phi'\\&\quad+r^2d\phi'^2+d\xi^2+dr^2,\end{aligned} \quad (31)$$

where we used $\phi'=\phi-\Omega_0\eta$. In this spacetime the event horizon is at $\xi=0$. In addition to the event horizon there is a stationary limit surface at $\xi=\Omega_0 r$, where the Killing vector ∂_η becomes null. That surface is the elliptic hypersurface [22]. In the interval $0<\xi<\Omega_0 r$, the Killing vector is spacelike. We can also show that the entropy in the comoving frame is the same form with Eq. (30). *These facts imply that the divergence of the thermodynamic quantities is deeply related to the stationary limit surface in the comoving frame rather than the event horizon.*

IV. ENTROPY OF A SCALAR FIELD IN A ROTATING BLACK HOLE

A. General formalism

Let us consider a scalar field with mass μ in thermal equilibrium at temperature $1/\beta$ in the rotating black hole background, of which line element is generally given by

$$\begin{aligned}ds^2&=g_{tt}(r,\theta)dt^2+2g_{t\phi}(r,\theta)dtd\phi+g_{\phi\phi}(r,\theta)d\phi^2\\&\quad+g_{rr}(r,\theta)dr^2+g_{\theta\theta}(r,\theta)d\theta^2.\end{aligned} \quad (32)$$

This metric has two Killing vector fields: the timelike Killing vector $\xi^\mu=(\partial_t)^\mu$ and the axial Killing vector $\psi^\mu=(\partial_\phi)^\mu$. The metrics we are concerned with of the Kaluza-Klein, the Sen, and the Kerr-Newman black holes are in the Appendix. The properties of those metrics are

$$g_{tt}g_{\phi\phi}-g_{t\phi}^2=-\Delta(r)\sin^2\theta\to 0 \quad (33)$$

and

$$(g_{tt}g_{\phi\phi}-g_{t\phi}^2)g_{rr}\to\text{finite} \quad (34)$$

as one approaches the horizon. Another property is that there are two important surfaces (the event horizon and the stationary limit surface), and the two surfaces do not coincide. On the stationary limit surface the Killing vector ξ^μ vanishes, and the Killing vector $\xi^\mu+\Omega_H\psi^\mu$ is null on the horizon, where Ω_H is the angular velocity of the horizon.

The equation of motion of the field with mass μ and arbitrarily coupled to the scalar curvature $R(x)$ is

$$[\nabla_\mu\nabla^\mu-\xi R-\mu^2]\Psi=0, \quad (35)$$

where ξ is an arbitrary constant. The $\xi=1/6$ and $\mu=0$ cases correspond to the conformally coupled one. We assume that the scalar field is rotating with a constant azimuthal angular velocity Ω_0. The associated conserved quantities are angular momentum J. The free energy of the system is then given by

$$F=\frac{1}{\beta}\sum_m\int_0^\infty d\mathscr{E}g(\mathscr{E},m)\ln(1-e^{-\beta(\mathscr{E}-m\Omega_0)}), \quad (36)$$

where $g(\mathscr{E},m)$ is the density of state for a given $\mathscr{E}$ and m.

To evaluate the free energy we will follow the brick wall method of 't Hooft [5]. Following the brick wall method we impose a small radial cutoff h such that

$$\Psi(x)=0 \quad \text{for } r\leq r_H+h, \quad (37)$$

where r_H denotes the coordinate of the event horizon. To remove the infrared divergence we also introduce another cutoff $L\gg r_H$ such that

$$\Psi(x)=0 \quad \text{for } r\geq L. \quad (38)$$

It is noted that the brick wall is spherically symmetric. In the WKB approximation with $\Psi=e^{-i\mathscr{E}t+im\phi+iS(r,\theta)}$, Eq. (35) yields the constraint [20]

$$p_r^2=\frac{1}{g^{rr}}[-g^{tt}\mathscr{E}^2+2g^{t\phi}\mathscr{E}m-g^{\phi\phi}m^2-g^{\theta\theta}p_\theta^2-V(x)], \quad (39)$$

where $p_r=\partial_rS$, $p_\theta=\partial_\theta S$, and $V(x)=\xi R(x)+\mu^2$. In the WKB approximation it is important to note that the number of states for a given $\mathscr{E}$ is determined by p_θ, p_r, and m. The number of modes with energy less than $\mathscr{E}$ and with a fixed m is obtained by integrating over p_θ in phase space:

$$\begin{aligned}\Gamma(\mathscr{E},m)&=\frac{1}{\pi}\int d\phi d\theta\int dr\int dp_\theta p_r(\mathscr{E},m,x)\\&=\frac{1}{\pi}\int d\phi d\theta\int dr\int dp_\theta\bigg[\frac{1}{g^{rr}}[-g^{tt}\mathscr{E}^2+2g^{t\phi}\mathscr{E}m\\&\quad-g^{\phi\phi}m^2-g^{\theta\theta}p_\theta^2-V(x)]\bigg]^{1/2}.\end{aligned} \quad (40)$$

The integration over p_θ must be carried over the phase space such that $p_r \geqslant 0$.

At this point we need some remarks. In a rotating system, in general, there is a superradiance effect, which occurs when $0 < \mathscr{E} < m\Omega_0$. For this range of the frequency the free energy F becomes a complex number. In the case $\mathscr{E} = m\Omega_0$ the free energy is divergent. Therefore to obtain a real finite value for the free energy F, we must require that $\mathscr{E} > m\Omega_0$. (For $0 < \mathscr{E} < m\Omega_0$ the free energy diverges. See below.) This requirement says that we must restrict the system to be in the region such that $g'_{tt} \equiv g_{tt} + 2\Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} < 0$. In this region $\mathscr{E} - m\Omega_0 > 0$, so the free energy is a finite real value. It is easily showed as follows. Let us define $E = \mathscr{E} - m\Omega_0$. Then it is written as

$$\begin{aligned} E &= \left(\frac{g^{t\phi}}{g^{tt}} - \Omega_0\right)m + \frac{1}{-g^{tt}}[(g^{t\phi}m)^2 + (-g^{tt})(V + g^{\phi\phi}m^2 \\ &\quad + g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2)]^{1/2} \\ &= (\Omega - \Omega_0)m + \frac{-\mathscr{D}}{g_{\phi\phi}}\left[\frac{1}{-\mathscr{D}}m^2 + \frac{g_{\phi\phi}}{-\mathscr{D}}\left(V + \frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}}\right)\right]^{1/2}, \end{aligned} \tag{41}$$

where we used

$$g^{tt} = \frac{g_{\phi\phi}}{\mathscr{D}}, \quad g^{t\phi} = \frac{-g_{t\phi}}{\mathscr{D}}, \quad g^{\phi\phi} = \frac{g_{tt}}{\mathscr{D}}, \tag{42}$$

and $\Omega = -g_{t\phi}/g_{\phi\phi}$. Here $-\mathscr{D} = g_{t\phi}^2 - g_{tt}g_{\phi\phi}$. From Eq. (41), for all m, p_r, and p_θ, one can see the condition such that $E > 0$ is

$$\frac{\sqrt{-\mathscr{D}}}{g_{\phi\phi}} \pm (\Omega - \Omega_0) > 0 \tag{43}$$

or

$$g'_{tt} \equiv g_{tt} + 2\Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} < 0. \tag{44}$$

Therefore in the region such that $-g'_{tt} > 0$ (called region I) the free energy is real, but in region such that $-g'_{tt} < 0$ (called region II) the free energy is complex. However in region I the integration over the momentum phase space is convergent. But in region II the integration over the momentum phase is divergent. These facts become more apparent if we investigate the momentum phase space. In region I the possible points of p_i satisfying $\mathscr{E} - \Omega_0 p_\phi = E$ for a given E are located on the surface

$$\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} + \frac{-g'_{tt}}{-\mathscr{D}}\left(p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{g'_{tt}}E^2 = \left(\frac{E^2}{-g'_{tt}} - V\right),\right. \tag{45}$$

which is the ellipsoid, *a compact surface*. Here $p_\phi = m$. So the density of state $g(E)$ for a given E is finite and the integrations over p_i give a finite value. But in region II the possible points of p_i are located on the surface

$$\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} - \frac{g'_{tt}}{-\mathscr{D}}\left(p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{g'_{tt}}E\right)^2 = -\left(\frac{E^2}{g'_{tt}} + V\right), \tag{46}$$

which is the hyperboloid, *a noncompact surface*. So $g(E)$ diverges and the integration over p_i diverges. In the case of $g'_{tt} = 0$, the possible points are given by the surface

$$\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} = \frac{p_\phi - (g_{\phi\phi}E^2/\mathscr{D} + V)/[(2g_{t\phi}/\mathscr{D})E]}{-\mathscr{D}/2g_{t\phi}E}, \tag{47}$$

which is elliptic paraboloid and also *noncompact*. Therefore the value of the p_i integration is divergent. Actually the surface such that $g'_{tt} = 0$ is the velocity of the light surface (VLS). Beyond VLS (in region II) the comoving observer must move more rapidly than the velocity of light. Thus we will assume that the system is in region I. (For the possible region I see Sec. IV B.) For example, in the case of $\Omega_0 = 0$ the points satisfying $g'_{tt} = 0$ are on the stationary limit surface. The region of the outside (inside) of the stationary limit surface corresponds to region I (II). In the rotating system in Sec. II region I is $r < 1/\Omega_0$ and $r > 1/\Omega_0$ corresponds to region II. In the Rindler spacetime with a rotation, $\xi > \Omega_0 r$ corresponds to region I, and $\xi < \Omega_0 r$ to region II.

With the assumption that the system is in region I we can obtain the free energy as

$$\begin{aligned} \beta F &= \sum_m \int_{m\Omega_0}^\infty d\mathscr{E} g(\mathscr{E}, m) \ln(1 - e^{-\beta(\mathscr{E} - m\Omega_0)}) \\ &= \int_0^\infty d\mathscr{E} \sum_m g(\mathscr{E} + m\Omega_0, m) \ln(1 - e^{-\beta\mathscr{E}}) \\ &= -\beta \int_0^\infty d\mathscr{E} \frac{1}{e^{\beta\mathscr{E}} - 1} \int dm \Gamma(\mathscr{E} + m\Omega_0, m), \end{aligned} \tag{48}$$

where we have integrated by parts and we assume that the quantum number m is a continuous variable. The integrations over m and p_θ yield

$$\begin{aligned} F &= -\frac{4}{3}\int d\phi d\theta \int_{r_H + h}^L dr \int_{V(x)\sqrt{-g'_{tt}}}^\infty \\ &\quad \times d\mathscr{E} \frac{1}{e^{\beta\mathscr{E}} - 1} \frac{\sqrt{g_4}}{\sqrt{-g'_{tt}}}\left(\frac{\mathscr{E}^2}{-g'_{tt}} - V(x)\right)^{3/2}. \end{aligned} \tag{49}$$

In particular when $\Omega_0 = 0$ and the nonrotating case $g_{t\phi} = 0$, the free energy (49) coincides with the expression in [5,8] and it is proportional to the volume of the optical space in the limit $V(x) = 0$ [23]. It is easy to see that the integrand diverges as $r_H + h$ or L approach the surface such that $g'_{tt} = 0$. In that case the contribution of the $V(x)$ can be negligible.

For a massless and minimally coupled scalar field case $(\mu = \xi = 0)$ the free energy reduces to

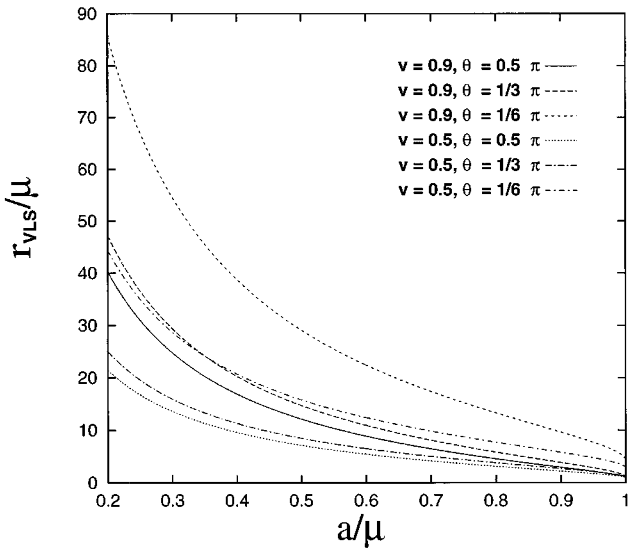

FIG. 1. The position of the outer velocity of light surface for the Kaluza-Klein black hole.

$$\beta F=-\frac{N}{\beta^3}\int d\theta d\phi\int_{r_H+h}^{L}dr\frac{\sqrt{g_4}}{(-g'_{tt})^2}$$
$$=-N\int_0^{\beta}d\tau\int d\theta d\phi\int_{r_H+h}^{L}dr\sqrt{g_4}\frac{1}{\beta_{\mathrm{local}}^4},\qquad(50)$$

where $\beta_{\mathrm{local}}=\sqrt{-g'_{tt}}\beta$ is the reciprocal of the local Tolman temperature [24] in the comoving frame. This form is just the free energy of a gas of massless particles at local temperature $1/\beta_{\mathrm{local}}$.

From this expression (50) it is easy to obtain expressions for the total energy U, angular momentum J, and entropy S of a scalar field

$$J=\langle m\rangle=-\frac{1}{\beta}\frac{\partial}{\partial\Omega_0}(\beta F)$$
$$=\frac{4N}{\beta^4}\int d\theta d\phi\int_{r_H+h}^{L}dr\frac{\sqrt{g_4}}{(-g'_{tt})^2}\frac{g_{\phi\phi}}{(-g'_{tt})}(\Omega_0-\Omega),\qquad(51)$$

$$U=\langle\mathcal{E}\rangle=\Omega_0J+\frac{\partial}{\partial\beta}(\beta F)$$
$$=\frac{N}{\beta^4}\int d\theta d\phi\int_{r_H+h}^{L}dr\frac{\sqrt{g_4}}{(-g'_{tt})^2}\times\left[3+4\frac{\Omega_0(\Omega_0-\Omega)g_{\phi\phi}}{(-g'_{tt})}\right],\qquad(52)$$

$$S=\beta^2\frac{\partial}{\partial\beta}F=\beta(U-F-\Omega_0J)$$
$$=4\frac{N}{\beta^3}\int d\theta d\phi\int_{r_H+h}^{L}dr\frac{\sqrt{g_4}}{(-g'_{tt})^2},\qquad(53)$$

which are also divergent as one approach the surface such that $g'_{tt}=0$.

B. The region such that $-g'_{tt}>0$

In this section we study where the possible region I is for three black holes, the Kaluza-Klein, and the Sen, the Kerr-Newman black holes, for $\Omega_0=\Omega_H,\Omega_0<\Omega_H$, and the extreme case with $\Omega_0=\Omega_H$.

1. The Kaluza-Klein black hole

(a) $\Omega_0=\Omega_H$ case: In the $\Omega_0=\Omega_H$ case the position of the light of velocity surface is exactly found. In such a case g'_{tt} can be written as

$$g'_{tt}=g_{tt}+2\Omega_Hg_{t\phi}+\Omega_H^2g_{\phi\phi}\qquad(54)$$

$$=\frac{\mu^2}{B\Sigma}(x-\overline{r_H})\left\{\frac{y^2\sin^2\theta}{4\overline{r_H^2}}(1-v^2)x^3+\frac{y^2\sin^2\theta}{4\overline{r_H^2}}[2-\overline{r_-}(1-v^2)]x^2+\left[-1+\frac{y^2\sin^2\theta}{4\overline{r_H^2}}[4+y^2(1-v^2)\cos^2\theta-2\overline{r_-}]\right]x+\left[\overline{r_-}+\frac{y^2\sin^2\theta}{4\overline{r_H^2}}[-4\overline{r_H}-\overline{r_-}y^2(1-v^2)\cos^2\theta]\right]\right.$$
$$\equiv\frac{\mu^2}{B\Sigma}(x-\overline{r_H})\frac{y^2\sin^2\theta}{4\overline{r_H^2}}(1-v^2)(x^3+a_1x^2+a_2x+a_3)\qquad(55)$$

for $\theta\neq0$, where $x=r/\mu$, $y=a/\mu$, $\overline{r_H}=r_H/\mu$, and $\overline{r_-}=r_-/\mu$. From this we can see that there are two VLS's. One is the horizon ($r=r_H$), and another light of velocity surface (call outer VLS) is given by [25]

$$r_{\mathrm{VLS}}=2\mu\sqrt{-Q}\cos\left(\frac{1}{3}\Theta\right)-\frac{1}{3}a_1\mu,\qquad(56)$$

where

$$\Theta=\arccos\left(\frac{P}{\sqrt{-Q^2}}\right)\qquad(57)$$

with

$$Q=\frac{3a_2-a_1^2}{9},\quad P=\frac{9a_1a_2-27a_3-2a_1^3}{54}.\qquad(58)$$

In the case of the slowly rotating black hole (a is small) the VLS is approximately given by

$$r_{\mathrm{VLS}}\sim2\mu\frac{r_H}{a\sqrt{1-v^2}\sin\theta}-\frac{1}{3}\left(\frac{2}{1-v^2}-\frac{r_-}{\mu}\right)\mu,\qquad(59)$$

which is an open, roughly, cylindrical surface. As $v\to1$ or $a\to0$ the VLS becomes more distant, which came from the

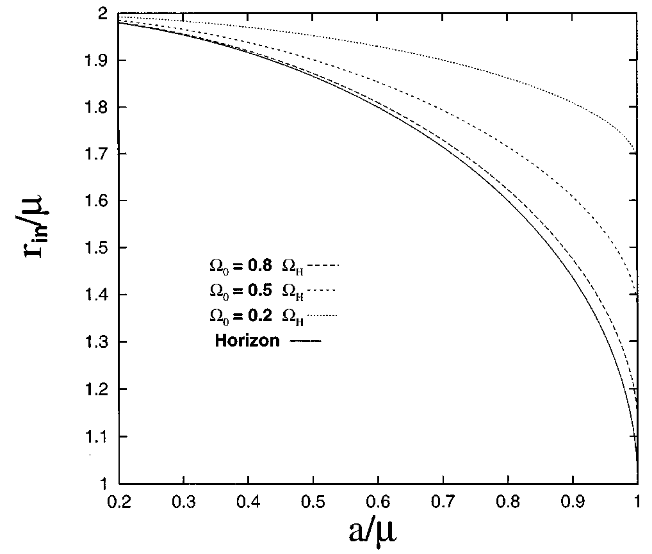

FIG. 2. The position of r_{in} at $\theta=0.5\pi$ for the Kaluza-Klein black hole. $v=0.5$.

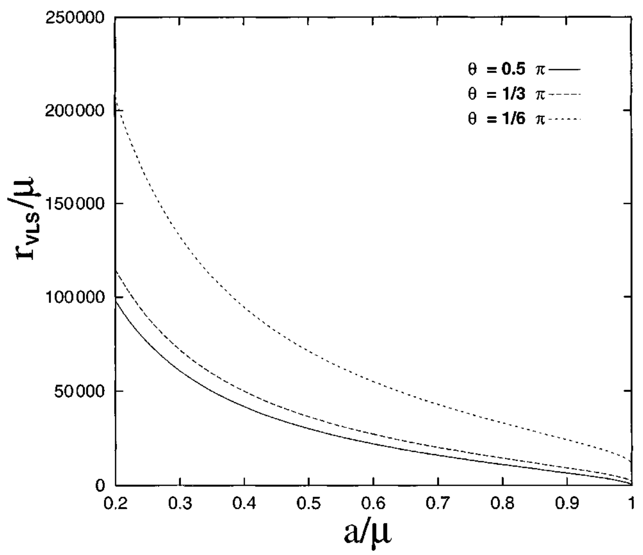

FIG. 3. The position of the outer velocity of light surface for the Sen black hole. $\gamma=5.0$.

fact that as $v\rightarrow 1$ or $a\rightarrow 0$ the coordinate angular velocity $d\phi/dt=-g_{t\phi}/g_{\phi\phi}$ vanishes. For $\theta=0$ it is always that $g'_{tt}<0$ for $r>r_H$. As $a\rightarrow\mu$ the outer VLS approaches the horizon. See Fig. 1.

(b) $\Omega_0<\Omega_H$ case. In this case $g'_{tt}=0$ is a fourth-order polynomial equation in r for a given θ. Region I corresponds to $r_{\mathrm{in}}<r<r_{\mathrm{VLS}}$. At $\theta=\pi/2$ r_{in} is between the stationary limit surface and the event horizon, and at $\theta=0$ r_{in} contacts with the event horizon. Actually the inner VLS r_{in} is between the stationary limit surface and the event horizon for all θ. The particular point is that as $\Omega_0\rightarrow\Omega_H$, r_{in} approaches the horizon. However it does attach to the horizon only when $\Omega_0=\Omega_H$. While the outer velocity of light surface is located at a very far distance from the horizon, it is a roughly cylindrical surface as in case $\Omega_0=\Omega_H$. For the position of the inner VLS see Fig. 2.

(c) The extreme black hole case with $\Omega_0=\Omega_H$. The extreme black hole for the Kaluza-Klein black hole occurs when $\mu^2=a^2$. In this case the inner horizon and outer horizon are at the same place. At $\theta=1/2\pi$, g'_{tt} is written as

$$g'_{tt}=\frac{\mu^2}{B\Sigma}(x-\overline{r_H})^2x\left(x+\frac{2}{1-v^2}\right)\frac{1-v^2}{4},\qquad(60)$$

which shows that the possible region such that $g'_{tt}<0$ does not exist at $\theta=1/2\pi$. Therefore in the extreme black hole case it is impossible to consider the brick wall model of 't Hooft.

2. The Sen black hole

(a) $\Omega_0=\Omega_H$ case. In $\Omega_0=\Omega_H$ case g'_{tt} can be written as

$$g'_{tt}=g_{tt}+2\Omega_H g_{t\phi}+\Omega_H^2 g_{\phi\phi}\qquad(61)$$

$$=\frac{\mu^2}{\Sigma}(x-\overline{r_H})\left\{\frac{y^2\sin^2\theta}{4\overline{r_H^2}\cosh^4\gamma}x^3+\frac{y^2\sin^2\theta}{4\overline{r_H^2}\cosh^4\gamma}(2\cosh2\gamma\right.$$

$$\left.-\overline{r_-})x^2+\left[-1+\frac{y^2\sin^2\theta}{\overline{r_H^2}}+\frac{y^2\sin^2\theta}{4\overline{r_H^2}\cosh^4\gamma}(y^2\cos^2\theta-2\overline{r_-}\cosh2\gamma)\right]x+\left[\overline{r_-}+\frac{y^2\sin^2\theta}{\overline{r_H}}-\frac{y^2\sin^2\theta}{4\overline{r_H^2}\cosh^4\gamma}(\overline{r_-}y^2\cos^2\theta)\right]\right\}$$

$$\equiv\frac{\mu^2}{\Sigma}(x-\overline{r_H})\frac{y^2\sin^2\theta}{4\overline{r_H^2}\cosh^4\gamma}(x^3+a_1x^2+a_2x+a_3)\qquad(62)$$

for $\theta\neq 0$, where $x=r/\mu$, $y=a/\mu$, $\overline{r_H}=r_H/\mu$, and $\overline{r_-}=r_-/\mu$. Then the exact position of the inner VLS and outer VLS are are given by

$$r_{\mathrm{in}}=r_H,\quad r_{\mathrm{VLS}}=2\mu\sqrt{-Q}\cos\left(\frac{1}{3}\Theta\right)-\frac{1}{3}a_1\mu.\qquad(63)$$

The position of the outer VLS for small a is approximately given by

$$r_{\mathrm{VLS}}\sim\frac{2\mu r_H\cosh^2\gamma}{a\sin\theta}-\frac{1}{3}\left(2\cosh(2\gamma)-\frac{r_-}{\mu}\right)\mu,\qquad(64)$$

which is an open, roughly, cylindrical surface. As $a\rightarrow 0$ the VLS goes to the infinity, and it disappears when $a=0$. As γ or a is increasing the VLS approaches the horizon. At $\theta=(1/2)\pi$, similarly to the Kaluza-Klein black hole, $g'_{tt}<0$ for $r>r_H$. See Fig. 3.

(b) $\Omega_0<\Omega_H$ case. In this case $g'_{tt}=0$ is also a fourth-order equation in r for a given θ. Similarly to the Kaluza-Klein black hole region I is $r_{\mathrm{in}}<r<r_{\mathrm{VLS}}$. At $\theta=0$ the inner VLS r_{in} is at the horizon, and at $\theta=\pi/2$ r_{in} is located between the stationary limit surface and the event horizon. See Fig. 4. As $\Omega_0\rightarrow\Omega_H$, r_{in} approaches the horizon. Only when $\Omega_0=\Omega_H$ it coincides with the event horizon. The outer ve-

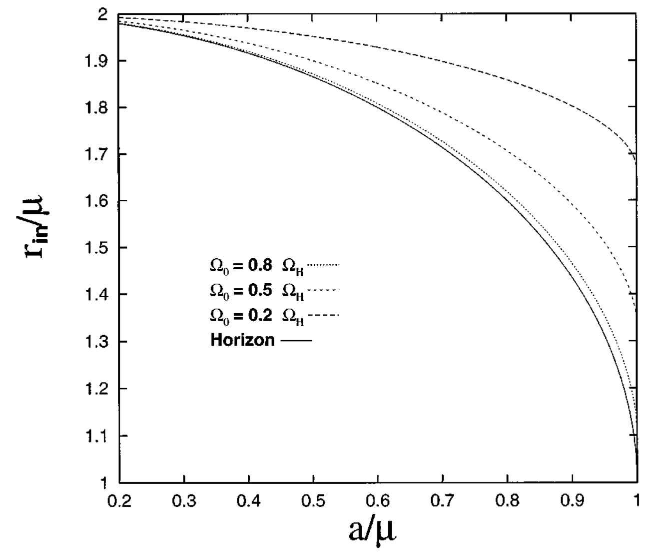

FIG. 4. The position of the inner velocity of light surface for the Sen black hole. $\gamma=5.0, \theta=0.5\pi$.

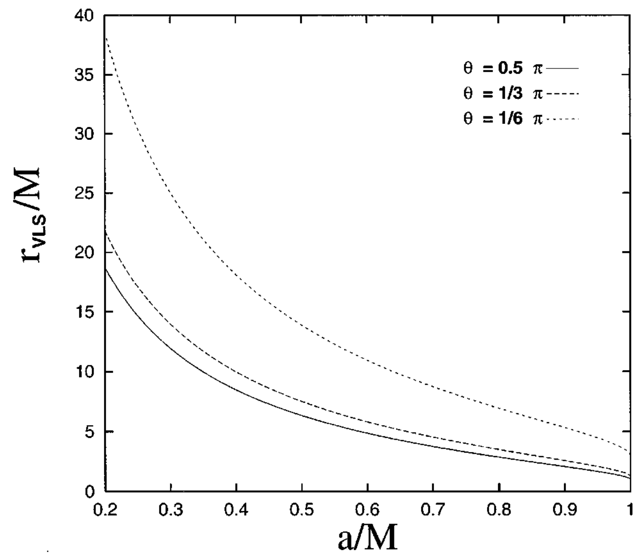

FIG. 5. The position of the outer light of velocity surface for the Kerr-Newman black hole. $e=0.0$.

locity of light surface, in the case of small a, locates at the very far distance from the horizon, and it is a roughly cylindrical surface.

(c) The extreme black hole case with $\Omega_0=\Omega_H$. The extreme black hole for the Kaluza-Klein black hole occurs when $\mu^2=a^2$. In this case the inner horizon and outer horizon are at the same place. At $\theta=(1/2)\pi$ g'_{tt} is written as

$$g'_{tt}=\frac{\mu^2}{\Sigma}(x-\overline{r_H})^2 x(x+2\cosh 2\gamma) \tag{65}$$

which shows that the possible region such that $g'_{tt}<0$ does not exist at $\theta=1/2\pi$. Therefore in the extreme black hole case it is impossible to consider the brick wall model of 't Hooft.

3. The Kerr-Newman black hole

(a) $\Omega_0=\Omega_H$ *case*. In $\Omega_0=\Omega_H$ case we can exactly find the position of the light of velocity surface. In such a case g'_{tt} can be written as

$$\begin{aligned}g'_{tt}&=g_{tt}+2\Omega_H g_{t\phi}+\Omega_H^2 g_{\phi\phi}\\&=\frac{M^2}{\Sigma}(x-\overline{r_H})\{\overline{\Omega}_H^2\sin^2\theta x^3+\overline{r_H}\overline{\Omega}_H^2\sin^2\theta x^2\\&\quad+[-1+\overline{\Omega}_H^2\sin^2\theta(y^2+y^2\cos^2\theta+\overline{r_H^2})]x \qquad (66)\\&\quad+[2(1-\overline{\Omega}_H y\sin^2\theta)^2-\overline{r_H}+\overline{r_H}\overline{\Omega}_H^2\sin^2\theta(\overline{r_H^2}+y^2\\&\quad+y^2\cos^2\theta)]\} \qquad (67)\end{aligned}$$

$$\equiv\frac{M^2}{\Sigma}(x-\overline{r_H})\overline{\Omega}_H^2\sin^2\theta(x^3+a_1x^2+a_2x+a_3) \tag{68}$$

for $\theta\neq 0$, where $x=r/M$, $y=a/M$, $z=e/M$, $\overline{\Omega}_H=M\Omega_H$, $\overline{r}_H=r_H/M$. Then the exact position of the outer light of velocity surface is given by

$$r_{\mathrm{VLS}}=2M\sqrt{-Q}\cos\left(\frac{1}{3}\Theta\right)-\frac{1}{3}a_1M. \tag{69}$$

For small a Eq. (69) is approximately given by

$$r_{\mathrm{VLS}}\sim\frac{1}{\Omega_H\sin\theta}-\frac{r_H}{3}, \tag{70}$$

which is an open, roughly, cylindrical surface. For $\theta=0$ it is always that $g'_{tt}<0$ for $r>r_H$. As $a\to 0$, r_{VLS} goes to infinity, and as $a\to\sqrt{M^2+e^2}$ it approaches the event horizon. See Fig. 5. The inner VLS r_{in} is the event horizon.

(b) $\Omega_0<\Omega_H$ case: In this case, similarly to other black holes, the inner VLS r_{in} approaches the horizon as $\Omega_0\to\Omega_H$. See Fig. 6. The inner VLS is a compact surface, which shrinks to the horizon as $\Omega_0\to\Omega_J$. See Fig. 7. The outer VLS is at far place, which disappears when $\Omega_0=0$.

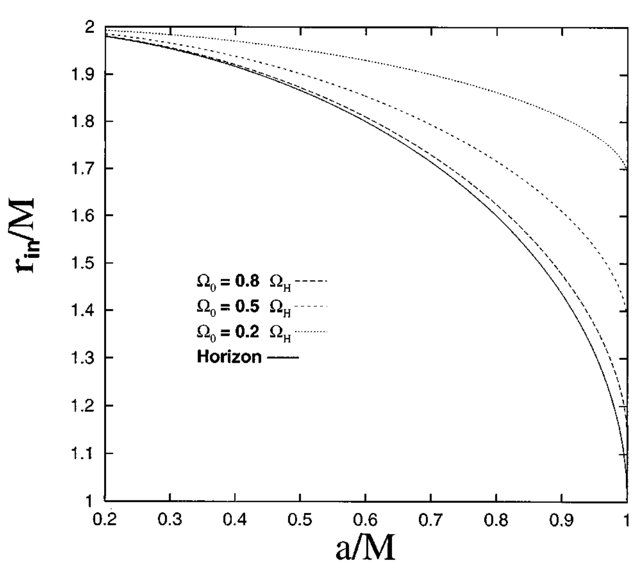

FIG. 6. The position of the inner light of surface for the Kerr-Newman black hole. $\theta=0.5\pi$.

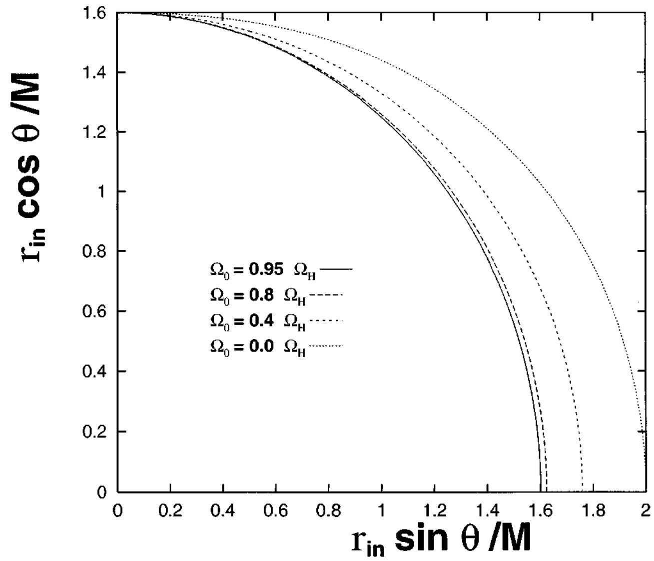

FIG. 7. The shape of the inner light of surface for the Kerr-Newman black hole. $a=0.8M, e=0$.

(c) The extreme black hole case with $\Omega_0=\Omega_H$. For the extreme Kerr-Newman black hole case, which occurs when $M^2=a^2+e^2$, g'_{tt} at $\theta=(1/2)\pi$ is written as

$$g'_{tt}=\frac{M^2}{\Sigma}\frac{y}{1+y^2}(x-1)^2\left(x+1-\frac{1}{y}\right)\left(x+1+\frac{1}{y}\right). \quad (71)$$

From this we obtain the position of VLS at $\theta=\pi/2$ as

$$r=M \quad \text{for } \frac{1}{2}M\leqslant a\leqslant M \text{ and } a=0, \quad (72)$$

$$r=\left(-1+\frac{M}{a}\right)M \quad \text{for } 0<a<\frac{1}{2}M. \quad (73)$$

The second case corresponds to the extreme black hole that is slowly rotating and has many charges. (In this case $e>\sqrt{3}/2M\approx 0.866M$). In particular, in the case of $e\leqslant\sqrt{3}/2M$ ($a=M$ for $e=0$) the horizon and the light of the velocity surface are at the same position. Therefore in case of the extreme black hole with $a\geqslant 1/2M$ it is impossible to consider the brick wall model of 't Hooft.

V. THE ENTROPY IN THE HARTLE-HAWKING VACUUM

The Hartle-Hawking vacuum state [29] is one that the angular velocity Ω_0 is equal to that of the event horizon, and the temperature β is equal to the Hawking temperature, where the Hawking temperature and the angular velocity of the horizon are defined as [26]

$$T_H=\frac{\kappa}{2\pi}, \quad \Omega_H=\lim_{r\to r_H}\left(-\frac{g_{t\phi}}{g_{\phi\phi}}\right). \quad (74)$$

Here κ is the surface gravity of the horizon.

First of all let us assume that $\Omega_0=\Omega_H$. In this case, as stated in Sec. IV, the possible region I is $r_H<r<L<r_{\rm VLS}$. The outer brick wall must be located inside the outer VLS. This fact was already pointed out by Frolov and Thorne [27] to remove the singular structure of the Hartle-Hawking vacuum and modify it. Now recall that in general $g'_{tt}|_{r=r_H}=0$. This came from that g'_{tt} is the same form as $\chi^\mu\chi_\mu=(\xi^\mu+\Omega_H\psi^\mu)(\xi_\mu+\Omega_H\psi_\mu)$, and χ^μ is null on the horizon. So it follows that $g'_{tt}=(r-r_H)G(r,\theta)$, where $G(r,\theta)$ is a nonvanishing function at $r=r_H$ except the extremal case. (We cannot consider the extreme black hole case.)

Therefore for the three black holes the leading behaviors of the free energy F for very small h are then given by

$$\beta F\approx -\frac{N}{\beta^3}\int d\theta d\phi\int_{r_H+h}^{L}dr\frac{\sqrt{g_4}}{(-g'_{tt})^2} \quad (75)$$

$$=-\frac{N}{\beta^3}\int d\theta d\phi\int_{r_H+h}^{L}dr\frac{D(r)}{(r-r_H)^2G^2(r,\theta)}, \quad (76)$$

where $D(r,\theta)=\sqrt{g_4}$. Since $D(r,\theta)$ and $G(r,\theta)$ are nonvanishing functions at $r=r_H$ we can expand it about $r=r_H$ as

$$D(r,\theta)=D(r_H,\theta)+D'(r_H,\theta)(r-r_H)+O((r-r_H)^2), \quad (77)$$

$$\frac{1}{G^2(r,\theta)}=\frac{1}{G^2(r_H,\theta)}+\left(\frac{1}{G^2(r_H,\theta)}\right)'+O((r-r_H)^2), \quad (78)$$

where a prime denotes the partial derivative for r. So the free energy is approximately given by

$$\begin{aligned}\beta F\approx &-\frac{N}{\beta^3}\int d\phi d\theta\int dr\left\{\frac{D(r_H,\theta)}{G^2(r_H,\theta)}\frac{1}{(r-r_H)^2}\right.\\&\left.+\left(\frac{D(r_H,\theta)}{G^2(r_H,\theta)}\right)'\frac{1}{(r-r_H)}+O((r-r_H)^0)\right\}\\=&-\frac{2\pi N}{\beta^3}\left\{\frac{1}{h}\int d\theta\frac{D(r_H,\theta)}{G^2(r_H,\theta)}\right.\\&\left.-\ln(h)\int d\theta\left(\frac{D(r_H,\theta)}{G^2(r_H,\theta)}\right)'+\cdots\right\},\end{aligned} \quad (79)$$

which show that generally, in addition to the linear divergence term in h, there is a logarithmic one in the case of rotating black hole. If we write the free energy in terms of the proper distance cutoff ϵ, it becomes in very simple form,

$$\begin{aligned}\beta F\approx &-\frac{N}{\beta^3}\int_{r=r_H}d\phi d\theta\sqrt{g_{\theta\theta}g_{\phi\phi}}\\&\times\int_{r_H+h}^{L}dr\sqrt{g_{rr}}\left(\frac{g_{\phi\phi}}{g_{t\phi}^2-g_{tt}g_{\phi\phi}}\right)^{3/2}\\\approx&-\frac{N}{2(\kappa\beta)^3}\frac{A_H}{\epsilon^2},\end{aligned} \quad (80)$$

where A_H is the area of the event horizon, and ϵ is the proper distance from the horizon to r_H+h:

$$\epsilon=\int_{r_H}^{r_H+h} dr\sqrt{g_{rr}}. \tag{81}$$

However the proper distance cutoff is dependent on the coordinate θ, which is the general property of the rotating black hole.

From the free energy F we obtain the leading behaviors of the entropy S as

$$S=\beta^2\frac{\partial}{\partial\beta}F\approx\frac{N}{\beta^3}\left(A\frac{1}{h}+B\ln(h)+\text{finite}\right), \tag{82}$$

where A and B are in c numbers in Eq. (79), or

$$S\approx\frac{4N}{2(\kappa\beta)^3}\frac{A_H}{\epsilon^2}. \tag{83}$$

The entropy S is linearly and logarithmically divergent as $h\to 0$. The divergences arise because the density of state for a given E diverges as h goes to zero.

Now we take T as the Hartle-Hawking temperature $T_H=\kappa/2\pi$. Then the entropy becomes

$$S_H\approx\frac{N8\pi^3}{\kappa^3}\left(A\frac{1}{h}+B\ln(h)+\text{finite}\right), \tag{84}$$

or

$$S_H\approx\frac{N}{4\pi^3}\frac{A_H}{\epsilon^2}. \tag{85}$$

The entropy of a scalar field in the Hartle-Hawking state diverges quadratically in ϵ^{-1} as the system approaches the horizon. Or it diverges in h^{-1} and $\ln(h)$. In the case $a=0$ our result (85) agrees with the result calculated by 't Hooft [5] and with one in [28]. These facts imply that the leading behaviors of entropy (85) are in general form.

VI. SUMMARY AND CONCLUSION

By using the brick wall method we have calculated the entropies of the rotating systems with a rotation Ω_0 at thermal equilibrium with temperature T in the rotating black holes. In the WKB approximation to get the real finite free energy and entropy the system must be in region I. As the system approaches the VLS ($r_{\rm in}$ and $r_{\rm VLS}$) the thermodynamic quantities become divergent. From this fact *we conclude that the divergence of the thermodynamic quantities including the entropy is related to the stationary limit surface in the comoving frame*. In the spherical symmetric black hole the stationary limit surface and the event horizon coincide. Only when $\Omega_0=\Omega_H$ can the system approach the horizon. The entropy for this case is linearly and logarithmically divergent as the ultraviolet cutoff goes to zero. To remove such a divergence, in addition to the renormalization of the gravitational constant, we need the renormalization of the curvature square term [27]. But after the renormalization the entropy is not proportional to the area of the event horizon. If we use the proper distance cutoff the entropy is proportional to the horizon area A_H. But the cutoff depends on the coordinate θ.

Another particular point is that in the extremal black hole case we cannot consider the brick wall method of 't Hooft except for the case $0<a<1/2M$ in Kerr-Newman black hole.

ACKNOWLEDGMENTS

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APPENDIX

For the three rotating black holes the metrics, the surface gravities, and the proper distances ϵ are given as follows.

(1) The Kaluza-Klein black hole [10]

$$\begin{aligned}ds^2=&-\frac{\Delta-a^2\sin^2\theta}{B\Sigma}dt^2-2a\sin^2\theta\frac{1}{\sqrt{1-v^2}}\frac{Z}{B}dt\,d\phi\\&+\left[B(r^2+a^2)+a^2\sin^2\theta\frac{Z}{B}\right]\sin^2\theta d\phi^2\\&+\frac{B\Sigma}{\Delta}dr^2+B\Sigma\,d\theta^2,\end{aligned} \tag{A1}$$

where

$$\Delta=r^2-2\mu r+a^2,\quad \Sigma=r^2+a^2\cos^2\theta,\quad Z=\frac{2\mu r}{\Sigma},$$

$$B=\left(1+\frac{v^2Z}{1-v^2}\right)^{\frac{1}{2}}. \tag{A2}$$

The physical mass M, the charge Q, the angular momentum J, and the horizon are expressed by the parameters v,μ, and a as

$$M=\mu\left[1+\frac{v^2}{2(1-v^2)}\right],\quad Q=\frac{\mu v}{1-v^2},$$

$$J=\frac{\mu a}{\sqrt{1-v^2}},r_H=\mu+\sqrt{\mu^2-a^2}. \tag{A3}$$

The surface gravity and proper distance are

$$\kappa_{\text{Kaluza Klein}}=\frac{\sqrt{(1-v^2)(\mu^2-a^2)}}{r_H^2+a^2}, \tag{A4}$$

$$\epsilon_{\text{Kaluza Klein}}=2\left(\frac{B(r_H)\Sigma(r_H)}{2r_H-2\mu}\right)^{1/2}\sqrt{h}. \tag{A5}$$

(2) The Sen black hole [17]

$$ds^2=-\frac{\Delta-a^2\sin^2\theta}{\Sigma}dt^2-\frac{4\mu ra\cosh^2\gamma\sin^2\theta}{\Sigma}dt\,d\phi \tag{A6}$$

$$+\frac{\Sigma}{\Delta}dr^2+\Sigma\,d\theta^2+\frac{\Lambda}{\Sigma}\sin^2\theta d\phi^2, \tag{A7}$$

where

$$\Delta=r^2-2\mu r+a^2,\quad \Sigma=r^2+a^2\cos^2\theta+2\mu r\sinh^2\gamma, \tag{A8}$$

$$\Lambda=(r^2+a^2)(r^2+a^2\cos^2\theta)+2\mu r a^2\sin^2\theta \tag{A9}$$

$$+4\mu r(r^2+a^2)\sinh^2\gamma+4\mu^2r^2\sinh^4\gamma. \tag{A10}$$

The mass M, the charge Q, the angular momentum J, and the horizon are given by parameters μ,β, and a as

$$M=\frac{\mu}{2}(1+\cosh 2\gamma),\quad Q=\frac{\mu}{\sqrt{2}}\sinh 2\gamma,$$

$$j=\frac{a\mu}{2}(1+\cosh 2\gamma),\quad r_H=\mu+\sqrt{\mu^2-a^2}. \tag{A11}$$

The surface gravity and proper distance are

$$\kappa_{\mathrm{Sen}}=\frac{\sqrt{(2M^2-e^2)^2-4J^2}}{2M[2M^2-e^2+\sqrt{(2M^2-e^2)^2-4J^2}]}, \tag{A12}$$

$$\epsilon_{\mathrm{Sen}}=2\left(\frac{r_H^2+a^2\cos^2\theta+2\mu r_H\sinh^2\gamma}{2r_H-2\mu}\right)^{1/2}\sqrt{h}. \tag{A13}$$

(3) The charged Kerr black hole [18]

$$ds^2=-\left(\frac{\Delta-a^2\sin^2\theta}{\Sigma}\right)dt^2-\frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dt\,d\phi+\left[\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2+\frac{\Sigma}{\Delta}dr^2+\Sigma d\theta^2, \tag{A14}$$

where

$$\Sigma=r^2+a^2\cos^2\theta,\quad \Delta=r^2+a^2+e^2-2Mr, \tag{A15}$$

and e,a, and M are charge, angular momentum per unit mass, and mass of the spacetime, respectively. The event horizon is

$$r_H=M+\sqrt{M^2-a^2-e^2}. \tag{A16}$$

The surface gravity and proper distance are

$$\kappa_{\mathrm{Kerr}}=\frac{\sqrt{M^2-a^2-e^2}}{2M[M+\sqrt{M^2-a^2-e^2}]-e^2}, \tag{A17}$$

$$\epsilon_{\mathrm{Kerr}}=2\left(\frac{r_H^2+a^2\cos^2\theta}{2r_H-2M}\right)^{1/2}\sqrt{h}. \tag{A18}$$

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