Black hole entropy: Statistical mechanics agrees with thermodynamics

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We discuss the connection between different entropies introduced for a black hole. It is demonstrated in the two-dimensional example that the (quantum) thermodynamical entropy of a hole coincides (including UV-finite terms) with its statistical-mechanical entropy calculated according to 't Hooft and regularized by the Pauli-Villars scheme. [S0556-2821(96)03718-6]

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Since Bekenstein introduced the thermodynamical analogy in black hole physics [1] and Hawking discovered [2] thermal radiation from a black hole confirming this analogy, it is an intriguing problem as to what degrees of freedom are counted by the entropy of a black hole. Equivalently, what (if any) statistical mechanics is responsible for the Bekenstein-Hawking entropy? Recently, this problem was attacked from different sides and a number of calculations were produced (see reviews in [3]). Every such calculation typically deals with a specific definition of entropy. Below I briefly list them.

According to 't Hooft [4] the statistical-mechanical entropy $S_{\rm SM}$ arises from a thermal bath of quantum fields propagating outside the horizon (see also [5]). An alternative interesting approach treats entropy as arising from entanglement [6,7]. Starting with the pure vacuum state one traces over modes of the quantum field propagating inside the horizon and obtains the density matrix ρ. The entanglement entropy then is defined by the standard formula $S_{\rm ent}=-{\rm Tr}\rho\ \ln\rho$ and is essentially due to correlations of modes propagating at different sides of the horizon. There are formal arguments [7,8] that $S_{\rm ent}$ coincides with the entropy $S_{\rm con}$ appearing in the conical approach [9–12]. According to this approach (see [8,12]) one considers the black hole system off shell by fixing only the temperature $T^{-1}=\int g_{00}^{1/2}d\tau=\beta g_{00}^{1/2}(L)$ on the external boundary and the topology of the black hole geometry. An arbitrary metric $g_{\mu\nu}(x)$ satisfying these conditions (the fixing of T imposes a boundary condition on the metric) typically has a conical singularity with deficit angle $\delta=2\pi(1-\beta/\beta_H)$ on the horizon. The free energy $F[g_{\mu\nu}(x),T]$ is then a functional of the metric $g_{\mu\nu}(x)$ inside the region and of the temperature T on the boundary. The equilibrium state of the system under T fixed is found from the extreme equation $\delta F|_T=0$ and is described by a regular (on-shell) metric with vanishing deficit angle ($\beta=\beta_H$). The conical entropy is defined as $S_{\rm con}=-\partial_T F$ where after taking the derivative ∂_T one considers $S_{\rm con}$ on the equilibrium configuration. The black hole entropy originally appeared within the thermodynamical framework and it is determined by total response of the (equilibrium) free energy F on variation of temperature: $dF=-S_{\rm TD}dT$. Remarkably, the conical method gives precisely the thermodynamical entropy, $S_{\rm con}=S_{\rm TD}$, for the equilibrium configuration.

For ordinary thermodynamical systems the thermodynamical and microscopical (statistical-mechanical) entropies are exactly the same. However for the black hole case, it was argued in [13] that black holes provide us with a unique example of a specific system for which these entropies do not necessarily coincide.

It should be noted that every calculation of statistical entropy encounters the problem of dealing with the very peculiar behavior of the physical quantities near the horizon where they typically diverge. To remove these divergences 't Hooft introduced the ''brick wall,'' a fixed boundary near the horizon within which the quantum field does not propagate. Essentially, this procedure (as it formulated in [4]) must be implemented in addition to the removing of standard ultraviolet divergences. Another important point is that this procedure changes the topology of the original black hole space-time by introducing an extra boundary near the horizon. There are only a few calculations that imply ''invariant'' regularization when these extra divergences on the horizon are eliminated simultaneously with the standard UV divergences. Such calculations preserve the original topology of the black hole space-time. An example is the conical method in which, after UV renormalization, the residual divergence of the free energy (at the tip of the cone) is proportional to the second power of the deficit angle and hence does not affect the quantities (energy and entropy) calculable for the equilibrium configuration.

The other calculation is a reformulation of the 't Hooft approach by using the Pauli-Villars (PV) regularization scheme [14]. It consists of introducing a number of fictitious fields (regulators) of different statistics and with very large masses. Remarkably, this procedure not only yields the standard UV regularization but automatically implements a cutoff for the entropy calculation allowing one to remove the ''brick wall.''

Both the conical and PV calculations give the ordinary UV divergences for the entropy when the regularization is removed. A comparison of the structure of these divergent

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terms[1] for the Reissner-Nordström black hole in four dimensions (see [10] and [14]) shows that they are really identical and take precisely the form to provide the correct renormalization of bare entropy in agreement with the suggestion of Susskind and Uglum [9,11]. This fact suggests that the two calculations really lead to the same result and the corresponding entropies $S_{\rm con}(S_{\rm TD})$ and $S_{\rm SM}$ are identical including UV-finite terms.

The present state of the theory makes it impossible to prove or refute this statement in four dimensions. However, recent intensive study of two-dimensional models shows that the black hole physics there looks rather similar to the four-dimensional one while the calculations are technically simpler. In this paper we carry out the precise calculations in two dimensions and demonstrate equality of the thermodynamical and (regularized by the Pauli-Villars scheme) statistical-mechanical entropies.

The black hole thermodynamical entropy at the one-loop level is really a sum

$$S_{\rm TD}=(S_{\rm cl}+S_1)[g^{qc}_{\mu\nu}], \tag{1}$$

where $S_{\rm cl}$ is entropy coming from the classical gravitational action and S_1 is a part coming from quantum one-loop term in the effective action. In principle, both entropies can be calculated off shell for an arbitrary black hole metric by the conical method assuming that $\beta\neq\beta_H$. We get then the thermodynamical entropy when we set $\beta=\beta_H$ at the end and the back reaction is taken into account by calculating the result for the quantum-corrected black hole metric $g^{qc}_{\mu\nu}(x)$ (and β_H^{-1} is also assumed to be the quantum-corrected Hawking temperature). The concrete form of $g^{qc}_{\mu\nu}(x)$ is not important for us since all the quantities can be calculated implicitly for arbitrary black hole metric.

The form of $S_{\rm cl}$ as function of the black hole geometry is determined from the classical gravitational action. For concreteness we may take it to be some kind of dilaton gravity,

$$W_{cl}=-\int [e^{\Phi}R+(\nabla\Phi)^2+V(\Phi)]-\kappa\int R, \tag{2}$$

where we included also the two-dimensional (2D) Einstein term with some ''gravitational constant'' κ. Then $S_{\rm cl}=4\pi e^{\Phi(x_+)}+4\pi\kappa$, where x_+ is position of the horizon.

Essentially, our consideration concerns only that part of $S_{\rm TD}$ that is due to quantum one-loop terms in the action. We have nothing to say about $S_{\rm cl}$. Its statistical interpretation requires additional considerations and, possibly, implementation of new ideas.

Consider now the quantum massless scalar field described by the action

$$W_{\rm sc}=\frac{1}{2}\int (\nabla\phi)^2. \tag{3}$$

[1]This concerns only contribution to the entropy due to the quantum matter minimally coupled to gravity. In the nonminimal case the situation is more complicated and at the present time we have no correspondence between the two methods [8,14].

In two dimensions it induces the one-loop effective action in the form of the Polyakov term

$$W_1=-\frac{1}{96\pi}\int R\Box^{-1}R-\frac{1}{24\pi}\ln(\Lambda\mu)\int R, \tag{4}$$

where we omitted the 2D cosmological constant that is irrelevant to our consideration. The last term in Eq. (4) gives the UV divergence in two dimensions; μ is an appropriate UV regulator (Λ is an infrared regulator). If we apply Pauli-Villars regularization (see below) then μ is precisely the PV-regulator. This term renormalizes the ''gravitational constant'' κ. Note that in this model there is no renormalization of e^{Φ}, which is the real gravitational coupling in Eq. (2).

For the metric written in the conformal gauge $g_{\mu\nu}=e^{2\sigma}\delta_{\mu\nu}$ the term (4) leads to the entropy in the form [15] (see also [10,11])

$$S_1=\tfrac{1}{6}\,\sigma(x_+)+\tfrac{1}{6}\ln(\Lambda\mu). \tag{5}$$

Let the black hole instanton be described by the 2D metric

$$ds^2_{\rm BH}=g(x)d\tau^2+\frac{1}{g(x)}dx^2, \tag{6}$$

where the metric function $g(x)$ has simple zero in $x=x_+$; $x_+\leqslant x\leqslant L$, $0\leqslant\tau\leqslant\beta_H$, $\beta_H=4\pi/g'(x_+)$. It is easy to see that Eq. (6) is conformal to the flat disk of radius z_0:

$$ds^2_{\rm BH}=e^{2\sigma}z_0^2(dz^2+z^2d\tilde{\tau}^2),$$

$$\sigma=\frac{1}{2}\ln g(x)-\frac{2\pi}{\beta_H}\int_L^x\frac{dx}{g}+\ln\frac{\beta_H}{2\pi z_0},$$

$$z=\exp\left(\frac{2\pi}{\beta_H}\int_L^x\frac{dx}{g}\right), \tag{7}$$

where $\tilde{\tau}=2\pi\tau/\beta_H$ $(0\leqslant\tilde{\tau}\leqslant2\pi)$, $0\leqslant z\leqslant1$. So, applying Eq. (5) we get, for S_1,

$$S_1=\frac{1}{12}\int_{x_+}^L\frac{dx}{g}\left(\frac{4\pi}{\beta_H}-g'\right)+\frac{1}{6}\ln[\mu\beta_H g^{1/2}(L)], \tag{8}$$

where we omitted the irrelevant term that is the function of (Λ,z_0) but not of parameters of the black hole and have retained dependence on UV regulator μ. The last term in Eq. (8) really contains logarithm of temperature $T^{-1}=\beta_H g^{1/2}(L)$ measured on the external boundary $x=L$. The UV divergence of the entropy (5), (8) renormalizes the ''gravitational constant'' κ in $S_{\rm cl}$ in agreement with the general statement of [9,11].

We now calculate the statistical-mechanical entropy $S_{\rm SM}$. Applying Pauli-Villars regularization in two dimensions one needs to introduce a set of fictitious fields with very large masses: two anticommuting scalar fields with mass $\mu_{1,2}=\mu$ and one commuting field with mass $\mu_3=\sqrt{2}\mu$. Consider the free energy of the ensemble of the original scalar field and regulators with an inverse temperature β:

$$\beta F=\sum_n \ln(1-e^{-\beta E_n}). \tag{9}$$

Note that energy E_n in Eq. (9) is defined with respect to Killing vector ∂_t $(\tau=\imath t)$ and fields are expanded as $\phi=e^{\imath Et}f(x)$. Therefore, β in Eq. (9) is related with temperature T measured at $x=L$ as $T^{-1}=\beta g^{1/2}(L)$. The relevant density matrix is $\rho=\Sigma_n \phi_n \phi_n^* e^{-\beta E_n}$, where $\{\phi_n\}$ is the basis of eigenvectors. One should take into account that for the regulator fields the Hilbert space has indefinite metric and hence a part of the regulators contributes with a minus sign.

The free energy (9) can be determined for the arbitrary black hole metric (6) without reference to the precise form of the metric function $g(x)$. Repeating the calculation of Ref. [14] in this 2D case and applying WKB approximation we finally get

$$F=-\frac{1}{\pi}\int_0^\infty \frac{dE}{e^{\beta E}-1}\int_{x_+ +h}^{L}\frac{dx}{g(x)}\{E-2[E^2-\mu^2 g(x)]^{1/2} +[E^2-2\mu^2 g(x)]^{1/2}\}. \tag{10}$$

It should be noted that the WKB approximation for the original massless scalar field is really exact. We introduced in Eq. (10) a ''brick wall'' cutoff h. In fact, one can see that divergences at small h are precisely canceled in Eq. (10) between the original scalar and the regulator fields. This is the 2D analogue of the mechanism discovered in [14]. So one can remove the cutoff in Eq. (10). However we will keep it arbitrarily small in the process of calculation of separate terms entering in Eq. (10).

It is straightforward to compute the contribution of the original massless field in Eq. (10). For computation of the regulator's contribution take the fixed E and consider the integral

$$I[\mu]=\int_{x_+ +h}^{L_E}\frac{dx}{g(x)}(E^2-\mu^2 g(x))^{1/2}, \tag{11}$$

where integration is done from the horizon $(x_+ +h)$ to distance L_E defined from equation $g(L_E)=E^2/\mu^2$. It is clear that when μ grows L_E becomes closer and closer to $(x_+ +h)$. So, considering the limit of large μ we conclude that integral (11) is concentrated near the horizon where we have $g(x)=(4\pi/\beta_H)(x-x_+)=(2\pi\rho/\beta_H)^2$, $dx/g=(\beta_H/2\pi)(d\rho/\rho)$ and the new radial variable ρ now runs from $\epsilon=\sqrt{\beta_H h/\pi}$ to $(E\beta_H/2\pi\mu)$. The integral (11) then reads

$$I[\mu]=\mu\int_\epsilon^{E\beta_H/(2\pi\mu)}\frac{d\rho}{\rho}\sqrt{\left(\frac{E\beta_H}{2\pi\mu}\right)^2-\rho^2} =\frac{(E\beta_H)}{2\pi}\left(\operatorname{arctanh}\sqrt{1-\left(\frac{\mu\epsilon 2\pi}{E\beta_H}\right)^2} -\sqrt{1-\left(\frac{2\pi\epsilon\mu}{E\beta_H}\right)^2}\right). \tag{12}$$

Using the asymptote $\operatorname{arctanh}(1-x)=-\frac{1}{2}\ln(x/2)+O(x)$ we finally get

$$I[\mu]=-\frac{(E\beta_H)}{2\pi}\left[1+\frac{1}{2}\ln 2+\ln\left(\frac{\mu\epsilon\pi}{E\beta_H}\right)\right]. \tag{13}$$

This is the key identity allowing computation of the free energy (10). Omitting details that are rather simple below the final result is

$$F=-\frac{1}{12}\left[\frac{\beta_H}{2\beta^2}\int_{x_+}^{L}\frac{dx}{g}\left(\frac{4\pi}{\beta_H}-g'\right)+\frac{\beta_H}{\beta^2}\ln(\mu\beta g^{1/2}(L))\right] +\frac{\beta_H}{\beta^2}\left[\frac{1}{2\pi^2}\int_0^\infty\frac{dx\ x\ \ln x}{e^x-1}-\frac{1}{12}(1-\ln 2)\right], \tag{14}$$

where we removed the brick wall cut-off and used that $\int_0^\infty (dx\ x)/(e^x-1)=\pi^2/6$. The statistical-mechanical free energy (14) is really an off-shell quantity (see [14]) defined for arbitrary metric (6) and β not necessarily equal to β_H.

Calculating now entropy $S_{\mathrm{SM}}=\beta^2\partial_\beta F$ and putting $\beta=\beta_H$ we obtain

$$S_{\mathrm{SM}}=\frac{1}{12}\int_{x_+}^{L}\frac{dx}{g}\left(\frac{4\pi}{\beta_H}-g'\right)+\frac{1}{6}\ln(\mu\beta_H g^{1/2}(L))+C, \tag{15}$$

where C is some numerical constant not depending on μ or metric $g(x)$. So, we see that S_{SM} exactly coincides with S_1 (8). We conclude that at least this part of the thermodynamical entropy has statistical meaning.

Various calculations of black hole entropy in two dimensions were recently considered in [16]. In particular, it was concluded that S_{SM} and S_{TD} are different: one has to subtract a (divergent) contribution of the fictitious Rindler particles from S_{SM} in order to get S_{TD}.

The correspondence of our results with that of [16] is seen from analysis of important interplay of two different limits $h\to 0$ (brick wall) and $\mu^{-1}\to 0$ (UV regularization). If one takes the limit $\mu^{-1}\to 0$ first one obtains that contribution of the regulators in the free energy (10) completely vanishes. One then gets the quantities that are functions of the brick wall parameter h and divergent in the limit $h\to 0$. These are the quantities calculated in [16]. Elimination of their divergence (with respect to limit $h\to 0$) might require some subtraction procedure proposed in [16]. Note, that in this regime the ''brick wall'' is treated as a real boundary staying at *macroscopical* distance h from the horizon with h being larger than any UV cutoff μ^{-1}.

The situation is different if we consider a ''brick wall'' as a fictitious imaginary boundary with h being smaller than any scale μ^{-1} of UV cutoff. Then the ''brick wall'' divergences are eliminated by the standard UV regularization and the UV regulators do contribute to the free energy and entropy.[2] This contribution is concentrated at the horizon. It leads to the appearance of additional terms in the entropy (15) that are finite after renormalization. It is worth noting

[2]This possibly means that we do not need a ''brick wall'' at all ($h=0$ from the very beginning) in order to formulate the statistical mechanics of quantum fields around a black hole. We must just impose some analyticity condition on quantum field wave functions on the horizon and deal with a continuous energy spectrum of the quantum field system.

that the mechanism of this phenomena is similar to that of the conformal anomaly. This similarity is not occasional since the result for the statistical entropy (15) appears to coincide with the thermodynamical expression (5), (8), which indeed originated from the conformal anomaly of the effective action (4).

We do not have this phenomena in statistical mechanics on space-time without horizons where the statistical entropy was proved to be conformal invariant and not dependent on UV cutoff (see [17]). This is easily seen from our analysis. Indeed, in this case where we have $g(x) \geqslant g_0 > 0$ everywhere and for large UV cutoff $\mu > \mu_0 = E/g_0^{1/2}$, contribution of the regulators disappears in the free energy (10).

Thus, in the presence of horizons the statistical mechanics of quantum fields depends on their UV behavior. The UV regulators lead to nontrivial contribution to statistical entropy that is finite after renormalization. Unfortunately, the straightforward generalization of this result on higher dimensions meets the still open problem of statistical description of nonminimally coupled conformal matter [14].

Concluding, we propose that there is some unification: all the entropies (S_{SM}, S_{ent}, S_{con}, S_{TD}) introduced for a black hole that arise due to quantum minimal matter are really identical. In two dimensions this statement can be supported by precise calculation. Recall that this does not concern the classical entropy of a hole for the statistical explanation for which a special consideration is required.

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