PHYSICAL REVIEW D VOLUME 54, NUMBER 6 15 SEPTEMBER 1996

Surface gravity in dynamical spherically symmetric spacetimes

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(Received 22 March 1996)

A definition of surface gravity at the apparent horizon of dynamical spherically symmetric spacetimes is proposed. It is based on a unique foliation by ingoing null hypersurfaces. The function parametrizing the hypersurfaces can be interpreted as the phase of a light wave uniformly emitted by some far-away static observer. The definition gives back the accepted value of surface gravity in the static case by virtue of its nonlocal character. Although the definition is motivated by the behavior of outgoing null rays, it turns out that there is a simple connection between the surface gravity, the acceleration of any radially moving observer, and the observed frequency change of the infalling light signal. In particular, this gives a practical and simple method of how any geodesic observer can determine surface gravity by measuring only the redshift of the infalling light wave. The surface gravity can be expressed as an integral of matter field quantities along an ingoing null line, which shows that it is a continuous function along the apparent horizon. A formula for the area change of the apparent horizon is presented, and the possibility of thermodynamical interpretation is discussed. Finally, concrete expressions of surface gravity are given for a number of four-dimensional and two-dimensional dynamical black hole solutions. [S0556-2821(96)05518-X]

PACS number(s): 04.70.Bw, 04.70.Dy

I. INTRODUCTION

There has been renewed interest in spherically symmetric spacetimes in the past half decade. The unexpected complexity of the problem is well illustrated by the large number of new ''dirty'' black hole solutions. An important result proved by Visser [1] is that these stationary matter fields always decrease the surface gravity compared to the same mass vacuum black hole. Dynamical spherically symmetric spacetimes were initially studied mainly to check the validity of the cosmic censorship hypothesis. More recently, supported by powerful numerical methods, considerable effort has been focused on near-critical collapsing solutions at the black hole formation threshold.

The fundamental question is, what physical quantities are describing these spherically symmetric collapses? Undoubtedly, local quantities such as a now well-defined gravitational mass function and densities belonging to the matter fields are essential. Because of their importance in the stationary case, we expect that generalizations of thermodynamical quantities will also play a major role. For stationary black holes, the surface gravity is proportional to the temperature of the Hawking radiation. On the other hand, considering the collapse of a spherical shell, Hiscock [2] proposed to identify one-quarter of the area of the apparent horizon as the gravitational entropy. Furthermore, Hajicek [3] suggested that the Hawking effect is associated with the apparent horizon rather than the event horizon, since the apparent horizon in spherically symmetric spacetimes acts as the boundary of negative energy states. Hence we expect that some naturally generalized surface gravity for apparent horizons will have a crucial role as a physical quantity in dynamical spacetimes.

It is possible to formulate a local dynamical analogue of black hole thermodynamics even for apparent horizons that are not spherically symmetric. Using a null tetrad formalism, Collins [4] has derived a formula for the area change of the apparent horizon, which can be interpreted as a generalized first law of thermodynamics. However, the temperature term in this equation is tetrad dependent, and no unique tetrad choice is made in the nonstationary case. This ambiguity reflects the difficulty of selecting an appropriate distance function along the apparent horizon. Hayward [5] has used the natural distance defined by the spacetime metric, although this choice is divergent in the stationary limit when the horizon becomes null. Hayward also presents the analogues of the zeroth and second laws of thermodynamics, and defines a dynamical counterpart of surface gravity, called trapping gravity. Unfortunately, when specializing to static spherically symmetric spacetimes again, the trapping gravity does not agree with the accepted value of surface gravity even for charged Reissner-Nordström black holes (see the Appendix).

There are two basic ways to introduce surface gravity in stationary spacetimes. The first, physically more direct method is in terms of the acceleration of stationary observers near the black hole horizon. This form of the definition proves to be very difficult to generalize. In dynamical spherically symmetric spacetimes, the observers moving on constant radius orbits are the most natural analogues of the static observers. However, their acceleration has a qualitatively different behavior, being proportional to the matter density instead of any possible generalization of surface gravity.

The second, mathematically more straightforward approach is to define surface gravity as the inaffinity of the Killing vector field along the black hole horizon. In the gen-

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eral dynamical case the apparent horizon ceases to be null, and there is no geodesic contained in the horizon. However, in spherical symmetry, the outgoing radial light rays are necessarily geodesics, and they are locally constant radius orbits when they cross the apparent horizon. Hence it is natural to attempt to generalize surface gravity as the inaffinity of these outgoing null orbits [6]. The concept of inaffinity is defined only with respect to a preferred parametrization of the curves. The main difficulty is how to choose this particular parametrization, considering that one has to get back the Killing time in the stationary limit. The normalization of the Killing vector is defined at spacelike infinity, which shows that our definition cannot be local either.

The most important idea in this paper is to parametrize the outgoing null geodesics using a natural spherically symmetric foliation of ingoing null hypersurfaces. We assume that the labeling of these hypersurfaces is defined by the proper time of a static observer at infinity. This foliation can be most easily observed by any dynamical observer, simply by observing a radio wave emitted uniformly by a far-away reference clock. We can interpret the function generating this foliation as a global advanced-time coordinate. We will see that for static observers in a static spherically symmetric spacetime, this advanced-time parameter agrees with the Killing time, which ensures that our surface gravity indeed reduces to the accepted value in the nondynamical case.

Our natural ingoing null foliation will allow us not only to give a clear physical interpretation of surface gravity, but also to prescribe the most practical way to measure it in any static or dynamical spherically symmetric spacetime. Any observer moving along a timelike orbit can precisely measure the apparent change of frequency of a standard radio or light signal falling in from the far-away asymptotically flat region. We can find an explicit relation between this frequency change and the acceleration of the observer. In particular, for any geodesic observer crossing the horizon, the proper time derivative of the redshift of the infalling wave is exactly equal to the surface gravity. This is particularly interesting, since it means that surface gravity can be determined by performing simple frequency measurements only.

Another physical approach, which may lead to a different (but still not local) definition of surface gravity, is by using a fully dynamical generalization of the Hartle-Hawking formula [7]. Assuming that the apparent horizon area corresponds to the entropy of a dynamical black hole [2], this formula may be interpreted as a generalized first law of black hole thermodynamics. It has been pointed out by Collins [4] that the temperature term appearing in this equation can correspond to some possible generalization of surface gravity only in the near-stationary limit. Furthermore, this temperature term can change in a noncontinuous way along the horizon whenever there is a jump in the matter field density. This happens, for example, at the surface of a collapsing star. In contrast, as we will see, our dynamical surface gravity is always continuous for regular matter fields.

Dynamical surface gravity is defined in Sec. II, using the inaffinity of outgoing null rays at the apparent horizon. In Sec. III, a method is described for how any observer, which crosses the horizon in an arbitrary way, can measure surface gravity by observing light signals falling in from infinity. In Sec. IV, the surface gravity is expressed as an integral of regular matter field quantities along an ingoing null curve coming from past null infinity. In Sec. V, an equation for the area change of the apparent horizon is presented. The possibility of interpreting it as a dynamical first law of black hole thermodynamics is discussed. In Sec. VI, the value of the surface gravity is calculated for several exact solutions. These include the charged Vaidya metric, self-similar scalar field solutions, $(1+1)$-dimensional dilaton gravity and homogenous dust ball collapse. In the Appendix, while examining the properties of null congruences, Hayward's definition of trapping gravity is reviewed in the spherically symmetric case, and the relation to our formulation is discussed. We use units in which the gravitational constant and the speed of light satisfy $G=c=1$.

II. DYNAMICAL SURFACE GRAVITY

The surface gravity κ of stationary spacetimes is defined by the behavior of the timelike Killing vector ξ^α at the event horizon. The definition has a nonlocal character. If ξ^α is a Killing vector field, then $b\xi^\alpha$ is also a Killing vector for any constant b. This changes the value of the surface gravity from κ to $b\kappa$. Therefore one must fix the normalization of ξ^α. The obvious way to do it is to require $\xi^\alpha\xi_\alpha=-1$ at spacelike infinity. To calculate surface gravity, either one has to know the Killing vector field globally or one has to perform an integration between the horizon and spacelike infinity to determine the ''anomalous redshift factor'' [1].

There are several equivalent expressions which can be used to define surface gravity in stationary spacetimes. The most appropriate for generalizing into dynamical spacetimes is

$$\xi^\beta \xi^\alpha_{;\beta} = \kappa \xi^\alpha. \tag{1}$$

Since the wave vector of a light signal is an affine null geodesic, κ has the physical meaning of determining the frequency decrease, or in other words the *redshift*, of an outgoing light signal moving along the horizon. Hence κ describes the ''energy loss'' of a photon trying to climb out of the black hole, but only able to move exactly along the constant radius horizon. No such frequency change occurs for a light signal moving exactly along the horizon of an extreme Reissner-Nordström black hole, although the redshift of a photon escaping from very close to the horizon to infinity can be still arbitrarily large.

In any spherically symmetric spacetime there is a natural foliation by ingoing spherically symmetric *null* hypersurfaces. Let us suppose that these hypersurfaces are parametrized by a function v. If the spacetime is asymptotically flat at past null infinity, we can make the function v *unique* (up to an additive constant) by requiring that $\xi^\alpha v_{;\alpha}=1$ at past null infinity, where ξ^α is the asymptotic Killing vector. This requirement means that v is fixed by the proper time of a far-away stationary observer. We can consider the function v as a global advanced-time coordinate. The parametrization of the null surfaces can be more conveniently fixed using the natural radial function ρ instead of ξ^α. At infinity $\xi^\alpha\xi_\alpha=-1$, $\rho^{;\alpha}\rho_{;\alpha}=1$, and $\xi^\alpha\rho_{;\alpha}=0$. Hence in place of $\xi^\alpha v_{;\alpha}=1$ we can equivalently require $\rho^{;\alpha}v_{;\alpha}=1$ at past null infinity.

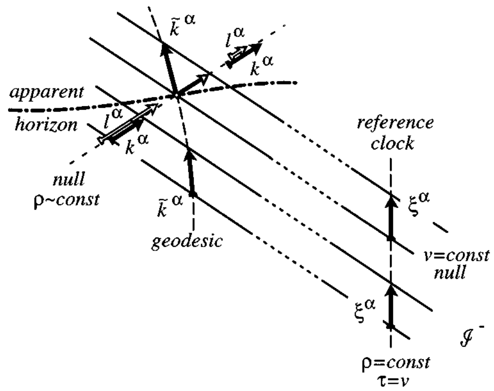

FIG. 1. An illustration of our dynamical surface gravity definition. The function v parametrizing the null foliation is fixed by the proper time τ of a far-away reference clock. The velocity vector of the clock agrees with the asymptotic Killing vector ξ^α, and $\xi^\alpha v_{;\alpha}=1$. The surface gravity describes the inaffinity of both k^α and $\widetilde{k}^\alpha$ (black arrows), as long as they are tangent to a geodesic and $k^\alpha v_{;\alpha}=\widetilde{k}^\alpha v_{;\alpha}=1$. The white arrow represents the affine geodesic wave vector l^α of some outgoing light signal crossing the horizon. The ''frequency'' $l^\alpha v_{;\alpha}$ is a decreasing function of the advanced time v.

In a static spherically symmetric spacetime the Killing vector ξ^α is defined everywhere, and it is easy to see that $\xi^\alpha v_{;\alpha}$ is constant along the ingoing constant v lines:

$$v^{;\beta}(\xi^\alpha v_{;\alpha})_{;\beta}=v^{;\alpha}v^{;\beta}\xi_{(\alpha;\beta)}+\frac{1}{2}\xi^\alpha(v^{;\beta}v_{;\beta})_{;\alpha}=0. \qquad (2)$$

Hence $\xi^\alpha v_{;\alpha}=1$ everywhere. This means that for static observers the advanced time v agrees with the Killing time, apart from an observer-dependent additive constant determining the time ''zero.''

In a dynamical spacetime the apparent horizon is not null anymore, and there is no geodesic contained in the horizon. However, outgoing radial null curves are always geodesic because of spherical symmetry. Furthermore, since the expansion of outgoing null rays vanishes at the apparent horizon, the outgoing null curves are locally constant radius orbits when they cross the horizon. Hence instead of the Killing vector, which is very problematic to generalize to dynamical spacetimes, we will use the inaffinity of an outgoing null vector field k^α to define surface gravity. The null condition and the spherical symmetry only fix the direction of k^α. The most difficult problem is how to fix the normalization of this vector field. Since we want our definition to give back the usual value for the surface gravity when specializing to static spacetimes, k^α should agree with the Killing vector ξ^α on a static horizon. We can assure this by requiring $k^\alpha v_{;\alpha}=1$ at every point of the spacetime (see Fig. 1). This determines k^α uniquely in a nonlocal way. Because k^α is geodesic everywhere, the relation $k^\beta k^\alpha_{;\beta}=\kappa k^\alpha$ can be used to define κ at every point of the spacetime which can be reached by an ingoing radial light ray coming from past null infinity. However, on physical grounds, we are interested in the value of the surface gravity only at the apparent horizon. Since $k^\alpha \rho_{;\alpha}=0$ only on the apparent horizon, the physical significance of κ is much less clear elsewhere.

Multiplying the formula $k^\beta k^\alpha_{;\beta}=\kappa k^\alpha$ by $v_{;\alpha}$,

$$\kappa=v_{;\alpha}k^\beta k^\alpha_{;\beta}=k^\beta(v_{;\alpha}k^\alpha)_{;\beta}-k^\alpha k^\beta v_{;\alpha\beta}=-k^\alpha k^\beta v_{;\alpha\beta}. \qquad (3)$$

Since $v^{;\beta}v_{;\alpha\beta}=v^{;\beta}v_{;\beta\alpha}=0$, for any scalar function a the vector $\widetilde{k}^\alpha=k^\alpha+av^{;\alpha}$ will also satisfy $\kappa=-\widetilde{k}^\alpha\widetilde{k}^\beta v_{;\alpha\beta}$. This shows that the fundamental structure is not the vector field k^α, but the function v determining the null foliation. We only have to assume that k^α points in a radial direction and $k^\alpha v_{;\alpha}=1$. Since no derivative of k^α appears, it is enough to choose any such vector at only one point and no need to construct a vector field. The vector k^α can be not only null but also timelike or spacelike.

Definition. Given a foliation by ingoing null hypersurfaces, parametrized by a function v which satisfies $v_{;\alpha}\rho^{;\alpha}=1$ (or $\xi^\alpha v_{;\alpha}=1$) at past null infinity, the *surface gravity* at some point of the spacetime is defined as

$$\kappa=-k^\alpha k^\beta v_{;\alpha\beta}, \qquad (4)$$

where k^α is a vector pointing in a radial direction and satisfying $k^\alpha v_{;\alpha}=1$.

Given any radially directed geodesic, we can parametrize it by the advanced time v. Then the tangent vector $\widetilde{k}^\alpha$ satisfies $\widetilde{k}^\alpha v_{;\alpha}=1$ and the geodesic equation $\widetilde{k}^\beta\widetilde{k}^\alpha_{;\beta}=\widetilde{\kappa}\widetilde{k}^\alpha$. Multiplying by $v_{;\alpha}$, we get $\widetilde{\kappa}=\kappa$ (see Fig. 1).

Consequence. For any radial geodesic with tangent vector k^α satisfying $k^\alpha v_{;\alpha}=1$, the surface gravity κ describes the *inaffinity* of the geodesic as

$$k^\beta k^\alpha_{;\beta}=\kappa k^\alpha. \qquad (5)$$

The physically most relevant case is when k^α is the unique outgoing null vector crossing the apparent horizon and satisfying $k^\alpha v_{;\alpha}=1$. At the horizon of static black holes this null vector agrees with the Killing vector, and our definition gives the standard value of surface gravity. The physical meaning of the dynamical κ is the same as in the stationary case. An outgoing light signal moves along a locally constant radius orbit when it crosses the apparent horizon. Since the parametrization v is not affine, κ determines the frequency decrease, that is, the *redshift* of the light signal at the horizon (see Fig. 1). Physically, the photon loses its ''energy'' because of the attractivity of the black hole.

What happens if we try to calculate the surface gravity using a different parametrization of the null hypersurfaces, a function $\widetilde{v}$ which is not asymptotically well behaving at past null infinity? Then we get $\widetilde{\kappa}=-\widetilde{k}^\alpha\widetilde{k}^\beta\widetilde{v}_{;\alpha\beta}$ for some $\widetilde{k}^\alpha$ satisfying $\widetilde{k}^\alpha\widetilde{v}_{;\alpha}=1$. The physical parametrization always can be obtained by a relabeling of the null surfaces, $v\equiv v(\widetilde{v})$. Since $\widetilde{k}^\alpha v_{;\alpha}=dv/d\widetilde{v}\equiv v'$, we have to rescale the vector $\widetilde{k}^\alpha$ and use $k^\alpha=\widetilde{k}^\alpha/v'$ to ensure that $k^\alpha v_{;\alpha}=1$. Then

$$\kappa=-k^\alpha k^\beta v_{;\alpha\beta}=-\frac{1}{(v')^2}\widetilde{k}^\alpha\widetilde{k}^\beta(v'\widetilde{v}_{,\alpha})_{;\beta}. \qquad (6)$$

Hence the physical surface gravity κ is related to the unphysical $\widetilde{\kappa}$ as

$$\kappa=\frac{1}{v'}\tilde{\kappa}-\frac{1}{(v')^2}v''=\frac{1}{v'}\tilde{\kappa}+\left(\frac{1}{v'}\right)', \tag{7}$$

where the primes denote derivatives with respect to $\tilde{v}$.

Given the function v, one can choose it as one of the coordinates in a null coordinate system $x^\alpha=(v,r,\theta,\phi)$. The metric takes the form

$$ds^2=-Fdv^2+2Gdvdr+\rho^2 d\Omega^2, \tag{8}$$

where F, G, and ρ are functions of v and r, and $G>0$. The only remaining freedom is in choosing the r coordinate. Using the Christoffel symbols in this coordinate system, from Eq. (4) we have $\kappa=k^\alpha k^\beta \Gamma^1_{\alpha\beta}$, and, since $\Gamma^1_{22}=\Gamma^1_{12}=0$,

$$\kappa=\Gamma^1_{11}=\frac{G_{,v}}{G}+\frac{F_{,r}}{2G}, \tag{9}$$

independently of the radius function ρ. If we choose r as an outgoing null coordinate, then we obtain a double-null coordinate system with $F=0$ and $\kappa=G_{,v}/G$. Another convenient choice is $r=\rho$, which we will use in most of the paper.

III. PHYSICAL IMPLICATIONS

The familiar method of determining the surface gravity of a stationary black hole is by measuring the acceleration of observers moving along the Killing orbits near the horizon. Using the coordinate system (8) where $r=\rho$, the most natural generalization of the Killing vector is $\xi^\alpha=(1,0,0,0)$, because it satisfies $\xi^\alpha\rho_{;\alpha}=0$, and reduces to the Killing vector in the static case. Since $\xi^\alpha\xi_\alpha=-F$, the velocity of the observers moving along these constant radius orbits is $u^\alpha=(1/\sqrt{F},0,0,0)$. Their acceleration is $a^\alpha=u^\beta u^\alpha_{;\beta}$ and

$$\xi^\alpha\xi_\alpha a^\beta a_\beta=\left(\frac{G_{,v}}{G}+\frac{F_{,r}}{2G}+\frac{F_{,v}}{2F}\right)^2. \tag{10}$$

In stationary spacetimes $F_{,v}=0$, and comparing with Eq. (9) we can see that this expression gives κ^2. In a dynamical spherically symmetric spacetime the derivative of the radius function $\rho=r$ vanishes in outgoing null directions at the points of the apparent horizon. Since $(2G,F,0,0)$ is such a null vector field, this means that $F=0$ there. However, from Einstein's equations we have $F_{,v}=-8\pi\rho G T_{vv}$, and hence $F_{,v}/F$ and $\xi^\alpha\xi_\alpha a^\beta a_\beta$ in general diverges at the apparent horizon. The only combination which is always finite is $(\xi^\alpha\xi_\alpha)^3 a^\beta a_\beta$. It is proportional to $T_{\alpha\beta}\xi^\alpha\xi^\beta$ instead of κ, and zero in the static case. From these arguments we can see that in dynamical spacetimes the acceleration of constant radius observers cannot be used to define any generalization of surface gravity at the apparent horizon. To illustrate the problem more concretely, let us consider the Vaidya spacetime, for which $G=1$, $F=1-2m(v)/r$, and $\rho=r$. Then $\kappa=m/r^2$, and

$$\xi^\alpha\xi_\alpha a^\beta a_\beta=\left(\frac{m}{r^2}-\frac{2m'}{r-2m}\right)^2. \tag{11}$$

This diverges at the apparent horizon $r=2m$, whenever $m'=dm/dv\neq 0$.

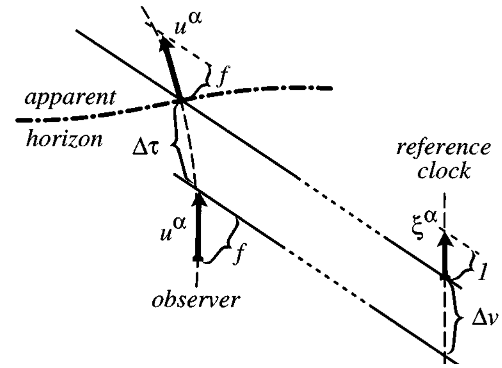

FIG. 2. Given an observer with velocity u^α, define the function $f=u^\alpha v_{;\alpha}$. If there is an infalling light wave emitted by a far-away static clock with frequency ω_∞, the observer measures $\omega=f\omega_\infty$ frequency. For a geodesic observer, the surface gravity κ is equal to the derivative of the redshift z with respect to the proper time τ, i.e., $\kappa=u^\alpha z_{;\alpha}$ where $z=1/f-1$.

There is a very intimate connection between the acceleration of radially moving observers, the advanced time v, and the dynamical surface gravity κ. Consider an arbitrary congruence of curves in the constant angle radial plane, generated by some vector field u^α. We do not assume that the congruence is geodesic, and it can be not only timelike but also spacelike or null. If we define the function f by $f=u^\alpha v_{;\alpha}$, the vector field $k^\alpha=u^\alpha/f$ satisfies the normalization condition $k^\alpha v_{;\alpha}=1$ needed in the definition of κ. We define the vector field $a^\alpha=u^\beta u^\alpha_{;\beta}$ which is just the acceleration when u^α is a normalized velocity vector. Then $a^\alpha=f^2 k^\beta k^\alpha_{;\beta}+fk^\alpha k^\beta f_{;\beta}$. Multiplying this by $v_{;\alpha}$, using the fact that the derivative of $k^\alpha v_{;\alpha}$ vanishes, and substituting the defining relation (4) of κ, we get

$$a^\alpha v_{;\alpha}=f^2\kappa+u^\alpha f_{;\alpha}. \tag{12}$$

If the congruence is geodesic, then $a^\alpha=0$ and $\kappa=u^\alpha(1/f)_{;\alpha}$. This is not very surprising, since we have seen in the previous section that κ describes the inaffinity of any geodesic parametrized by v. The important thing is that for timelike u^α the function f has a simple physical interpretation and can be very easily measured. Since $f=u^\alpha v_{;\alpha}$, the value of f gives the ratio of the global advanced time change Δv and the observer's proper time change $\Delta\tau$ along the orbit (see Fig. 2):

$$f=\frac{dv}{d\tau}. \tag{13}$$

Considering a light signal emitted by a static observer at infinity with frequency ω_∞, the observed frequency is $\omega=f\omega_\infty$. Introducing the redshift factor

$$z=\frac{\omega_\infty}{\omega}-1=\frac{1}{f}-1, \tag{14}$$

the surface gravity is

$$\kappa=u^\alpha z_{;\alpha}=\frac{dz}{d\tau}. \tag{15}$$

This shows that for any geodesic observer, the proper-time derivative of the observed redshift of a standard light or radio signal is equal to the surface gravity κ. Since such frequency changes can be very easily and most precisely determined, this is the most practical method of measuring surface gravity in spherically symmetric spacetimes, even in the static case.

The proper time is measured by a clock carried by the observer. Actually, since we have not used that the norm of u^α is -1, τ does not even have to be proper time; it is enough if it is proportional to it. But to get the physical κ, the frequency ω_∞ of the light signal must be determined by the proper time of a static clock at infinity. To measure the surface gravity of the apparent horizon, the observer must actually cross the horizon. If the apparent horizon is spacelike, the observer is unable to send the result of the measurement back to infinity. The utmost an observer far from the black hole can know is the approximate value of κ at the event horizon, even if the physical meaning of κ is not clear there. However, we expect that dynamical surface gravity will play the most important role in the case of evaporating black hole models, when the apparent horizon is timelike and located outside the event horizon.

Equation (12) provides the most practical way of measuring surface gravity for nongeodesic radially moving observers as well. In our null coordinate system let

$$u^\alpha_\mp=\left(f,\frac{f^2F\mp1}{2fG},0,0\right), \tag{16}$$

assuming $f\geqslant0$. Since $g_{\alpha\beta}u^\alpha_\mp u^\beta_\mp=\mp1$ and $g_{\alpha\beta}u^\alpha_- u^\beta_+=0$, a general observer moving in a radial direction can be described by the velocity u^α_-, while u^α_+ is the outward-pointing normal vector to the orbits. Defining $a^\alpha_\mp=u^\beta_\mp u^\alpha_{\mp;\beta}$ we have

$$a^\alpha_\mp=\frac{1}{z+1}(\kappa-u^\beta_\mp z_{;\beta})u^\alpha_\pm. \tag{17}$$

This follows from the fact that $a^\alpha_\mp$ has to be parallel to $u^\alpha_\pm$, and that by contracting with $v_{;\alpha}$ we get back Eq. (12). The norm of u^α_+ is 1, and the acceleration $|a^\alpha_-|$ can be directly measured, while $u^\beta_- z_{;\beta}$ can be determined by observing the frequency change of light signals falling in from infinity.

All observers who measure a constant redshift z have acceleration proportional to the surface gravity κ. Hence, to find the natural generalization of the static observers, one has to look for those solutions of the equation $u^\alpha_\mp f_{;\alpha}=0$ which reduce to the Killing orbits in the static limit. Unfortunately, in general, this equation is too difficult to solve analytically. One has the freedom to specify the value of f as initial data on some surface, for example, on the apparent horizon. A natural choice to fix f is by requiring $u^\alpha_\mp$ to be tangent to the horizon. In general, each orbit determined by $u^\alpha_\mp f_{;\alpha}=0$ will be tangent to the horizon at only one point, and since they never cross into the other side, the apparent horizon will emerge as the ''envelope'' of these orbits. The acceleration of the observers at the moment of touching the horizon is $f_h\kappa$. The function f_h can be interpreted as a generalized redshift factor. It is finite for dynamical spacetimes, but always diverges in the static limit.

IV. INTEGRAL FORMULA

In the coordinate system $ds^2=-Fdv^2+2Gdvdr+r^2(d\theta^2+\sin^2\theta d\varphi^2)$, the independent components of the Einstein's equations are

$$8\pi T_{rr}=\frac{2G_{,r}}{rG}, \tag{18}$$

$$8\pi T_{vr}=-\frac{2G}{r^2}M_{,r}, \tag{19}$$

$$8\pi T_{vv}=\frac{2}{r^2}(FM_{,r}+GM_{,v}), \tag{20}$$

$$8\pi\left(T_{\theta\theta}-\frac{r^2}{4}T_\alpha{}^\alpha\right)=\frac{r^2}{2G}\kappa_{,r}+\frac{M}{r}, \tag{21}$$

where

$$M=\frac{r}{2}\left(1-\frac{F}{G^2}\right),\quad \kappa=\frac{G_{,v}}{G}+\frac{F_{,r}}{2G}, \tag{22}$$

and $T_{\alpha\beta}$ is the stress tensor of matter fields. Using the radius function ρ, the local mass M can be expressed in a coordinate-system-invariant form as $M=\rho(1-\rho^{;\alpha}\rho_{;\alpha})/2$.

Defining the vectors $\xi^\alpha=(1,0,0,0)$ and $\ell^\alpha=(0,-1,0,0)$, we have $T_{vv}=\xi^\alpha\xi^\beta T_{\alpha\beta}$, $T_{vr}=-\xi^\alpha\ell^\beta T_{\alpha\beta}$, and $T_{rr}=\ell^\alpha\ell^\beta T_{\alpha\beta}$. The vector ℓ^α can be easily constructed in any coordinate system, since it is future directed null and $\ell^\alpha\rho_{;\alpha}=-1$. However, for ξ^α, only its direction is fixed locally by $\xi^\alpha\rho_{;\alpha}=0$ and the normalization $\xi^\alpha\xi_\alpha=-F$ is known only after globally constructing the asymptotically well-behaving foliation given by v. The best one can do locally is to define the Kodama vector [8] ζ^α by $\zeta^\alpha\rho_{;\alpha}=0$ and $\zeta^\alpha\zeta_\alpha=2M/\rho-1=F/G^2$. Then $\xi^\alpha=G\zeta^\alpha$. Since the cotangent vector $\rho_{;\alpha}=(0,1,0,0)$ can be easily constructed in any coordinate system, it is useful to write the more covariant combination of Eqs. (19) and (20):

$$8\pi T_r{}^r=\frac{F_{,r}}{rG^2}-\frac{2M}{r^3}. \tag{23}$$

Using Eq. (18) and assuming that G approaches 1 at past null infinity, G can be expressed as an integral of local quantities along an ingoing radial null line:

$$\ln G=-4\pi\int_r^\infty rT_{rr}dr. \tag{24}$$

This shows that $G=1$ in the whole outer vacuum region. If the matter fields satisfy the weak energy condition, then G is nonincreasing in ingoing null directions and $0<G\leqslant1$. After calculating G, even F can be determined locally from the expression (22) of the local mass, $F=G^2(1-2M/r)$.

From Eq. (21), κ can be expressed as an integral along an ingoing radial null curve:

$$\kappa=\int_r^\infty G\left(\frac{2M}{r^3}-\frac{16\pi}{r^2}T_{\theta\theta}+4\pi T_\alpha{}^\alpha\right)dr. \tag{25}$$

Since G is not a local quantity, before calculating κ one has to evaluate the integral (24) to get G at every point of the null line. Using the expression for T_{vr} from

$$T_\alpha{}^\alpha=\frac{2}{G}T_{vr}+\frac{F}{G^2}T_{rr}+\frac{2}{r^2}T_{\theta\theta}, \tag{26}$$

we can get

$$\left(G\frac{M}{r^2}\right)_{,r}=2G\left(\pi T_{rr}-\pi T_\alpha{}^\alpha+\frac{2\pi}{r^2}T_{\theta\theta}-\frac{M}{r^3}\right), \tag{27}$$

which gives another integral formula for κ:

$$\kappa=G\frac{M}{r^2}+\int_r^\infty 2\pi G\left(T_{rr}+T_\alpha{}^\alpha-\frac{6}{r^2}T_{\theta\theta}\right)dr. \tag{28}$$

This latter expression is especially useful if there are vacuum regions.

Using Eq. (23),

$$\frac{F_{,r}}{2G}=G\left(4\pi rT_r{}^r+\frac{M}{r^2}\right). \tag{29}$$

The best we can do for the $G_{,v}/G$ term in the expression (9) of κ is to take the derivative of Eq. (24). We obtain

$$\kappa=G\left(4\pi rT_r{}^r+\frac{M}{r^2}\right)-4\pi\int_r^\infty r\frac{\partial T_{rr}}{\partial v}dr. \tag{30}$$

After G is already known, the partial derivative can be written in a coordinate-system-invariant form, as a Lie derivative along the vector field $\xi^\alpha=G\zeta^\alpha$:

$$\frac{\partial T_{rr}}{\partial v}=\ell^\alpha\ell^\beta\mathcal{L}_{G\zeta}T_{\alpha\beta}. \tag{31}$$

In the static case ξ^α is the Killing vector and the integral term vanishes.

It is instructive to introduce a basis carried by observers moving along the constant radius orbits. Setting $n_\alpha=(G/\sqrt{F})\rho_{;\alpha}$ and $t^\alpha=(1/\sqrt{F})\xi^\alpha$, we have $n^\alpha n_\alpha=1$, $u^\alpha u_\alpha=-1$, and $u^\alpha n_\alpha=0$. The measured energy density is $\varepsilon=T_{\alpha\beta}t^\alpha t^\beta$, the radial energy flow is $S=T_{\alpha\beta}t^\alpha n^\beta$, and the radial pressure is $P=T_{\alpha\beta}n^\alpha n^\beta$. We have the ingoing null vector $t^\alpha-n^\alpha=(\sqrt{F}/G)\ell^\alpha$. Then

$$T_{rr}=T_{\alpha\beta}\ell^\alpha\ell^\beta=\frac{G^2}{F}T_{\alpha\beta}(t^\alpha-n^\alpha)(t^\beta-n^\beta)=\frac{\varepsilon-2S+P}{1-(2M/r)}. \tag{32}$$

Since ℓ^α is regular, this shows that $\varepsilon-2S+P$ must approach zero at the horizon. Similarly we can get $T_{vr}=G(S-\varepsilon)$, $T_{vv}=F\varepsilon$, and $T_r{}^r=P-S$. Substituting into Eq. (24),

$$\ln G=-4\pi\int_r^\infty\frac{r(\varepsilon-2S+P)}{1-(2M/r)}dr. \tag{33}$$

In the static case $S=0$, and this reduces to the formula determining the "anomalous redshift" $\phi=-\ln G$ given by Visser [1]. From Eq. (30), using the fact that on the horizon $r=2M$ and $\varepsilon-2S+P=0$, we get the surface gravity of static black holes [1]:

$$\kappa=\frac{G}{2r}(1-8\pi r^2\varepsilon). \tag{34}$$

V. AREA LAW

Trying to obtain a dynamical analogue of the second law of thermodynamics, we calculate the advanced time derivative of the apparent horizon area. The radius and the local mass of the apparent horizon are related by $r_h=2M_h$. Since

$$\frac{dr_h}{dv}=2\frac{dM_h}{dv}=2\left(M_{,v}+M_{,r}\frac{dr_h}{dv}\right), \tag{35}$$

we have

$$\frac{dr_h}{dv}=\frac{2M_{,v}}{1-2M_{,r}}. \tag{36}$$

Since at the horizon $F=0$ and $T_{vr}=GT_r{}^r$ from Eqs. (19) and (20) we get

$$M_{,r}=-4\pi r_h^2T_r{}^r,\quad M_{,v}=\frac{4\pi r_h^2}{G_h}T_{vv}. \tag{37}$$

The change of the horizon area is

$$\frac{dA_h}{dv}=8\pi r_h\frac{dr_h}{dv}=\frac{4\pi r_h^2 8\pi T_{vv}}{G_h(4\pi r_hT_r{}^r+1/2r_h)}. \tag{38}$$

Using Eq. (30), we get

$$\Theta\frac{dA_h}{dv}=4\pi r_h^2T_{vv}, \tag{39}$$

where

$$\Theta=G_h\left(4\pi r_hT_r{}^r+\frac{1}{2r_h}\right)=\frac{1}{8\pi}\left(\kappa+4\pi\int_{r_h}^\infty r\frac{\partial T_{rr}}{\partial v}dr\right). \tag{40}$$

In the quasistationary limit the integral term becomes negligible, and we obtain an expression corresponding to the Hartle-Hawking formula [7]. If we identify one-quarter of the apparent horizon area as the gravitational entropy [2], then we may interpret Eq. (39) as a generalized first law of black hole thermodynamics.

One of the problems with the temperature term Θ is that it can be a noncontinuous function along the apparent horizon if there is a sudden change in the matter density. Whenever there is a jump in T_{rr}, the integrand in Eq. (40) becomes unbounded and Θ stops being continuous too. On the other hand, since every quantity remains regular in the integral form (25) of κ, our dynamical surface gravity is always continuous when the energy densities are bounded. Another difficulty is that Θ is not necessarily positive. Since the radius function ρ is always constant in the outgoing null direc-

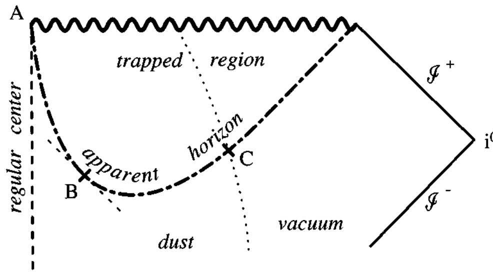

FIG. 3. Conformal diagram describing the collapse of an inhomogeneous dust ball. The apparent horizon is timelike inner between the points A and B, spacelike outer between B and C, and null in the vacuum region. If there is a sudden change in the density, for example, at the surface of the ball, then the direction of the horizon changes noncontinuously there. Depending on the initial density distribution, a null singularity may appear at the center, and near it the apparent horizon has to become spacelike outer again.

tion at the horizon, dr_h/dv and hence dA_h/dv are always positive for spacelike and negative for timelike apparent horizons. If the weak energy condition holds, then $T_{vv}\geqslant 0$ on the right-hand side of Eq. (39), and $\Theta\geqslant 0$ for spacelike while $\Theta\leqslant 0$ for timelike horizons. Furthermore, under the weak energy condition, spacelike horizons are outer, while timelike horizons are inner according to the classification of Hayward [5] (see the Appendix). Hence one would expect that horizons, separating an asymptotically flat region from the black hole region, are always spacelike. However, as we will see in the next section when studying the example of pressureless dust collapse, near the center the apparent horizon can become timelike (see Fig. 3). This timelike region is separated form the outer spacelike region by points where the horizon is ingoing null. In black hole evaporation models the energy condition is violated, and if $T_{vv}<0$, then $\Theta>0$ for timelike apparent horizons.

One would expect that the change of the black hole mass appears in the first law of thermodynamics. Instead, the right-hand side of Eq. (39) describes the ingoing energy flux across the apparent horizon. Unfortunately, there is no direct relation between this energy flux and the change of the local mass along the horizon. Because $r_h=2M_h$ always holds on the horizon,

$$\frac{1}{8\pi}\frac{1}{2r_h}\frac{dA_h}{dv}=\frac{dM_h}{dv}, \tag{41}$$

which is independent of the surface gravity. From Eq. (20) we can see that the horizon value of T_{vv} is proportional to the derivative of the local mass in the constant radius outgoing null direction. Hence

$$\Theta\frac{dA_h}{dv}=G_hM_{,v}\,. \tag{42}$$

Unfortunately the derivatives in the two sides of the equation are taken in different spacetime directions.

When the horizon is spacelike, there is a unique outgoing unit-vector z^α tangent to the horizon, satisfying $z^\alpha z_\alpha=1$. Although z^α determines a natural local specification of distance along the horizon, it has the disadvantage of diverging in the static limit. Using Eq. (30), (37), and (40) we get

$$\sqrt{\Theta}z^\alpha A_{h;\alpha}=4\pi r_h\sqrt{\frac{r_h}{G_h}T_{vv}}. \tag{43}$$

The fact that the temperature term appears under a square root follows from the unnatural normalization of the vector z^α. Substituting from Eq. (A13) in the Appendix, we get the form of the first law given by Hayward [5].

VI. EXAMPLES

The simplest spherically symmetric dynamical spacetime for which we can calculate surface gravity is the charged Vaidya metric [9], describing a massless, charged null fluid falling into a charged black hole. In the coordinate system (8) we have $G=1$, $F=1-2m(v)/r+e(v)^2/r^2$, and $\rho=r$, and it follows from Eq. (9) that $\kappa=m/r^2-e^2/r^3$. Since the radius of the outer and inner apparent horizons is $r_\pm=m\pm\sqrt{m^2-e^2}$, the horizon surface gravity is

$$\kappa_\pm=\frac{1}{2r_\pm}\left(1-\frac{e^2}{r_\pm^2}\right)=\pm\frac{1}{r_\pm^2}\sqrt{m^2-e^2}, \tag{44}$$

in local agreement with the Reissner-Nordström value. The surface gravity is always positive for the outer and negative for the inner horizon. Taking a partial derivative of Eq. (44), we can see that charging this type of black hole always decreases its outer-horizon surface gravity. If the infalling matter is not charged and satisfies the energy conditions, then $\partial m/\partial v\geqslant 0$, and the inner-horizon surface gravity is always a decreasing function of time. The outer-horizon surface gravity also decreases for not very strongly charged black holes which satisfy $4e^2<3m^2$.

Our next example is the Roberts solution [10], describing the self-similar collapse of a massless scalar field. The metric is $ds^2=-h\,du\,dv+\rho^2d\Omega^2$, where $h=1$ and $\rho^2=[(1-p^2)v^2-2vu+u^2]/4$. For parameter values $p>1$, this solution describes the formation of an unboundedly increasing mass black hole, with a spacelike apparent horizon at $u=(1-p^2)v$. Since from Eq. (9) we have $\kappa=h_{,v}/h=0$ everywhere, the surface gravity is zero all along the apparent horizon. This indicates that the Roberts solution describes an extreme black hole, analogously to the $e=m$ Reissner-Nordström metric.

However, not all self-similar black holes have vanishing surface gravity. There is a conformally coupled scalar counterpart of the Roberts solution [11]. The two metrics are related by a conformal transformation $d\tilde{s}^2=-\tilde{h}\,du\,dv+\tilde{\rho}^2d\Omega^2$, where $\tilde{\rho}^2=\tilde{h}\rho^2$ and $\tilde{h}$ is a function of u, v, and p. The new apparent horizon is determined by $\tilde{\rho}_{,v}=0$. There, using Eq. (9),

$$\kappa=\frac{1}{\tilde{h}}\tilde{h}_{,v}=-\frac{1}{\rho^2}(\rho^2)_{,v}=\frac{1}{2\rho^2}[u-(1-p^2)v]. \tag{45}$$

The horizon exists for $p>1$, and it is a spacelike self-similarity surface given by $u=c(1-p^2)v/4$, where c is a constant weakly depending on p: $2.535<c<2.6667$. Substituting into the surface gravity formula,

$$\kappa = \frac{8(4-c)}{v[8(c-2)+(p^2-1)c^2]} > 0. \tag{46}$$

After the moment of black hole formation, the surface gravity gradually decreases to zero from an infinitely big initial value, as the mass increases unboundedly.

Our third example is the (1+1)-dimensional dilaton gravity proposed by Callan, Giddings, Harvey, and Strominger (CGHS) [12]. It is defined by the action

$$S = \frac{1}{2\pi}\int dx^2 \sqrt{-g}\left\{ e^{-2\phi}[R+4(\nabla\phi)^2+4\lambda^2] - \frac{1}{2}\sum_{i=1}^{N} (\nabla f_i)^2 \right\}, \tag{47}$$

where $g_{\alpha\beta}$ is the two-dimensional metric, ϕ is the dilaton field, f_i are matter fields, and λ is a cosmological constant. It is convenient to use a double-null coordinate system (x^+,x^-), and denote the only nonvanishing component of the metric by $g_{+-} = -\frac{1}{2}e^{2\overline{\rho}}$. Considering $(1/\lambda)e^{-\phi}$ as a radius function, the apparent horizon is located at $\partial_+\phi = 0$. It follows from the field equations that one can always introduce a coordinate system where $\overline{\rho} = \phi$. The static vacuum solution of the model is given by $e^{-2\overline{\rho}} = e^{-2\phi} = m/\lambda - \lambda^2 x^+ x^-$, where m is a parameter giving the mass of the black hole. The asymptotically normed Killing vector of this solution is $\xi^\alpha = (\lambda x^+, -\lambda x^-)$, and using Eq. (1), we get $\kappa = \lambda$, independently of the black hole mass. We will see shortly that our dynamical surface gravity always agrees with the cosmological constant, even for nonvacuum dynamical solutions. Suppose that we are given any asymptotically flat nonvacuum dynamical solution of the field equations. Since the solution approaches the static vacuum solution at large distances, the parametrization defined by $e^{\lambda v} = \lambda x^+$ has to be asymptotically well behaving for any solution. In the (v,x^-) coordinate system the nonvanishing metric component is $g_{v-} = e^{\lambda v} g_{+-}$. Since on the apparent horizon $\partial_+\phi = 0$, that is, $\partial_v g_{+-} = 0$, from Eq. (9) the dynamical surface gravity is

$$\kappa = \frac{1}{g_{v-}}\partial_v g_{v-} = \lambda. \tag{48}$$

There exists a semiclassical model proposed by Russo, Susskind, and Thorlacius (RST) [13], which reduces to the previously discussed CGHS theory at the classical level. The vacuum solutions have identical forms in the two models. By the same argument as in the previous paragraph, one can show that the surface gravity of any asymptotically flat dynamical black hole solution of this theory is equal to the cosmological constant, $\kappa = \lambda$, on the apparent horizon.

As our last example, we calculate the apparent horizon surface gravity of a uniform density dust ball collapsing from the rest. The internal region is equivalent to a part of the Friedmann cosmology,

$$ds^2 = a^2(-d\eta^2 + d\psi^2 + \sin^2\psi d\Omega^2), \tag{49}$$

where

$$a = c\left(\cos\frac{\eta}{2}\right)^2 \tag{50}$$

and c is some constant. The proper time is $\tau = c(\eta + \sin\eta)/2$. We match to a Schwarzschild solution at the world line of a dust particle at $\psi = \psi_0$. The mass parameter of the external solution is $m = (c/2)\sin^3\psi_0$, and the maximal radius of the ball is $r_0 = c\sin\psi_0$. The apparent horizon is represented by the timelike surface $\eta = \pi - 2\psi$. This timelike horizon is a future inner trapping horizon in Hayward's classification [5] (see the Appendix). Using Eq. (9), the surface gravity belonging to the null foliation $\tilde{v} = \eta + \psi$ is $\tilde{\kappa} = -\cot\psi$. Unfortunately, the parametrization $\tilde{v}$ is not asymptotically well behaving when continued into the vacuum region. Hence we will have to use the transformation formula (7) to get the physical surface gravity κ, where $v = t + r + 2m\ln(r/2m - 1)$ is the regular null coordinate in the external Schwarzschild region. Looking from the vacuum region, the matching boundary is generated by radial null geodesics with maximal radius r_0. The radius and proper time of such geodesic can be expressed using a parameter η as

$$r = r_0\left(\cos\frac{\eta}{2}\right)^2, \quad \tau = \frac{1}{2}\sqrt{\frac{r_0^3}{2m}}(\eta + \sin\eta). \tag{51}$$

Since r and τ have to agree at both sides of the boundary, the parameter η also agrees with the inner time coordinate η. At the boundary $\tilde{v} = \eta + \psi_0$, and, hence,

$$v' \equiv \frac{dv}{d\tilde{v}} = \frac{dv}{d\eta} = \frac{dv}{dr}\frac{dr}{d\eta} + \frac{dv}{dt}\frac{dt}{d\tau}\frac{d\tau}{d\eta} = \frac{r_0(\cos\eta/2)^3}{\sin\psi_0\cos(\psi_0 - \eta/2)}. \tag{52}$$

Since $\tilde{v} = (\pi - 2\psi) + \psi$ at the horizon and $\tilde{v} = \eta + \psi_0$ at the matching surface, we have $\eta = \pi - \psi - \psi_0$. The final result for the surface gravity is

$$\kappa = \frac{\sin\psi_0}{4r_0\sin\psi[\sin(\psi+\psi_0)/2]^4}[\cos(\psi-\psi_0) + 2\cos(2\psi_0) - 3\cos(\psi+\psi_0)]. \tag{53}$$

At the surface of the ball $\kappa = 1/4m$, which shows that our dynamical surface gravity indeed changes continuously along the apparent horizon. At the central singularity the surface gravity diverges to minus infinity.

VII. SUMMARY AND DISCUSSION

In this paper we have proposed a definition of surface gravity on the apparent horizon of spherically symmetric dynamical black holes. Since in stationary spacetimes the surface gravity is not a local quantity, our definition cannot be local either. The necessary nonlocal structure is an asymptotically regular foliation by ingoing null hypersurfaces. The resulting dynamical surface gravity is proportional not only to the frequency decrease of the outgoing light rays, but also to the acceleration of some special family of observers. Furthermore, any observer can easily measure it by observing the apparent redshift of standard light signals falling in from infinity.

We have also seen that the area change of the apparent horizon, which may be essential in possible thermodynamical interpretations, becomes directly proportional to the surface gravity only in the stationary limit. On the other hand, although it is well known that stationary black holes emit Hawking radiation with temperature proportional to their surface gravity, it is unclear whether or not a dynamical analogue of this statement can be formulated. There have been attempts to define the dynamical temperature only at the (approximate) event horizon of Vaidya spacetime [14]. If there was discrepancy between the temperature and the surface gravity, it might be linked with the nonthermal nature of the Hawking radiation.

Based on the study of the examples in the previous section, we can have a number of conjectures on the general dynamical behavior of surface gravity in spherically symmetric spacetimes. It is natural to expect that the surface gravity of evaporating black holes is always a positive and nondecreasing function of time. This case is especially important, since if similarly to the quasistatic limit there was a close relation between the temperature of the (thermal part of the) Hawking radiation and the surface gravity, then this would be the strongest support in favor of our definitions. Although these kinds of calculations are extremely difficult to perform in four-dimensional Einstein theory, it is very encouraging that the dynamical surface gravity of black holes in the exactly solvable two-dimensional dilaton gravity models (CGHS and RST) is a positive mass-independent constant, in accordance with the Hawking temperature calculations [15].

If the matter fields falling into the black hole satisfy the energy conditions, then according to the classification of Hayward [5] (see the Appendix), the apparent horizon is either spacelike outer or timelike inner. From the examples it seems very likely that spacelike outer horizons always have non-negative surface gravity which decreases in the outgoing direction in most of the physically relevant cases. The surface gravity of timelike inner horizons is probably always a decreasing function of time if the singularity is in the future. Since the surface gravity is continuous, it is initially positive even in the inner region. However, if this inner region is large enough, then the surface gravity can become negative there.

It is possible that the positivity of κ may be proved somehow by the integral formula (25), if one uses the energy conditions and that $2M<r$ outside the horizon. Although Eq. (21) gives us the derivative of κ in the ingoing null direction, there are no similar relations for the other directional derivatives. Substituting Eqs. (26) and (19) into Eq. (21), we get

$$\kappa_{,r}=\frac{8\pi G}{r^2}T_{\theta\theta}-\frac{4\pi F}{G}T_{rr}+\frac{2G}{r}\left(\frac{M}{r}\right)_{,r}. \tag{54}$$

At the apparent horizon $F=0$ and $r=2M$. Since outside of the horizon $r>2M$, the third term is negative for spacelike outer and positive for timelike inner horizons (see Fig 3). At the boundary of these two regions, where the horizon is ingoing null, $\kappa_{,r}$ is exactly the horizon directional derivative, and its signature is determined only by the signature of the angular directional pressure $(1/r^2)T_{\theta\theta}$. In particular, for collapsing dust $T_{\theta\theta}=0$ and $\kappa_{,r}=0$. This indicates that in case of dust collapse the surface gravity takes its maximal value exactly where the apparent horizon becomes an ingoing null hypersurface.

ACKNOWLEDGMENTS

We would like to thank Dr. J. Soda for suggesting that we should calculate the surface gravity of black holes in the exactly solvable two-dimensional dilaton gravity models. One of the authors (G.F.) would like to thank the members of the Cosmology and Gravitation Laboratory of Nagoya University for their kind hospitality and acknowledges the support of OTKA Grant No. T 017176.

APPENDIX

In this appendix we review some important properties of radial null congruences in spherically symmetric spacetimes. We mostly follow the approach of Hayward [5]. Denote the future-directed null vector fields generating the congruences by k_+^α and k_-^α pointing in the outgoing and ingoing future radial null directions, respectively. Define the normalization function f by $k_+^\alpha k_{-\alpha}=-e^{-f}$. Because of spherical symmetry, $k_\pm^\alpha$ are geodesics, although they are not necessarily affinely parametrized:

$$k_+^\beta k_{+;\beta}^\alpha=b_+k_+^\alpha,\quad k_-^\beta k_{-;\beta}^\alpha=b_-k_-^\alpha. \tag{A1}$$

Given two intersecting ingoing and outgoing foliations of null hypersurfaces determined by constant values of ξ^+ and ξ^-, there are two obvious ways to define the null vector fields. The first is to set $k_{\pm\alpha}=-\xi^{\mp}_{;\alpha}$. Then $b_\pm=0$ and $k_\pm^\alpha$ are affine geodesics. The other way is to define the null vector fields by

$$k_\pm^\alpha=\left(\frac{\partial}{\partial\xi^\pm}\right)^\alpha. \tag{A2}$$

In this case the inaffinity parameters are $b_\pm=-k_\pm^\alpha f_{;\alpha}$.

The tensor

$$h_{\alpha\beta}=g_{\alpha\beta}+e^f(k_{+\alpha}k_{-\beta}+k_{-\alpha}k_{+\beta}) \tag{A3}$$

acts as a projection operator into the two-spheres. Since $B_{\alpha\beta}^\pm=h_\alpha^\gamma h_\beta^\delta k_{\pm\gamma;\delta}$ is symmetric and its trace-free part vanishes, the twist and the shear are zero. The expansion is

$$\Theta_\pm=B_\alpha^{\pm\alpha}=\frac{2}{\rho}k_\pm^\alpha\rho_{;\alpha}, \tag{A4}$$

where ρ is the natural radius function.

Using the Einstein's equations, one can derive two useful expressions for the directional derivatives of the expansions. The formula corresponding to the Raychaudhuri equation is

$$k_\pm^\alpha\Theta_{\pm;\alpha}=-\tfrac{1}{2}\Theta_\pm^2+b_\pm\Theta_\pm-8\pi T_{\alpha\beta}k_\pm^\alpha k_\pm^\beta, \tag{A5}$$

where $T_{\alpha\beta}$ is the stress tensor of the matter fields. The cross-focusing equation gives the derivative in the another null direction:

$$k_{\mp}^{\alpha}\Theta_{\pm;\alpha}=-\Theta_{+}\Theta_{-}-(k_{\mp}^{\alpha}f_{;\alpha}+b_{\mp})\Theta_{\pm}-\frac{1}{\rho^{2}}e^{-f}+8\pi T_{\alpha\beta}k_{+}^{\alpha}k_{-}^{\beta}. \tag{A6}$$

If either of the two expansions Θ_+ or Θ_- vanishes on a sphere of symmetry, the sphere is called a *marginal sphere*. The closure of a hypersurface foliated by marginal spheres is called a *trapping horizon*. A marginal sphere on the horizon with $\Theta_+=0$ is called *future* if $\Theta_-<0$ and *past* if $\Theta_->0$. It is called *outer* if $k_-^{\alpha}\Theta_{+;\alpha}<0$ and *inner* if $k_-^{\alpha}\Theta_{+;\alpha}>0$. If the weak energy condition holds, outer horizons are spacelike or null, while inner horizons are timelike or null. In both cases, they are null in the k_+^{α} direction if and only if $T_{\alpha\beta}k_+^{\alpha}k_+^{\beta}=0$.

The name ''inner'' for horizons satisfying $k_-^{\alpha}\Theta_{+;\alpha}>0$ can be misleading though. These inner horizons can separate a trapped region from an asymptotically flat untrapped region. A future horizon which is a smooth connected hypersurface can be outer in one region and change to be inner in another region, simply by becoming timelike through *ingoing* null directions. For example, in certain cases of pressureless dust collapse, the horizon can be timelike inner in a region close to the regular center, analogously to the cosmological horizon in a collapsing universe. Going outwards, this horizon becomes ingoing null at a two-sphere, and then it is spacelike outer. Asymptotically, in the Scwarzschild region, the horizon becomes null again, but then in the outgoing direction (see Fig. 3).

Following Hayward [5], we define the *trapping gravity* of an outer trapping horizon by the formula

$$\kappa_H=\tfrac{1}{2}\sqrt{-e^{f}k_-^{\alpha}\Theta_{+;\alpha}}. \tag{A7}$$

Changing k_+^{α} and k_-^{α} one can see that κ_H is invariantly defined only on the trapping horizon, where $\Theta_+=0$. Using the cross-focusing equation (A6),

$$\kappa_H=\frac{1}{2}\sqrt{\frac{1}{\rho^2}-8\pi e^{f}T_{\alpha\beta}k_+^{\alpha}k_-^{\beta}}. \tag{A8}$$

In the vacuum case we get the familiar $1/2\rho$ value, agreeing with surface gravity of the Schwarzschild solution. While κ_H is defined on the trapping horizon of dynamical spacetimes, the surface gravity is defined on the event horizon of stationary solutions. For stationary spacetimes the two kinds of horizons coincide. However, in general, the value of the trapping gravity κ_H is different from the value of the surface gravity. This is the case even for the Reissner-Nordström solution, the surface gravity of which is

$$\kappa=\frac{1}{r_h^2}\sqrt{m^2-e^2}=\frac{1}{2r_h}\left(1-\frac{e^2}{r_h^2}\right), \tag{A9}$$

where m is the mass, e is the charge parameter, and $r_h=m+\sqrt{m^2-e^2}$. The value of the trapping gravity is

$$\kappa_H=\frac{1}{2r_h^2}\sqrt{r_h^2-e^2}. \tag{A10}$$

For fixed m, κ is a monotonically decreasing function of e, while κ_H is not monotonic. $\kappa=\kappa_H$ only for $e=0$ and $e=m$.

Working in the null coordinate system (8) where $r=\rho$, we can choose

$$k_+^{\alpha}=\left(1,\frac{F}{2G},0,0\right),\quad k_-^{\alpha}=(0,-1,0,0). \tag{A11}$$

Then $e^{-f}=G$ and $\Theta_+=F/rG$. Since $F=0$ on the horizon,

$$\kappa_H=\frac{1}{2G}\sqrt{\frac{1}{r_h}F_{,r}}. \tag{A12}$$

Comparing with Eqs. (29) and (30), we obtain the relation between our dynamical surface gravity κ, the trapping gravity κ_H, and the temperature term Θ in Eq. (39):

$$\kappa+4\pi\int_{r_h}^{\infty}r\frac{\partial T_{rr}}{\partial v}dr=2r_hG\kappa_H^2=8\pi\Theta, \tag{A13}$$

where the integral is calculated along a constant v ingoing null line. We can see that the surface gravity agrees with the trapping gravity only in some special cases.

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