Conformally dressed black hole in 2 + 1 dimensions

Cristián Martínez*
Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile and Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile

Jorge Zanelli†
Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile and Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago, Chile

(Received 10 April 1996)

A three-dimensional black hole solution of Einstein equations with a negative cosmological constant coupled to a conformal scalar field is given. The solution is static, circularly symmetric, asymptotically anti–de Sitter–type and nonperturbative in the conformal field. The curvature tensor is singular at the origin while the scalar field is regular everywhere. The condition that the Euclidean geometry be regular at the horizon fixes the temperature to be $T=9\,r_+/16\pi l^2$. Using the Hamiltonian formulation including boundary terms of the Euclidean action, the entropy is found to be $\frac{2}{3}$ of the standard value $(\frac{1}{4}A)$, and in agreement with the first law of thermodynamics. [S0556-2821(96)02516-7]

PACS number(s): 04.20.Jb, 04.70.Bw, 97.60.Lf

I. INTRODUCTION

In the last ten years, three-dimensional gravity has become a popular laboratory to understand the fundamentals of classical and quantum gravity [1]. Thus, the discovery of a black hole solution in 2+1 dimensions [2] has further contributed to the interest in three-dimensional gravity. A complete review about this black hole can be found in [3].

Several generalizations of this solution have been constructed. For instance, minimally and nonminimally coupled dilaton fields with various black holes (charged and uncharged, spinning and nonspinning) [4–6] are known. For other interesting extensions see [3] and references therein.

The purpose of this article is to report on an exact black hole solution conformally coupled to a massless scalar field in 2+1 dimensions. The solution is static, circularly symmetric, and asymptotically anti–de Sitter–type and it possesses a curvature singularity at the origin. The scalar field is regular everywhere, has a fixed form, and cannot be obtained as a perturbation around a matter-free massive black hole. The system can be shown to have a well-defined thermodynamic behavior.

Here, we consider gravity with cosmological constant conformally coupled to a massless scalar field in D dimensions. The action is

$$I=I_G+I_C \tag{1}$$

with

$$I_G=\frac{1}{2\kappa}\int d^Dx\sqrt{-g}[R+2l^{-2}] \tag{2}$$

and

$$I_C=-\frac{1}{2}\int d^Dx\sqrt{-g}[g^{\mu\nu}\nabla_\mu\Psi\nabla_\nu\Psi+\xi_D R\Psi^2], \tag{3}$$

where R is the scalar curvature and $\xi_D=\frac{1}{4}(D-2)/(D-1)$. The value of ξ_D is chosen so that I_C be invariant under conformal transformations

$$g_{\mu\nu}\rightarrow\Omega^2(x)g_{\mu\nu},\Psi\rightarrow\Omega^{1-D/2}(x)\Psi. \tag{4}$$

This coupling, including electromagnetism but without cosmological constant in four dimensions, was previously considered by Bocharova, Bronnikov, and Melnikov [7] and Bekenstein (BBMB) [8,9] (see also [10]). The uncharged BBMB black hole solution is static, spherically symmetric, and asymptotically flat (there is no cosmological constant). The metric is the extreme Reissner-Nordström metric solution and the scalar field is unbounded at horizon. In [9] it is shown that this divergence is not physically troublesome.

Recently, the uniqueness of the BBMB black hole has been established [11] and it was shown to be the only static, spherically symmetric, asymptotically flat black hole solution of the Einstein-conformal field equations in four spacetime dimensions [12].

Below, we present a black hole solution for the system described in Eq. (1) in three dimensions.

II. BLACK HOLE SOLUTIONS

In three dimensions, the action reads

$$I=\int d^3x\sqrt{-g}\left[\frac{R+2l^{-2}}{2\kappa}-\tfrac{1}{2}g^{\mu\nu}\nabla_\mu\Psi\nabla_\nu\Psi-\tfrac{1}{16}R\Psi^2\right], \tag{5}$$

where $-l^{-2}$ is the cosmological constant and Ψ is the massless conformal scalar field.

The field equations are

$$G_{\mu\nu}-l^{-2}g_{\mu\nu}-\kappa T_{\mu\nu}=0 \tag{6}$$

*Electronic address: martinez@cecs.cl
†Electronic address: jz@cecs.cl

and

$$\Box\Psi-\tfrac{1}{8}R\Psi=0, \qquad (7)$$

where $\Box\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplace-Beltrami operator in the metric $g_{\mu\nu}$ and the matter stress tensor is

$$T_{\mu\nu}=\nabla_\mu\Psi\nabla_\nu\Psi-\tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\Psi\nabla_\beta\Psi + \tfrac{1}{8}[g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+G_{\mu\nu}]\Psi^2. \qquad (8)$$

It is straightforward to check that by virtue of Eq. (7), the stress tensor is traceless. This, in turn, implies that the geometry has a constant scalar curvature:

$$R=-6l^{-2}. \qquad (9)$$

We look for static, circularly symmetric, three-dimensional metrics whose expression in polar coordinates takes the form

$$ds^2=-N^2(r)F(r)dt^2+F^{-1}(r)dr^2+r^2d\theta^2, \qquad (10)$$

where $0\leqslant r<\infty$ is the proper radial coordinate and $0\leqslant\theta\leqslant 2\pi$. The solution is easily obtained fixing the time scale so that $N(r)=1$. Working with the advanced time coordinate $v=t+\int F^{-1}(r)dr$, the $r-r$ equation of Eq. (6) imposes the constraint

$$0=(\Psi')^2-\tfrac{1}{8}(\Psi^2)'', \qquad (11)$$

where prime denotes radial derivative. The above equation can be written as $0=\Psi^4(\Psi^{-2})''$ whose general solution is

$$\Psi(r)=\frac{A}{\sqrt{r+B}} \quad A,B=\text{ const.} \qquad (12)$$

Comparing the curvature for the metric (10) with Eq. (9), one obtains directly

$$F(r)=\frac{r^2}{l^2}-a-\frac{b}{r} \quad a,b=\text{const.} \qquad (13)$$

The ${}_v{}^v$ equation imposes the following relations among the constants of integration

$$a=3B^2l^{-2}, \quad b=2B^3l^{-2}, \quad A=\sqrt{\frac{8B}{\kappa}}, \quad B\geqslant 0. \qquad (14)$$

Thus, we obtain the black hole solution

$$F(r)=\frac{1}{l^2}\left[r^2-3B^2-\frac{2B^3}{r}\right]=\frac{(r+B)^2(r-2B)}{rl^2}, \qquad (15)$$

together with the matter field configuration

$$\Psi(r)=\sqrt{\frac{8B}{\kappa(r+B)}}, \qquad (16)$$

which can be explicitly checked to solve Eq. (7). It is easily shown in the advanced time coordinates that the surface where F vanishes ($r=2B\equiv r_+$) is null [13].

The asymptotic behavior of the metric is truly anti–de Sitter [i.e., $g_{00}\sim r^2+O(r^0)$, without terms linear in r]. Therefore, as shown in [14], the asymptotic symmetry group is the conformal one, which contains the anti–de Sitter group as a subgroup.

The Riemann tensor is singular at the origin as can be shown by evaluating the Kretschmann scalar

$$R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}=\frac{12(r^6+2B^6)}{l^4r^6}. \qquad (17)$$

This is the only singularity and is hidden by the event horizon.

The massless conformal scalar field Ψ is regular everywhere. Although one might expect the scalar field to endow the black hole with a hair, it should be noted that the solution is characterized by only one constant which, as we will show below, is related with the mass. Therefore, the presence of the scalar field does not generate an independent additional charge to the black hole, i.e., the scalar field produces no new hair. Furthermore, the solution presented here does not differ in the asymptotic region from a matter-free black hole.

III. THERMODYNAMICS

The Hamiltonian form of the action (5) is given by

$$I=\int[\pi^{ij}\dot{g}_{ij}+P\dot{\Psi}-N\mathscr{H}-N^i\mathscr{H}_i]d^2x\,dt+B_H, \qquad (18)$$

where B_H is a surface term.

In order to study the thermodynamics of this system we consider the minisuperspace of static, circularly symmetric geometries as described by (10), and scalar fields that depend only on the radial coordinate. The equations of motion obtained in this way are the same as Eqs. (6), and (7) after imposing the above restrictions. Thus, reducing the Hamiltonian action (18) to the minisuperspace gives

$$I=-2\pi(t_2-t_1)\int N(r)\mathscr{H}(r)dr+B_H \qquad (19)$$

with

$$\mathscr{H}=\frac{1}{2\kappa}\{F'(1-\zeta)-2Fr[\zeta''-\zeta^{-1}(\zeta')^2] \qquad (20)$$

$$-(2F+F'r)\zeta'-2rl^{-2}\}, \qquad (21)$$

$$\zeta\equiv\frac{\kappa}{8}\Psi^2. \qquad (22)$$

The partition function for a thermodynamical ensemble is identified with the Euclidean path integral in the saddle-point approximation around the Euclidean continuation of the classical solution [15]. In this approximation the Euclidean action is related to the thermodynamic functions (in units where $\hbar=k_B=1$ and $\kappa=8\pi$) by

$$I_E=\frac{\text{free energy}}{T}=\frac{M}{T}-S, \qquad (23)$$

where T, M, S denote temperature, mass, entropy, respectively and the Euclidean action I_E is related to the Lorentzian action by

$$I_E=-iI, \quad \tau=it. \tag{24}$$

The Euclidean continuation of the metric is

$$ds_E^2=N^2(r)F(r)d\tau^2+F(r)^{-1}dr^2+r^2d\theta^2 \tag{25}$$

with $\tau_1\leqslant\tau\leqslant\tau_2$ periodic, $r\geqslant r_+$, and the scalar field unchanged.

The condition that the geometries allowed in the variation should contain no conical singularities at the horizon implies

$$(\tau_2-\tau_1)F'|_{r=r_+}=4\pi, \tag{26}$$

which directly yields the temperature $[N(r)=1]$

$$T\equiv\beta^{-1}=\frac{1}{\tau_2-\tau_1} \tag{27}$$

$$=\frac{9r_+}{16\pi l^2}. \tag{28}$$

We now turn to the evaluation of the Euclidean action at the Euclidean solution. The classical solution is static and satisfies the constraint $\mathscr{H}=0$ and, therefore, the action at the classical solution is given by a boundary term B_E. This boundary term must be such that the geometry (25) be a true extremum among the class of metrics satisfying the right boundary conditions [16,17].

At infinity, we demand that the variations of the fields behave as

$$\delta N=0, \tag{29}$$

$$\delta F\rightarrow-\delta\frac{3r_+^2}{4l^2}, \tag{30}$$

$$\delta\zeta\rightarrow\frac{\delta r_+}{2r}. \tag{31}$$

At the horizon, we impose the regularity condition (26),

$$\beta F'|_{r=r_+}=4\pi \tag{32}$$

and

$$(\delta F)_{r_+}+F'|_{r=r_+}\delta r_+=0, \tag{33}$$

which is required by the definition of the horizon $F(r_+)=0$. And, $(\delta N)_{r_+}=0$.

The variation of the scalar field at the horizon is obtained by varying it with respect to r_+, maintaining the functional form of the classical solution, $\zeta=r_+/(2r+r_+)$. Hence,

$$\delta\zeta=\frac{2}{9r_+}\delta r_+. \tag{34}$$

The variation of the Euclidean action is

$$\delta I_E=\frac{\beta}{8}[(1-\zeta-r\zeta')\delta F+(F'r+4Fr\zeta^{-1}\zeta')\delta\zeta -2Fr\,\delta\zeta']_{r_+}^{\infty}+\delta B_E+\text{ terms vanishing on shell.} \tag{35}$$

For convenience, we write $B_E=B_E(\infty)+B_E(r_+)$. The contribution from infinity is

$$\delta B_E(\infty)=\beta\delta\left(\frac{3r_+^2}{32l^2}\right). \tag{36}$$

One can note here that the scalar field does not contribute to surface term at infinity. This is yet another indication of the nonexistence of charges associated to the conformal scalar field.

At the horizon, we have

$$\delta B_E(r_+)=\beta[\tfrac{1}{9}\,\delta F+\tfrac{1}{36}\,F'\,\delta r_+], \tag{37}$$

which, in view of Eqs. (32) and (33), can be written as

$$\delta B_E(r_+)=-\frac{\pi}{3}\delta r_+. \tag{38}$$

Combining Eqs. (36) and (38), the Euclidean action is found to be

$$I_E=\beta\frac{3r_+^2}{32l^2}-\frac{\pi}{3}r_++B_0, \tag{39}$$

where B_0 is an arbitrary constant independent of the fields at the boundaries. Imposing that $I_E=0$ for $r_+\rightarrow 0$, one finds that $B_0=0$. If we compare the above expression for I_E with Eq. (23), we learn that the energy and entropy are

$$M=\frac{3r_+^2}{32l^2}, \tag{40}$$

$$S=\frac{\pi r_+}{3}, \tag{41}$$

respectively. With these expressions, one can check that the first law of thermodynamics

$$dM=TdS \tag{42}$$

is satisfied.

IV. CONCLUDING REMARKS

The inclusion of the cosmological constant is absolutely necessary for obtaining the black hole solution. In spite of the fact that the matter coupling in Eq. (1) is of the same form as that of the BBMB theory, the resulting black holes are entirely different: The BBMB solution is asymptotically flat and is an extreme Reissner-Nordström hole, whereas the solution introduced here is asymptotically anti–de Sitter and nonextreme. Furthermore, one can readily see that the ansatz (10) $(N=1)$ does not yield an extension of the BBMB solution with cosmological constant in four dimensions.

The question of whether this solution represents a hairy black hole depends on the definition of hair one uses. In a very broad sense, any matter field that can be sustained by a black hole could be regarded as some kind of hair, as it is the case at hand. However, in a more strict sense, it is necessary for the matter field to carry an independently conserved charge, which does not occur in our case.

Another point of interest consists in looking for time-dependent solutions. The existence of these solutions could show that a black hole can be regarded as the result of collapsed matter fields [18–22]. However, the system of massless conformal scalar matter field coupled to gravity, assuming a stationary, spherically symmetric geometry [i.e., $F=F(r,t)$ and $\Psi=\Psi(r,t)$], gives rise to the same *static* solution (Birkhoff's theorem).

A related question is whether the static solution presented here is stable under linear perturbations. The question can be addressed for the case of circularly symmetric perturbations and will be discussed elsewhere [23].

We note that the entropy differs by a factor of $\frac{2}{3}$ from the "area law" $(\pi/2)\,r_+$. This deviation from the area law was also found in other systems of matter fields coupled to gravity [24]. In [25,26] this deviation is also shown to arise in "dirty" black hole and in systems of black holes coupled to strings.

Note added. A family of solutions of scalar-tensor fields coupled to gravity in 2+1 dimensions was recently reported [27]. It seems that the solution presented here might be obtained as a special, particularly simple, case among many others, but this is not completely clear to these authors at the moment.

ACKNOWLEDGMENTS

Useful discussions with M. Bañados, C. Teitelboim, and R. Troncoso are gratefully acknowledged. This work was supported in part by Grant Nos. 2940012/94, 4950004/95, 1940203, and 1960229 of FONDECYT (Chile) and Grant No. 27-953/ZI (USACH). The institutional support of a group of Chilean private companies (EMPRESAS CMPC, CGE, COPEC, MINERA ESCONDIDA, NOVAGAS Transportadores de Chile, BUSINESS DESIGN ASS., XEROX Chile) is also recognized.

[1] For a recent review, see S. Carlip, "Lectures on (2+1)-Dimensional Gravity," Davis Report No. UCD-95-6, gr-qc/9503024, 1995 (unpublished).

[2] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Bañados, C. Teitelboim, M. Henneaux, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).

[3] S. Carlip, Class. Quantum Grav. **12**, 2853 (1995).

[4] K.C.K. Chan and R.B. Mann, Phys. Rev. D **50**, 6385 (1994); **52**, 2600(E) (1995).

[5] K. C. K. Chan and R.B. Mann, Phys. Lett. B **371**, 199 (1996).

[6] P.M. Sá, A. Kleber, and J. P. S. Lemos, Class. Quantum Grav. **13**, 125 (1996).

[7] N. Bocharova, K. Bronnikov, and V. Melnikov, Vestn. Mosk. Univ. Fiz. Astron. **6**, 706 (1970).

[8] J. D. Bekenstein, Ann. Phys. (N.Y.) **82**, 535 (1974).

[9] J. D. Bekenstein, Ann. Phys. (N.Y.) **91**, 75 (1975).

[10] J. Frøyland, Phys. Rev. D **25**, 1470 (1982); K. S. Virbhadra and J. C. Parikh, Phys. Lett. B **331**, 302 (1994); **340**, 265(E) (1994).

[11] B. C. Xanthopoulos and T. Zannias, J. Math. Phys. **32**, 1875 (1991).

[12] B.C. Xanthopoulos and T. E. Dialynas, J. Math. Phys. **33**, 1463 (1992).

[13] In these coordinates, the metric reads $ds^2=-Fdv^2+2dvdr+r^2d\theta^2$. The horizon at $r=2B\equiv r_+$ is generated by the geodesics $r(\lambda)=r_+ d\theta/d\lambda=0$. This surface is marginally trapped since any of these geodesics satisfies $(dv)/(d\lambda dr/d\lambda)=-(r_+^2/2)(d\theta/d\lambda)^2\leqslant 0$, so that r decreases or remains constant with increasing v.

[14] J. D. Brown, and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).

[15] G. W. Gibbons and S.W. Hawking, Phys. Rev. D **15**, 2752 (1977).

[16] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).

[17] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **72**, 957 (1994).

[18] R. B. Mann and S. F. Ross, Phys. Rev. D **47**, 3319 (1993).

[19] V. Husain, Phys. Rev. D **50**, 2361 (1994).

[20] K.S. Virbhadra, "Exact Solutions of the Einstein-Scalar Equations in 2+1 Dimensions," Report No. gr-qc/9408035, 1994 (unpublished).

[21] V. Husain, Phys. Rev. D **52**, 6860 (1995).

[22] N. Cruz and J. Zanelli, Class. Quantum Grav. **12**, 957 (1995).

[23] C. Martínez and J. Zanelli, "Stability of the 2+1 conformal scalar black hole" (in preparation).

[24] J. D. Creighton and R. Mann, Phys. Rev. D **52**, 4569 (1995).

[25] M. Visser, Phys. Rev. D **48**, 5697 (1993).

[26] F. Englert, L. Houart, and P. Windey, Phys. Lett. B **372**, 29 (1996); Nucl. Phys. **B458**, 231 (1996).

[27] K. C. K. Chan "Modifications of the BTZ black hole by dilaton/scalar," Report No. gr-qc/9603038 (unpublished).