Bethe-Salpeter equation in QCD in a Wilson loop context

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We give a nonperturbative derivation of the Bethe-Salpeter equation in QCD based on the Feynman-Schwinger path integral representation of the one particle propagator in an external field. We obtain a path integral representation for a second order quark-antiquark amplitude in which the gauge field appears only through an appropriate Wilson loop integral W. Then, for such a quantity we derive a $q\overline{q}$ BS equation assuming that $i\ln W$ can be written as the sum of a perturbative contribution and an area term as in the derivation of the heavy quark potential. We also show that, by standard approximations, an effective meson mass operator can be obtained from our BS kernel. From this the corresponding Wilson loop potential is recovered, by $1/m^2$ expansion, spin-dependent and velocity-dependent terms included. On the contrary, neglecting spin-dependent terms, the relativistic flux tube model is reproduced. The method is illustrated also on the simplified case of two spinless particles interacting via a scalar field and on a one-dimensional potential model. [S0556-2821(96)02115-7]

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I. INTRODUCTION

Various approaches to the relativistic bound state problem in QCD have been considered in the literature in terms of numerical simulations on a lattice [1,2], relativistic Hamiltonians [3], and effective Lagrangians. In particular, attempts have been made to apply the Bethe-Salpeter (BS) equation to a treatment of the quark-antiquark bound states involving light quarks as well as heavy ones, hoping to obtain a unified and consistent description of the spectrum and of the properties of the mesons.

In all BS attempts, to our knowledge, the choice of the long range part of the kernel was purely conjectural and only made in such a way that the successful static heavy quark potential could be recovered in the semirelativistic limit. The most usual assumptions were those of the so-called scalar confinement and vectorial confinement. Both choices, in addition to being not theoretically motivated, meet with conceptual and phenomenological difficulties. A scalar-type confinement kernel fails in reproducing straight Regge trajectories [5], yields unstable variational solutions [6], and generates wrong velocity-dependent potentials in the semirelativistic limit [7,8]. Similarly a vectorial-type confinement kernel generates a wrong spin orbit and velocity-dependent potential [6,9].

For the above reasons a derivation of a $q\overline{q}$ BS equation from QCD first principles would be highly desirable. One of the main difficulties along this line is that the usual justification of the BS equation works in terms of a resummation of the perturbative expansion and such a method cannot be applied to the case of a confining theory. To overcome the difficulty it is necessary to develop an intrinsically nonperturbative method. We try to do this in the present paper, modifying the approach already followed in the derivation of the semirelativistic potential for heavy quarks.

To obtain the $q\overline{q}$ potential the basic object is the Wilson loop integral

$$W=\frac{1}{3}\left\langle \mathrm{Tr}P\exp\left(ig \oint_{\Gamma} dx^{\mu}A_{\mu}\right)\right\rangle, \tag{1.1}$$

where Γ denotes an appropriate closed loop in space-time, $A_{\mu}=\frac{1}{2}A_{\mu}^{a}\lambda^{a}$ is a color matrix, P the ordering prescription along Γ, and the expectation value stands for a functional integration on the gauge field alone. By the use of the path integral representation for a Pauli-type quark propagator in an external field, the evaluation of the potential is reduced to an evaluation of W. For such an evaluation the simplest assumption is to write $i\ln W$ as the sum of a perturbative term and an area term

$$i\ln W=i(\ln W)_{\mathrm{pert}}+\sigma S_{\mathrm{min}}, \tag{1.2}$$

σ being the so-called string tension and S_{min} the minimal area enclosed by Γ. In Eq. (1.2) ordinary perturbative theory and lattice theory results are combined. The first quantity is supposed to describe the short range forces, the second one to correspond to the long range ones. A term CP proportional to the Γ perimeter P can be also included; we do not consider it explicitly here, since such a term turns out with the same form of the kinetical terms and apparently it can be reabsorbed in a redefinition of them. More sophisticated expressions than Eq. (1.2), based on a perturbative theory in a nonperturbative background [4], are possible but will not be considered here. Obviously, in the above, confinement is introduced through the second term in Eq. (1.2).

In this paper we shall start from Eq. (1.2) and, using the covariant Feynman-Schwinger representation for the full quark propagator in an external field (rather than the semirelativistic one used in [10,11]), we shall arrive at a BS equation for a "second order" quark-antiquark Green function. The need to resort to the second order formalism is related to the use of a path integral representation. In a sense the final equation corresponds to the ordinary "first order" BS equation as the iterated Dirac equation corresponds to the ordinary one.

Eventually, in the center of mass frame we obtain a kernel of the form

$$\hat{I}(p_1,p_2;p_1',p_2')=\hat{I}_{\rm pert}(p_1,p_2;p_1',p_2')+\hat{I}_{\rm conf}(p_1,p_2;p_1',p_2') \tag{1.3}$$

$(p_1'+p_2'=p_1+p_2)$, with

$$\begin{aligned}\hat{I}_{\rm pert}=16\pi\frac{4}{3}\alpha_s\Big\{&D_{\rho\sigma}(Q)q_1^\rho q_2^\sigma-\frac{i}{4}\sigma_1^{\mu\nu}(\delta_\mu^\rho Q_\nu-\delta_\nu^\rho Q_\mu)\\&\times q_2^\sigma D_{\rho\sigma}(Q)+\frac{i}{4}\sigma_2^{\mu\nu}(\delta_\mu^\sigma Q_\nu-\delta_\nu^\sigma Q_\mu)q_1^\rho D_{\rho\sigma}(Q)\\&+\frac{1}{16}\sigma_1^{\mu_1\nu_1}\sigma_2^{\mu_2\nu_2}(\delta_{\mu_1}^\rho Q_{\nu_1}-\delta_{\nu_1}^\rho Q_{\mu_1})\\&\times(\delta_{\mu_2}^\sigma Q_{\nu_2}-\delta_{\nu_2}^\sigma Q_{\mu_2})D_{\rho\sigma}(Q)\Big\}+\cdots\end{aligned} \tag{1.4}$$

and

$$\hat{I}_{\rm conf}=\int d^3\mathbf{r}e^{i\mathbf{Q}\cdot\mathbf{r}}J(\mathbf{r},q_1,q_2), \tag{1.5}$$

with

$$\begin{aligned}J(\mathbf{r},q_1,q_2)=\frac{2\sigma r}{q_{10}+q_{20}}\bigg[&q_{20}^2\sqrt{q_{10}^2-\mathbf{q}_T^2}\\&+q_{10}^2\sqrt{q_{20}^2-\mathbf{q}_T^2}+\frac{q_{10}^2q_{20}^2}{|\mathbf{q}_T|}\\&\times\left(\arcsin\frac{|\mathbf{q}_T|}{q_{10}}+\arcsin\frac{|\mathbf{q}_T|}{q_{20}}\right)\bigg]+2\sigma\frac{\sigma_1^{k\nu}q_{20}q_{1\nu}r^k}{r\sqrt{q_{10}^2-\mathbf{q}_T^2}}\\&-2\sigma\frac{\sigma_2^{k\nu}q_{10}q_{2\nu}r^k}{r\sqrt{q_{20}^2-\mathbf{q}_T^2}}+\cdots.\end{aligned} \tag{1.6}$$

In such equations α_s denotes the strong coupling constant, $D_{\rho\sigma}(Q)$ denotes the free gluon propagator,[1] and we set

$$q_1=\frac{p_1+p_1'}{2},\quad q_2=\frac{p_2+p_2'}{2},\quad Q=p_1-p_1'=p_2'-p_2, \tag{1.7}$$

with $\mathbf{q}_1=-\mathbf{q}_2=\mathbf{q}$, $\quad q_T^h=(\delta^{hk}-\hat{r}^h\hat{r}^k)q^k$.

Equations (1.3)–(1.7) are the main result of the present paper.

They have to be considered as the lowest order terms in a mixed expansion in the constant α_s and in the quantity σa^2 (a being a typical length, like the radius of the meson). As we shall see, there are indications that the term in $(\sigma a^2)^2$ in the expansion is small.

[1]In the Coulomb gauge we take $D_{\rho\sigma}$ with the form

$$D_{00}(x)=-\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\delta(x_0),$$

$$D_{0k}=D_{k0}=0,$$

$$D_{hk}=-\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\delta^{hk}-\frac{k^hk^k}{\mathbf{k}^2}\right)e^{-ikx}.$$

It should be mentioned that our formalism is strictly related to that developed by Simonov and collaborators to establish an effective relativistic Hamiltonian [3]. Notice also that a preliminary account of our results has been already presented in [11] and [12].

From the above kernel, in the limit of large quark masses, we can obtain a semirelativistic quark-antiquark potential by standard techniques. Such potential is obviously identical to the one derived in [10,11], spin- and velocity-dependent terms included.

On the contrary, if we neglect the spin-dependent terms and perform an appropriate instantaneous approximation, but maintain full relativistic kinematics, we obtain the Hamiltonian of the so-called relativistic flux tube model [13–15].

We stress that, even if we have not yet performed any explicit calculation using the full expression (1.3)–(1.6), the results obtained in [3,13,8] (which should be considered as referring to limit cases), suggest that our kernel (1.3)–(1.7) should overcome all the difficulties encountered with the phenomenological kernels.

The plan of the paper is the following one. In Sec. II we illustrate our nonperturbative method of derivation of the BS equation in the model case of two spinless particles interacting through a scalar field, to which also the usual perturbative derivation applies: In this way we have in particular the opportunity of discussing the meaning and the limitations of the various approximations we have performed. In Sec. III we discuss the evaluation of $i\ln W$. In Sec. IV we introduce the Feynman-Schwinger representation and obtain a corresponding path integral representation for the second order $q\bar{q}$ Green function. In Sec. V we derive the inhomogeneous BS equation and obtain the kernel given by Eqs. (1.3)–(1.7). In Sec. VI we write the homogeneous BS equation and discuss the instantaneous and the semirelativistic potential. Finally in Sec. VII we draw some conclusions and make some additional remarks. The appendixes are devoted to technical details and to an illustration of the derivation method for a one-dimensional potential model.

II. PROPAGATOR AND THE BETHE-SALPETER EQUATION FOR SCALAR PARTICLES

Let us consider two scalar ''material'' fields ϕ_1 and ϕ_2 interacting through a third scalar field A with the coupling $\frac{1}{2}(g_1\phi_1^2A+g_2\phi_2^2A)$ (typically we may think of $g_1=-g_2=g$). After integration over ϕ_1 and ϕ_2, the full one-particle propagator can be written as

$$\begin{aligned}G_2^{(j)}(x-y)&=\langle 0|T[\phi_j(x)\phi_j(y)]|0\rangle=\langle i\Delta^{(j)}(x,y;A)\rangle\\&\equiv\frac{\int\mathcal{D}Ae^{iS_0(A)}M(A)i\Delta^{(j)}(x,y;A)}{\int\mathcal{D}Ae^{iS_0(A)}M(A)},\end{aligned} \tag{2.1}$$

where $\Delta^{(j)}(x,y;A)$ is the propagator for the particle j in the external field A, $S_0(A)$ is the free action for the field A, and the determinantal factor $M(A)$ comes from the integration on the fields ϕ_j:

$$M(A)=\prod_{j=1,2}\left[\frac{\det(\partial^\mu\partial_\mu+m_j^2-g_jA)}{\det(\partial^\mu\partial_\mu+m_j^2)}\right]^{-1/2}=1+\frac{1}{2}\sum_{j=1,2}\left\{-g_j\int d^4xA(x)\Delta_F^{(j)}(0)+\frac{1}{2}g_j^2\int d^4xd^4yA(x)\Delta_F^{(j)}(x-y)\right.$$
$$\left.\times A(y)\Delta_F^{(j)}(y-x)+\cdots\right\}+\frac{1}{8}\left(\sum_{j=1,2}g_j\int d^4xA(x)\triangle_F^{(j)}(0)+\cdots\right)^2+\cdots, \tag{2.2}$$

where $\Delta_F^{(j)}$ denotes the usual free scalar particle propagator.

The covariant Feynman-Schwinger representation for $\Delta^{(j)}(x,y;A)$ reads

$$\Delta^{(j)}(x,y;A)=-\frac{i}{2}\int_0^\infty ds\int_y^x\mathcal{D}z\mathcal{D}p\exp\left\{i\int_0^s d\tau\left[-p_\mu\dot{z}^\mu+\frac{1}{2}p_\mu p^\mu-\frac{1}{2}m_j^2+\frac{1}{2}g_jA(z)\right]\right\}$$
$$=-\frac{i}{2}\int_0^\infty ds\int_y^x\mathcal{D}z\exp\left\{-i\int_0^s d\tau\left[\frac{1}{2}(\dot{z}^2+m_j^2)-\frac{1}{2}g_jA(z)\right]\right\}, \tag{2.3}$$

where the path integral is assumed to be extended over all paths $z^\mu=z^\mu(\tau)$ connecting y with x $[0\leqslant\tau\leqslant s;\ z(0)=y,$ $z(s)=x]$. Obviously in Eq. (2.3) $\dot{z}$ stands for $dz(\tau)/d\tau$ and the ''functional measures'' are defined as

$$\mathcal{D}z=\left(\frac{1}{2\pi i\varepsilon}\right)^{2N}d^4z_1\cdots d^4z_{N-1},\quad \mathcal{D}p=\left(\frac{i\varepsilon}{2\pi}\right)^{2N}d^4p_1\cdots d^4p_{N-1}d^4p_N,$$
$$\mathcal{D}z\mathcal{D}p=\left(\frac{1}{2\pi}\right)^{4N}d^4p_1d^4z_1\cdots d^4p_{N-1}d^4z_{N-1}d^4p_N \tag{2.4}$$

(ε being the time lattice spacing and $\varepsilon\to 0$ being understood).

Replacing Eq. (2.3) in Eq. (2.1) we obtain

$$G_2^{(j)}(x-y)=\frac{1}{2}\int_0^\infty ds\int_y^x\mathcal{D}z\exp\left\{-\frac{i}{2}\int_0^s d\tau(\dot{z}^2+m_j^2)\right\}\left\langle\exp\left\{\frac{ig_j}{2}\int_0^s d\tau A(z)\right\}\right\rangle, \tag{2.5}$$

and, if we take simply $M(A)=1$ (quenched approximation),

$$\left\langle\exp\left\{\frac{ig_j}{2}\int_0^s d\tau A(z)\right\}\right\rangle=\exp\left\{-\frac{ig_j^2}{4}\int_0^s d\tau\int_0^\tau d\tau' D_F(z-z')\right\}, \tag{2.6}$$

$D_F(x)$ being now the free propagator for the field A given by

$$D_F(x)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2}.$$

Similarly, for the two-particle propagator,

$$G_4(x_1,x_2;y_1,y_2)=\langle 0|T[\phi_1(x_1)\phi_2(x_2)\phi_1(y_1)\phi_2(y_2)]|0\rangle=\langle i\Delta^{(1)}(x_1,y_1;A)i\Delta^{(2)}(x_2,y_2;A)\rangle$$
$$=\left(\frac{1}{2}\right)^2\int_0^\infty ds_1\int_0^\infty ds_2\int_{y_1}^{x_1}\mathcal{D}z_1\int_{y_2}^{x_2}\mathcal{D}z_2\exp\left\{-\frac{i}{2}\int_0^{s_1}d\tau_1(\dot{z}_1^2+m_1^2)-\frac{i}{2}\int_0^{s_2}d\tau_2(\dot{z}_2^2+m_2^2)\right\}$$
$$\times\left\langle\exp\left\{\frac{ig_1}{2}\int_0^{s_1}d\tau_1A(z_1)+\frac{ig_2}{2}\int_0^{s_2}d\tau_2A(z_2)\right\}\right\rangle \tag{2.7}$$

and [always for $M(A)=1$]

$$\left\langle\exp\left\{\frac{ig_1}{2}\int_0^{s_1}d\tau_1A(z_1)+\frac{ig_2}{2}\int_0^{s_2}d\tau_2A(z_2)\right\}\right\rangle=\exp\left\{\sum_{j=1,2}\frac{-ig_j^2}{4}\int_0^{s_j}d\tau_j\int_0^{\tau_j}d\tau_j' D_F(z_j-z_j')\right.$$
$$\left.-i\frac{g_1g_2}{4}\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2D_F(z_1-z_2)\right\} \tag{2.8}$$

[z_j stands for $z_j(\tau_j)$, z_j' for $z_j(\tau_j')$]. In conclusion we have

$$G_4(x_1,x_2;y_1,y_2)=(\tfrac{1}{2})^2\int_0^\infty ds_1\int_0^\infty ds_2\int_{y_1}^{x_1}\mathcal{D}z_1\int_{y_2}^{x_2}\mathcal{D}z_2\exp\left\{-\frac{i}{2}\int_0^{s_1}d\tau_1(\dot z_1^2+m_1^2)-\frac{ig_1^2}{4}\int_0^{s_1}d\tau_1\int_0^{\tau_1}d\tau_1' D_F(z_1-z_1')\right\}$$
$$\times\exp\left\{-\frac{i}{2}\int_0^{s_2}d\tau_2(\dot z_2^2+m_2^2)-\frac{ig_2^2}{4}\int_0^{s_2}d\tau_2\int_0^{\tau_2}d\tau_2' D_F(z_2-z_2')\right\}$$
$$\times\exp\left\{-\frac{ig_1g_2}{4}\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2 D_F(z_1-z_2)\right\}. \tag{2.9}$$

Equation (2.9), using the identities

$$\exp\left\{-\frac{ig_1g_2}{4}\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2 D_F(z_1-z_2)\right\}=1-\frac{ig_1g_2}{4}\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2 D_F(z_1-z_2)\exp\left\{-\frac{ig_1g_2}{4}\int_0^{\tau_1}d\tau_1'\int_0^{s_2}d\tau_2' D_F(z_1'-z_2')\right\}, \tag{2.10}$$

$$\int_{y_j}^{x_j}\mathcal{D}z_j\cdots=\int d^4\xi_j\int_{z_j(\tau_j)=\xi_j}^{x_j}\mathcal{D}z_j\int_{y_j}^{z_j(\tau_j)=\xi_j}\mathcal{D}z_j\cdots,$$

becomes, after some manipulation,

$$G_4(x_1,x_2;y_1,y_2)=G_2(x_1-y_1)G_2(x_2-y_2)-\frac{ig_1g_2}{4}\left(\frac{1}{2}\right)^2$$
$$\times\int_0^\infty d\tau_1\int_{\tau_1}^\infty ds_1\int_0^\infty d\tau_2\int_{\tau_2}^\infty ds_2\int d^4\xi_1\int d^4\xi_2\int_{z_1(\tau_1)=\xi_1}^{x_1}\mathcal{D}z_1\int_{z_2(\tau_2)=\xi_2}^{x_2}\mathcal{D}z_2\int_{y_1}^{z_1(\tau_1)=\xi_1}\mathcal{D}z_1$$
$$\times\int_{y_2}^{z_2(\tau_2)=\xi_2}\mathcal{D}z_2 D_F(\xi_1-\xi_2)\exp\left\{\int_{\tau_1}^{s_1}d\tau_1'\left[-\frac{i}{2}(\dot z_1'^2+m_1^2)-\frac{ig_1^2}{4}\int_{\tau_1}^{\tau_1'}d\tau_1'' D_F(z_1'-z_1'')\right]\right.$$
$$\left.+\int_{\tau_2}^{s_2}d\tau_2'\left[-\frac{i}{2}(\dot z_2'^2+m_2^2)-\frac{ig_2^2}{4}\int_{\tau_2}^{\tau_2'}d\tau_2'' D_F(z_2'-z_2'')\right]\right\}$$
$$\times\exp\left\{\int_0^{\tau_1}d\tau_1'\left[-\frac{i}{2}(\dot z_1'^2+m_1^2)-\frac{ig_1^2}{4}\int_0^{\tau_1'}d\tau_1'' D_F(z_1'-z_1'')\right]\right.$$
$$\left.+\int_0^{\tau_2}d\tau_2'\left[-\frac{i}{2}(\dot z_2'^2+m_2^2)-\frac{ig_2^2}{4}\int_0^{\tau_2'}d\tau_2'' D_F(z_2'-z_2'')\right]\right\}\exp\left\{-\frac{ig_1g_2}{4}\int_0^{\tau_1}d\tau_1'\int_0^{\tau_2}d\tau_2' D_F(z_1'-z_2')\right\}$$
$$\times\exp\left\{-\frac{ig_1^2}{4}\int_{\tau_1}^{s_1}d\tau_1'\int_0^{\tau_1}d\tau_1'' D_F(z_1'-z_1'')-\frac{ig_2^2}{4}\int_{\tau_2}^{s_2}d\tau_2'\int_0^{\tau_2}d\tau_2'' D_F(z_2'-z_2'')\right.$$
$$\left.-\frac{ig_1g_2}{4}\int_0^{\tau_1}d\tau_1'\int_{\tau_2}^{s_2}d\tau_2' D_F(z_1'-z_2')\right\}. \tag{2.11}$$

If we replace the last exponential with 1, we obtain immediately the Bethe-Salpeter equation

$$G_4(x_1,x_2,y_1,y_2)=G_2(x_1-y_1)G_2(x_2-y_2)-i\int d^4\xi_1\int d^4\xi_2\int d^4\eta_1\int d^4\eta_2 G_2(x_1-\xi_1)$$
$$\times G_2(x_2-\xi_2)I(\xi_1,\xi_2;\eta_1,\eta_2)G_4(\eta_1,\eta_2,y_1,y_2), \tag{2.12}$$

with the *ladder approximation* kernel

$$I(\xi_1,\xi_2,\eta_1,\eta_2)=g_1g_2D_F(\xi_1-\xi_2)\,\delta^4(\xi_1-\eta_1)\,\delta^4(\xi_2-\eta_2). \tag{2.13}$$

On the other hand, had we considered even the second term in the expansion of the last exponential in Eq. (2.11), we would have obtained (see Fig. 1 and Appendix B)

$$\begin{aligned} I(\xi_1,\xi_2,\eta_1,\eta_2)=&g_1g_2D_F(\xi_1-\xi_2)\delta^4(\xi_1-\eta_1)\delta^4(\xi_2-\eta_2)-ig_1^3g_2\int d^4\zeta_1 D_F(\xi_1-\eta_1)G_2(\xi_1-\zeta_1)G_2(\zeta_1-\eta_1)D_F(\zeta_1-\eta_2)\\ &\times\delta^4(\xi_2-\eta_2)-ig_1g_2^3\int d\zeta_2\delta^4(\xi_1-\eta_1)D_F(\xi_2-\eta_2)G_2(\xi_2-\zeta_2)G_2(\zeta_2-\eta_2)D_F(\eta_1-\zeta_2)\\ &-ig_1^2g_2^2D_F(\xi_1-\eta_2)G_2(\xi_1-\eta_1)D_F(\xi_2-\eta_1)G_2(\xi_2-\eta_2)+\cdots. \end{aligned} \tag{2.14}$$

Finally one can go beyond the quenched approximation and take into account additional terms in Eq. (2.2). After eliminating tadpole terms this would amount to replacing Eq. (2.8) with

$$\begin{aligned} \left\langle \exp\left\{\frac{ig_1}{2}\int_0^{s_1}d\tau_1A(z_1)+\frac{ig_2}{2}\int_0^{s_2}d\tau_2A(z_2)\right\}\right\rangle =&\exp\left\{\sum_{j=1,2}-\frac{ig_j^2}{4}\int_0^{s_j}d\tau_j\int_0^{\tau_j}d\tau_j'\left[D_F(z_j-z_j')+\sum_{i=1,2}\frac{g_i^2}{2}\int d^4\xi\int d^4\eta\right.\right.\\ &\left.\times D_F(z_j'-\xi)\times[\Delta_F^{(i)}(\xi-\eta)]^2D_F(\eta-z_j')+\cdots\right]-\frac{ig_1g_2}{4}\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2\\ &\times\left[D_F(z_1-z_2)+\sum_{i=1,2}\frac{g_i^2}{2}\int d^4\xi\int d^4\eta D_F(z_1-\xi)[\Delta_F^{(i)}(\xi-\eta)]^2\right.\\ &\left.\left.\times D_F(\eta-z_2)+\cdots\right]\right\} \end{aligned} \tag{2.15}$$

and so to insert $\phi_1\phi_1$ and $\phi_2\phi_2$ loops in all possible ways inside the graph.

Notice that the final form of the kernel we have obtained is expressed as an expansion in the coupling constants g_1 and g_2 as in the ordinary derivation. However, in the method described, once we have written Eq. (2.8) and set the last exponential in Eq. (2.11) equal to 1, Eqs. (2.12) and (2.13) follow exactly. So perturbative expansion appears only at the level of successive corrections and not in the basic approximation. This is the reason for which the method applies even to QCD when we replace Eq. (2.8) with Eq. (1.2). Actually, as we have already mentioned, we shall see that in such a case the kernel I is obtained as an expansion both in α_s and σa^2. Obviously by Fourier transformation of Eq. (2.12) we may pass from this to the more usual momentum counterpart.

III. QUARK-ANTIQUARK PROPAGATOR

Let us consider now the case of QCD. The QCD Lagrangian is

$$\mathcal{L}=\sum_{f=1}^{N_f}\overline{\psi_f}(i\gamma^\mu D_\mu-m_f)\psi_f-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathcal{L}_{\rm GF} \tag{3.1}$$

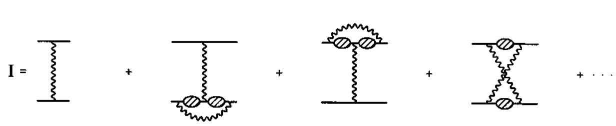

FIG. 1. Graphical representation of Eq. (2.14).

(where $D_\mu=\partial_\mu-igA_\mu$ and $L_{\rm GF}$ is the gauge-fixing term), in which the quarks are fermions and have spin and f is the flavor index.

In this case we find it convenient to work in the second order formalism.

As usual the gauge-invariant quark-antiquark Green function is given by

$$\begin{aligned} G_4^{\rm GI}(x_1,x_2,y_1,y_2)=&\frac{1}{3}\langle 0|T[\psi_2^c(x_2)U(x_2,x_1)\psi_1(x_1)\\ &\times\overline{\psi_1}(y_1)U(y_1,y_2)\overline{\psi_2^c}(y_2)]|0\rangle\\ =&\frac{1}{3}{\rm Tr}\langle U(x_2,x_1)S^{(1)}(x_1,y_1;A)\\ &\times U(y_1,y_2)C^{-1}S^{(2)}(y_2,x_2;A)C\rangle, \end{aligned} \tag{3.2}$$

where c denotes the charge-conjugate fields, C is the charge-conjugation matrix, U the path-ordered gauge string,

$$U(b,a)=P_{ba}\exp\left\{ig\int_a^b dx^\mu A_\mu(x)\right\} \tag{3.3}$$

(the integration path being an arbitrary line joining a to b), $S^{(1)}$ and $S^{(2)}$ the quark propagators in the external gauge field A^μ, and obviously

$$\langle f[A]\rangle=\frac{\int\mathcal{D}[A]M_f(A)f[A]e^{iS[A]}}{\int\mathcal{D}[A]M_f(A)e^{iS[A]}}, \tag{3.4}$$

$S[A]$ being the pure gauge field action and $M_f(A)$ the determinant resulting from the explicit integration on the fermionic fields [in practice, however, $M_f(A)=1$ in the adopted approximation].

The propagators $S^{(1)}$ and $S^{(2)}$ are supposed to be defined by the equation (for simplicity from here on we suppress the particle index j when there is no ambiguity)

$$(i\gamma^\mu D_\mu - m)S(x,y;A)=\delta^4(x-y) \tag{3.5}$$

and the appropriate boundary conditions.

Alternatively by putting

$$S(x,y;A)=(i\gamma^\nu D_\nu + m)\Delta_\sigma(x,y;A), \tag{3.6}$$

we have

$$(D_\mu D^\mu + m^2 - \tfrac{1}{2} g\sigma^{\mu\nu}F_{\mu\nu})\Delta^\sigma(x,y;A)=-\delta^4(x-y) \tag{3.7}$$

($\sigma^{\mu\nu}=(i/2)[\gamma^\mu,\gamma^\nu]$). Then, after replacing Eq. (3.6) in Eq. (3.2), we can check that we can take the differential operator out of the angle brackets by taking into account gauge invariance and we can write

$$G_4^{\rm GI}(x_1,x_2;y_1,y_2)=(i\gamma_1^\mu\overline{\partial_{x_1\mu}}+m_1)(i\gamma_2^\nu\overline{\partial_{x_2\nu}}+m_2)H_4^{\rm GI}(x_1,x_2;y_1,y_2), \tag{3.8}$$

with

$$H_4^{\rm GI}(x_1,x_2;y_1,y_2)=-\frac{1}{3}{\rm Tr}\langle U(x_2,x_1)\Delta_\sigma^{(1)}(x_1,y_1;A)U(y_1,y_2)\widetilde{\Delta}_\sigma^{(2)}(x_2,y_2;-\widetilde{A})\rangle. \tag{3.9}$$

Here the tilde denotes transposition on the color indices alone, while $\overline{\partial_{x_{j\mu}}}$ are derivatives appropriately defined.[2]

Now, notice that for Eq. (3.7) the Feynman-Schwinger representation can be written

$$\begin{aligned}\Delta_\sigma(x,y;A)&=-\frac{i}{2}\int_0^\infty ds P_{xy}T_{xy}\exp\left\{\frac{is}{2}(-D_\mu D^\mu - m^2+\tfrac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu})\right\}\\&=-\frac{i}{2}\int_0^\infty ds\int_y^x \mathcal{D}z P_{xy}T_{xy}\exp\left\{i\int_0^s d\tau\left[-\frac{1}{2}(m^2+\dot z^2)+gA_\rho(z)\dot z^\rho+\frac{g}{4}\sigma^{\mu\nu}F_{\mu\nu}(z)\right]\right\},\end{aligned} \tag{3.10}$$

P_{xy} and T_{xy} prescribing the ordering along the path of the color and of the spin matrices, respectively.

Furthermore, as a consequence of a variation in the path $z^\mu(\tau)\to z^\mu(\tau)+\delta z^\mu(\tau)$ respecting the extreme points, one has [remember that $z\equiv z(\tau)$, $z'\equiv z(\tau')$]

$$\delta\left(P_{xy}\exp\left\{ig\int_0^s d\tau \dot z^\mu A_\mu(z)\right\}\right)=ig\int_0^s \delta S^{\mu\nu}(z)P_{xy}\left(-F_{\mu\nu}(z)\exp\left\{ig\int_0^s d\tau'\dot z^{\mu\prime}A_\mu(z')\right\}\right), \tag{3.11}$$

with $\delta S^{\mu\nu}(z)=\dfrac{1}{2}(dz^\mu\delta z^\nu - dz^\nu\delta z^\mu)$. So one can write

$$T_{xy}\exp\left(-\frac{1}{4}\int_0^s d\tau\sigma^{\mu\nu}\frac{\delta}{\delta S^{\mu\nu}(z)}\right)\left(P_{xy}\exp\left\{ig\int_0^s d\tau'\dot z^{\mu\prime}A_\mu(z')\right\}\right)=T_{xy}P_{xy}\exp\left\{ig\int_0^s d\tau[\dot z^\mu A_\mu(z)+\tfrac{1}{4}\sigma^{\mu\nu}F_{\mu\nu}(z)]\right\}, \tag{3.12}$$

and Eq. (3.10) becomes

$$\Delta_\sigma(x,y;A)=-\frac{i}{2}\int_0^\infty ds\int_y^x\mathcal{D}zP_{xy}T_{xy}\mathcal{S}_0^s\exp\left\{i\int_0^s d\tau[-\tfrac{1}{2}(m^2+\dot z^2)+g\overline{\dot z^\mu}A_\mu(\overline{z})]\right\}, \tag{3.13}$$

with

$$\mathcal{S}_0^s=\exp\left[-\frac{1}{4}\int_0^s d\tau\sigma^{\mu\nu}\frac{\delta}{\delta S^{\mu\nu}(\overline{z})}\right]. \tag{3.14}$$

[2]We notice that, given a functional $\Phi[\gamma_{ab}]$ of the curve γ_{ab} with ends a and b, under general regularity condition, the variation of Φ consequent to an infinitesimal modification of the curve $\gamma\to\gamma+\delta\gamma$ can be expressed as the sum of various terms proportional respectively to δa, δb, and the single elements $\delta S_{\rho\sigma}(x)$ of the surface swept by the curve. Then, the derivatives ∂_{a^ρ}, ∂_{b^ρ} and $\delta/\delta S^{\rho\sigma}(x)$ are defined by the equation $\delta\Phi=\delta a^\rho\partial_{a^\rho}\Phi+\delta b^\rho\partial_{b^\rho}\Phi+\int_\gamma\delta S^{\rho\sigma}(x)\,\delta\Phi/\delta S^{\rho\sigma}(x)$. In our case this would amount to putting naively $\partial_a{}^\rho Pf[\int_a^b dx^\mu A_\mu(x)]=-Pf'[\int_a^b dx^\mu A_\mu(x)]A_\rho(a)$ and $\partial_{b^\rho}\int_a^b dx^\mu A_\mu(x)=A_\rho(b)Pf'[\int_a^b dx^\mu A_\mu(x)]$.

In Eq. (3.13) it is understood that $\overline{z^\mu}(\tau)$ has to be put equal to $z^\mu(\tau)$ after the action of $\mathcal{S}_0^s$. Alternatively, it is convenient to write $\overline{z}=z+\zeta$, to assume that $\mathcal{S}_0^s$ acts on $\zeta(\tau)$ with $\delta S^{\mu\nu}(z)=\frac{1}{2}(dz^\mu\delta\zeta^\nu-dz^\nu\delta\zeta^\mu)$, and to set eventually $\zeta=0$.

Replacing Eq. (3.13) in Eq. (3.9) we obtain

$$H_4^{\rm GI}(x_1,x_2;y_1,y_2)=(\tfrac{1}{2})^2\int_0^\infty ds_1\int_0^\infty ds_2\int_{y_1}^{x_1}\mathcal{D}z_1\int_{y_2}^{x_2}\mathcal{D}z_2T_{x_1y_1}T_{x_2y_2}\mathcal{S}_0^{s_1}\mathcal{S}_0^{s_2}\exp\left\{-\frac{i}{2}\int_0^{s_1}d\tau_1(m_1^2+\dot z_1^2)-\frac{i}{2}\int_0^{s_2}d\tau_2(m_2^2+\dot z_2^2)\right\}$$
$$\times\frac{1}{3}\left\langle {\rm Tr}P_\Gamma\exp\left\{ig\oint_\Gamma d\overline{z^\mu}A_\mu(\overline{z})\right\}\right\rangle, \tag{3.15}$$

where now Γ is the closed loop made by the quark world line Γ_1, the antiquark world line Γ_2 followed in the reverse direction, and the two lines x_1x_2 and y_2y_1; $\overline{z}$ is set equal to $\overline{z_j}=z_j+\zeta_j$ on Γ_j or simply equal to z on x_1x_2 and y_2y_1; again the final limit $\zeta_j\to0$ is understood.

Apart from the spin complications (3.15) is analogous to Eq. (2.7) and the Wilson loop integral appearing there is analogous to the right-hand side of Eq. (2.8). For this reason before going ahead we need to elaborate further Eq. (1.2). In QCD such an equation plays the same role as Eq. (2.8) in the scalar model.

IV. WILSON LOOP INTEGRAL

Let us come back to Eq. (1.2). To make real sense, this equation has to be understood in the quenched approximation $[M_f(A)=1]$; furthermore, the perturbative term has to be kept at the lowest order. Otherwise, perturbative and nonperturbative contributions should be untangled and a simple additivity assumption would be no longer correct.

So for the first term in Eq. (1.2) we can write

$$i(\ln W)_{\rm pert}=i\ln\left\langle\frac{1}{3}{\rm Tr}P\exp\left\{ig\oint_\Gamma dz^\mu A_\mu(z)\right\}\right\rangle_{\rm pert}=\frac{4}{3}g^2\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2D_{\mu\nu}(z_1-z_2)\dot z_1^\mu\dot z_2^\nu-\frac{2}{3}g^2\int_0^{s_1}d\tau_1\int_0^{s_1}d\tau_1'D_{\mu\nu}(z_1-z_1')$$
$$\times\dot z_1^\mu\dot z_1'^\nu-\frac{2}{3}g^2\int_0^{s_2}d\tau_2\int_0^{s_2}d\tau_2'D_{\mu\nu}(z_2-z_2')\dot z_2^\mu\dot z_2'^\nu+\cdots, \tag{4.1}$$

while for the second term in general we should write

$$S_{\rm min}=\int_{t_i}^{t_f}dt\int_0^1 d\lambda\left[-\left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial t}\right)\left(\frac{\partial u^\nu}{\partial\lambda}\frac{\partial u_\nu}{\partial\lambda}\right)+\left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial\lambda}\right)^2\right]^{1/2}, \tag{4.2}$$

$x^\mu=u^\mu(\lambda,t)$ being the equation of the minimal surface with contour Γ.

Let us assume that for fixed t we have

$$u^\mu(1,t)=z_1^\mu(\tau_1(t)),\quad u^\mu(0,t)=z_2^\mu(\tau_2(t)). \tag{4.3}$$

Since Eq. (4.2) is invariant under reparametrization, *a priori* the parameter t could be everything. However, if Γ_1 and Γ_2 never go backwards in time, t can be chosen as the ordinary time, $u^0(s,t)\equiv t$. Then $\tau_1(t)$ and $\tau_2(t)$ are specified by the equation

$$z_1^0(\tau_1)=z_2^0(\tau_2)=t \tag{4.4}$$

and we shall set

$$L=\int_0^1 d\lambda\left[-\left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial t}\right)\left(\frac{\partial u^\nu}{\partial\lambda}\frac{\partial u_\nu}{\partial\lambda}\right)+\left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial\lambda}\right)^2\right]^{1/2}. \tag{4.5}$$

Obviously L cannot depend on the extremal points $z_1(\tau_1)$ and $z_2(\tau_2)$ alone, but it has to depend even on the shape of the world lines, at least in a neighborhood of such points. So we shall write[3] in general $L=L(z_1,z_2,\dot z_1,\dot z_2,\Gamma)$ and

$$S_{\rm min}=\int_{y_1^0}^{x_1^0}dz_1^0\int_{y_2^0}^{x_2^0}dz_2^0\,\delta(z_1^0-z_2^0)L(z_1,z_2,\dot z_1,\dot z_2,\Gamma)$$
$$=\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2\,\delta(z_1^0-z_2^0)\dot z_1^0\dot z_2^0L(z_1,z_2,\dot z_1,\dot z_2,\Gamma). \tag{4.6}$$

In principle this expression can be considered a good approximation even if the world lines contain pieces going backwards in time. In fact, in such a case, if we fix, e.g., τ_1, Eq. (4.4) has more than one solution in τ_2 and, if Γ_1 and Γ_2 are not too much irregular in space (otherwise $S_{\rm min}$ is large and the weight of the loop is small), the minimal surface can be reconstructed as the algebraic sum of various pieces of surface.

[3]Reparametrization invariance and cumulant expansion [16] suggest the form $L=L_1(z_1,z_2,\dot z_1,\dot z_2)+\int d\tau_1'\int d\tau_2'L_2(z_1,z_2,\dot z_1,\dot z_2;z_1',z_2',\dot z_1',\dot z_2')+\cdots$ with L_1, $L_2,\ldots$ homogeneous functions of degree 0 in $\dot z_1,\dot z_2$ and of degree 1 in $\dot z_1',\dot z_2',\dot z_1'',\dot z_2'',\ldots$.

In practice, to have an explicit expression for L, we shall adopt the straight line approximation consisting of replacing $S_{\min}$ with the surface spanned by the straight lines connecting two equal points on Γ_1 and Γ_2. This amounts to taking in Eqs. (4.2) and (4.5),

$$u^0(\lambda,t)=t, \quad u^k(\lambda,t)=\lambda z_1^k(\tau_1(t))+(1-\lambda)z_2^k(\tau_2(t)). \tag{4.7}$$

Then we have

$$\dot{z}_1^0\dot{z}_2^0 L=|\mathbf{z}_1-\mathbf{z}_2|\int_0^1 d\lambda\{\dot{z}_{10}^2\dot{z}_{20}^2 -[\lambda\dot{\mathbf{z}}_{1T}\dot{z}_{20}+(1-\lambda)\dot{\mathbf{z}}_{2T}\dot{z}_{10}]^2\}^{1/2}, \tag{4.8}$$

where T denotes the transverse component $\dot{z}_{j\,\mathrm{T}}^h=(\delta^{hk}-\hat{r}^h\hat{r}^k)\dot{z}_j^h$ with $\mathbf{r}=\mathbf{z}_1-\mathbf{z}_2$ and $\hat{r}^h=r^h/r$. Notice that Eqs. (4.7) and (4.8) are exact equations in two limiting situations, the case in which the two world lines lie on a plane and the case in which they are trajectories of the double helycoid type. In the semirelativistic limit they are also exact up to second order terms in the ordinary velocities [10]. So the approximation seems to be a sensible one.

Notice also that in Eq. (4.1) we have neglected the contribution coming from the two strings x_1x_2 and y_1y_2, while in writing Eq. (4.6) we have not taken into account two border contributions to $S_{\min}$ corresponding to $x_1^0\neq x_2^0$ and $y_1^0\neq y_2^0$. This is correct for $x_1^0-y_1^0$ and $x_2^0-y_2^0$ large with respect to $|\mathbf{x}_1-\mathbf{x}_2|$, $|\mathbf{y}_1-\mathbf{y}_2|$, $x_1^0-x_2^0$, and $y_1^0-y_2^0$. So, strictly, we are going to obtain a BS equation for a quantity H_4 which coincides with H_4^{GI} only in the above limit. This is immaterial for what concerns bound states or asymptotic states but actually H_4 is no longer a gauge-invariant quantity.

Obviously Eq. (4.8) is not covariant, and so we assume it in the center of mass frame. Finally we stress that such an equation shall be imposed only *after* the application of the operators $\mathcal{S}_0^{s_1}$ and $\mathcal{S}_0^{s_2}$.

V. BETHE-SALPETER EQUATION IN QCD

Substituting Eqs. (4.1) and (4.6) into Eq. (3.15) we obtain

$$\begin{aligned} H_4(x_1,x_2;y_1,y_2)=&(\tfrac{1}{2})^2\int_0^\infty ds_1\int_0^\infty ds_2\int_{y_1}^{x_1}\mathcal{D}z_1\int_{y_2}^{x_2}\mathcal{D}z_2 T_{x_1y_1}T_{x_2y_2}\mathcal{S}_0^{s_1}\mathcal{S}_0^{s_2}\exp\bigg\{-\frac{i}{2}\int_0^{s_1}d\tau_1(m_1^2+\dot{z}_1^2)-\frac{i}{2}\int_0^{s_2}d\tau_2(m_2^2+\dot{z}_2^2)\\ &+\frac{2}{3}ig^2\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_1' D_{\mu\nu}(\overline{z_1}-\overline{z_1'})\dot{\bar{z}}_1'^{\,\nu}\dot{\bar{z}}_1^{\nu\prime}+\frac{2}{3}ig^2\int_0^{s_2}d\tau_2\int_0^{s_2}d\tau_2' D_{\mu\nu}(\overline{z_2}-\overline{z_2'})\dot{\bar{z}}_2^{\mu}\dot{\bar{z}}_2^{\nu\prime}\\ &-i\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2 E(\overline{z_1},\overline{z_2},\dot{\bar{z}}_1,\dot{\bar{z}}_2,\Gamma)\bigg\}, \end{aligned} \tag{5.1}$$

where we have set

$$E(z_1,z_2,\dot{z}_1,\dot{z}_2,\Gamma)=\frac{4}{3}g^2D_{\mu\nu}(z_1-z_2)\dot{z}_1^\mu\dot{z}_2^\nu+\sigma\delta(z_{10}-z_{20})\dot{z}_{10}\dot{z}_{20}L(z_1,z_2,\dot{z}_1,\dot{z}_2,\Gamma). \tag{5.2}$$

Equation (5.1) is analogous to Eq. (2.9); however, the occurrence of $\dot{z}_1$ and $\dot{z}_2$ in the interaction term requires a slight modification of the method used in Sec. II and specifically a change from the configurational path integral representation to the phase space one. This can be achieved by performing a Legendre transformation on the quantity occurring in curly bracket (without the factor i). Denoting such quantity by Φ and introducing the momenta $p_{\mu j}=-\delta\Phi/\delta\dot{z}_j^\mu$ we have

$$p_{\mu 1}=\dot{z}_{\mu 1}-\frac{4}{3}g^2\int_0^{s_1}d\tau_1' D_{\mu\nu}(\overline{z_1}-\overline{z_1'})\dot{\bar{z}}_1'^{\,\nu}+\int_0^{s_2}d\tau_2'\frac{\partial E(\overline{z_1},\overline{z_2'},\dot{\bar{z}}_1,\dot{\bar{z}}_2',\Gamma)}{\partial\dot{z}_1^\mu}+\cdots,$$

$$p_{\mu 2}=\dot{z}_{\mu 2}-\frac{4}{3}g^2\int_0^{s_2}d\tau_2' D_{\mu\nu}(\overline{z_2}-\overline{z_2'})\dot{\bar{z}}_2'^{\,\nu}+\int_0^{s_1}d\tau_1'\frac{\partial E(\overline{z_1'},\overline{z_2},\dot{\bar{z}}_1',\dot{\bar{z}}_2,\Gamma)}{\partial\dot{z}_2^\mu}+\cdots, \tag{5.3}$$

where the dots stand for possible additional terms due to the explicit dependence on Γ (cf. footnote 1). Equation (5.3) cannot be inverted in a closed form with respect to $\dot{z}_1$ and $\dot{z}_2$; however, we can do this by an expansion in $\alpha_s = g^2/4\pi$ and σa^2. At the lowest order we have

$$\dot{z}_1^\mu = p_1^\mu + \frac{4}{3}g^2 \int_0^{s_1} d\tau_1' D_{\mu\nu}(\overline{z_1} - z_1')\overline{p_1'}^\nu - \int_0^{s_2} d\tau_2' \frac{\partial E(\overline{z_1}, \overline{z_2'}, \overline{p_1}, \overline{p_2'}, \Gamma)}{\partial p_1^\mu} + \cdots,$$

$$\dot{z}_2^\mu = p_2^\mu + \frac{4}{3}g^2 \int_0^{s_2} d\tau_2' D_{\mu\nu}(\overline{z_2} - \overline{z_2'})\overline{p_2'}^\nu - \int_0^{s_1} d\tau_1' \frac{\partial E(\overline{z_{e1}'}, \overline{z_2}, \overline{p_1'}, \overline{p_2}, \Gamma)}{\partial p_2^\mu} + \cdots, \tag{5.4}$$

with

$$\overline{p_j^\mu} = p_j^\mu + \dot{\zeta}_j^\mu . \tag{5.5}$$

In conclusion we find (up to a determinantal factor that in this approximation can be set equal to 1)

$$H_4(x_1,x_2,y_1,y_2) = (\tfrac{1}{2})^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{y_1}^{x_1} \mathcal{D}z_1 \mathcal{D}p_1 \int_{y_2}^{x_2} \mathcal{D}z_2 \mathcal{D}p_2 T_{x_1y_1} T_{x_2y_2} \mathcal{S}_0^{s_1} \mathcal{S}_0^{s_2}$$
$$\times \exp\left\{ i\int_0^{s_1} d\tau_1 K_1 + i\int_0^{s_2} d\tau_2 K_2 - i\int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(\overline{z_1}, \overline{z_2}, \overline{p_1}, \overline{p_2}, \Gamma) + \cdots \right\}, \tag{5.6}$$

where

$$K_j = -p_j \cdot \dot{z}_j + \frac{1}{2}(p_j^2 - m_j^2) + \frac{2}{3}g^2 \int_0^{s_j} d\tau_j' D_{\mu\nu}(\overline{z_j} - \overline{z_j'})\overline{p_j^\mu}\,\overline{p_j^\nu}' + \cdots \tag{5.7}$$

includes the self-interaction term. Notice that here in $\mathcal{S}_0^{s_j}$ it must be understood that $\delta S^{\mu\nu}(z_j) = \frac{1}{2}\, d\tau_j (p_j^\mu \delta\zeta_j^\nu - p_j^\nu \delta\zeta_j^\mu) + \cdots$.

Now, using the identity

$$\exp\left\{ \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(\overline{z_1}, \overline{z_2}, \overline{p_1}, \overline{p_2}, \Gamma) \right\} = 1 + \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(\overline{z_1}, \overline{z_2}, \overline{p_1}, \overline{p_2}, \Gamma) \exp\left\{ \int_0^{\tau_1} d\tau_1' \int_0^{s_2} d\tau_2' E(\overline{z_1'}, \overline{z_2'}, \overline{p_1'}, \overline{p_2'}, \Gamma) \right\}, \tag{5.8}$$

corresponding to Eq. (2.10), we have

$$H_4(x_1,x_2;y_1,y_2) = (\tfrac{1}{2})^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{y_1}^{x_1} \mathcal{D}z_1 \mathcal{D}p_1 \int_{y_2}^{x_2} \mathcal{D}z_2 \mathcal{D}p_2 T_{x_1y_1} T_{x_2y_2} \mathcal{S}_0^{s_1} \mathcal{S}_0^{s_2} \left\{ \exp\left[i\int_0^{s_1} d\tau_1 K_1 + i\int_0^{s_2} d\tau_2 K_2 \right] \right.$$
$$- i\int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(\overline{z_1}, \overline{z_2}, \overline{p_1}, \overline{p_2}, \Gamma)$$
$$\left. \times \exp\left[i\int_0^{s_1} d\tau_1 K_1 + i\int_0^{s_2} d\tau_2 K_2 - i\int_0^{\tau_1} d\tau_1' \int_0^{s_2} d\tau_2' E(\overline{z_1'}, \overline{z_2'}, \overline{p_1'}, \overline{p_2'}, \Gamma) \right] \right\}. \tag{5.9}$$

To obtain from this an equation analogous to Eq. (2.11) we need to commute $\mathcal{S}_0^{s_1}\mathcal{S}_0^{s_2}$ with E. To this aim, bearing in mind Eqs. (5.2) and (5.7) and going back to the original form (4.5) for L, we find first (see Appendix C)

$$\frac{\delta}{\delta S^{\mu\nu}(z_1)} \int_0^{s_1} d\tau_1' \int_0^{s_2} d\tau_2' E(z_1', z_2', p_1', p_2', \Gamma) = \int_0^{s_2} d\tau_2' \left[\frac{4}{3} g^2 [\partial_\nu D_{\mu\sigma}(z_1 - z_2') - \partial_\mu D_{\nu\sigma}(z_1 - z_2')] p_2^\sigma + \sigma\delta(z_{10} - z_{20}') \right.$$
$$\left. \times \frac{p_{1\nu}(z_{1\mu} - z_{2\mu}') - p_{1\mu}(z_{1\nu} - z_{2\nu}')}{\sqrt{(p_{10}^2 - \mathbf{p}_1^2)(\mathbf{z}_1 - \mathbf{z}_2')^2 + [\mathbf{p}_1 \cdot (\mathbf{z}_1 - \mathbf{z}_2')]^2}} + \cdots \right] \tag{5.10}$$

and a similar result, with a minus sign in front, for the derivative $\delta/\delta S^{\mu\nu}(z_2)$. Furthermore,

$$\frac{\delta^2}{\delta S^{\mu\nu}(z_1)\,\delta S^{\rho\sigma}(z_1')}\int_0^{s_1}d\tau_1''\int_0^{s_2}d\tau_2''E=\frac{\delta^2}{\delta S^{\mu\nu}(z_2)\,\delta S^{\rho\sigma}(z_2')}\int_0^{s_1}d\tau_1''\int_0^{s_2}d\tau_2''E=0, \tag{5.11}$$

but

$$\frac{\delta^2}{\delta S^{\mu_1\nu_1}(z_1)\,\delta S^{\mu_2\nu_2}(z_2)}\int_0^{s_1}d\tau_1''\int_0^{s_2}d\tau_2''E=\frac{4}{3}g^2(\delta^\rho_{\mu_1}\partial_{\nu_1}-\delta^\rho_{\nu_1}\partial_{\mu_1})(\delta^\sigma_{\mu_2}\partial_{\nu_2}-\delta^\sigma_{\nu_2}\partial_{\mu_2})D_{\rho\sigma}(z_1-z_2). \tag{5.12}$$

Then, taking into account the relation

$$e^A B e^{-A}=\sum_{n=0}^{\infty}\frac{1}{n!}[A,[A,\ldots[A,B]\ldots]], \tag{5.13}$$

we have

$$\begin{aligned}
&\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2\mathcal{S}^{\tau_1+\varepsilon}_{\tau_1-\varepsilon}\mathcal{S}^{\tau_2+\varepsilon}_{\tau_2-\varepsilon}E(\overline{z_1},\overline{z_2},\overline{p_1},\overline{p_2},\Gamma)(\mathcal{S}^{\tau_1+\varepsilon}_{\tau_1-\varepsilon}\mathcal{S}^{\tau_2+\varepsilon}_{\tau_2-\varepsilon})^{-1}\\
&\quad=\left(1-\frac{1}{4}\int_0^{s_1}d\tau_1'\sigma_1^{\mu_1\nu_1}\frac{\delta}{\delta S^{\mu_1\nu_1}(\overline{z'_1})}\right)\left(1-\frac{1}{4}\int_0^{s_2}d\tau_2'\sigma_2^{\mu_2\nu_2}\frac{\delta}{\delta S^{\mu_2\nu_2}(\overline{z'_2})}\right)\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2E(\overline{z_1},\overline{z_2},\overline{p_1},\overline{p_2},\Gamma)\\
&\quad=R(z_1,z_2,p_1,p_2),
\end{aligned} \tag{5.14}$$

with

$$R=R_{\rm pert}+R_{\rm conf}, \tag{5.15}$$

$$\begin{aligned}
R_{\rm pert}=&-\frac{4}{3}g^2\Bigg\{D_{\rho\sigma}(z_1-z_2)p_1^\rho p_2^\sigma-\frac{1}{4}\sigma_1^{\mu\nu}(\delta^\rho_\mu\partial_{1\nu}-\delta^\rho_\nu\partial_{1\mu})D_{\rho\sigma}(z_1-z_2)p_2^\sigma-\frac{1}{4}\sigma_2^{\mu\nu}(\delta^\sigma_\mu\partial_{2\nu}-\delta^\sigma_\nu\partial_{2\mu})D_{\rho\sigma}(z_1-z_2)p_1^\rho\\
&+\frac{1}{16}\sigma_1^{\mu_1\nu_1}\sigma_2^{\mu_2\nu_2}(\delta^\rho_{\mu_1}\partial_{1\nu_1}-\delta^\rho_{\nu_1}\partial_{1\mu_1})(\delta^\sigma_{\mu_2}\partial_{2\nu_2}-\delta^\sigma_{\nu_2}\partial_{2\mu_2})D_{\rho\sigma}(z_1-z_2)\Bigg\}
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
R_{\rm conf}=&\,\sigma\delta(z_{10}-z_{20})\Bigg\{\left|\mathbf{z}_1-\mathbf{z}_2\right|\int_0^1 d\lambda\sqrt{p_{10}^2p_{20}^2-[\lambda\mathbf{p}_{1T}p_{20}+(1-\lambda)\mathbf{p}_{2T}p_{10}]^2}-\frac{1}{4}p_{20}\sigma_1^{\mu\nu}\frac{p_{1\nu}(z_{1\mu}-z_{2\mu})-p_{1\mu}(z_{1\nu}-z_{2\nu})}{|\mathbf{z}_1-\mathbf{z}_2|\sqrt{p_{10}^2-\mathbf{p}_{1\mathrm{T}}^2}}\\
&+\frac{1}{4}p_{10}\sigma_2^{\mu\nu}\frac{p_{2\nu}(z_{1\mu}-z_{2\mu})-p_{2\mu}(z_{1\nu}-z_{2\nu})}{|\mathbf{z}_1-\mathbf{z}_2|\sqrt{p_{20}^2-\mathbf{p}_{2\mathrm{T}}^2}}\Bigg\}.
\end{aligned} \tag{5.17}$$

Notice that in Eq. (5.14) we have eventually suppressed reference to the loop Γ and this amounts to adopting the straight line approximation.

Finally setting

$$H_2(x-y)=-\frac{i}{2}\int_0^\infty ds\int_y^x\mathcal{D}z\mathcal{D}p\,T_{xy}\mathcal{S}_0^s\exp\left\{i\int_0^s d\tau K\right\}, \tag{5.18}$$

we can write Eq. (5.9) as

$$\begin{aligned}
H_4(x_1,x_2;y_1,y_2)=&\,H_2^{(1)}(x_1-y_1)H_2^{(2)}(x_2-y_2)\\
&-\frac{i}{4}\int_0^\infty ds_1\int_0^\infty ds_2\int_{y_1}^{x_1}\mathcal{D}z_1\mathcal{D}p_1\int_{y_2}^{x_2}\mathcal{D}z_2\mathcal{D}p_2T_{x_1y_1}T_{x_2y_2}\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2R(z_1,z_2,p_1,p_2)\mathcal{S}_0^{s_1}\mathcal{S}_0^{s_2}\\
&\times\exp\left\{i\int_0^{s_1}d\tau_1'K_1'+i\int_0^{s_2}d\tau_2'K_2'-i\int_0^{\tau_1}d\tau_1'\int_0^{s_2}d\tau_2'E(z_1',z_2',p_1,p_2,\Gamma)\right\}.
\end{aligned} \tag{5.19}$$

At this point, to go ahead it is necessary to take explicitly into account the discrete form of Eq. (5.19). If we set

$$P\exp\left\{ig\oint_{\Gamma}dz^{\mu}A_{\mu}(z)\right\}=P\prod_{\Gamma}U(z_n,z_{n-1})=P\exp\left\{ig\sum_{\Gamma}(z_n^{\mu}-z_{n-1}^{\mu})A_{\mu}\left(\frac{z_n+z_{n-1}}{2}\right)\right\} \tag{5.20}$$

by manipulation in part similar to those performed on Eq. (2.11), we obtain the following the Bethe-Salpeter equation for H_4 (see Appendix D for details and Appendix A for illustration of a potential model)

$$H_4(x_1,x_2;y_1,y_2)=H_2^{(1)}(x_1-y_1)H_2^{(2)}(x_2-y_2)-i\int d^4\xi_1 d^4\xi_2 d^4\eta_1 d^4\eta_2 H_2^{(1)}(x_1-\xi_1)H_2^{(2)}(x_2-\xi_2)I(\xi_1,\xi_2;\eta_1,\eta_2)$$
$$\times H_4(\eta_1,\eta_2;y_1,y_2), \tag{5.21}$$

with

$$I(\xi_1,\xi_2,\eta_1,\eta_2)=-4\int\frac{d^4k_1d^4k_2}{(2\pi)^8}R\left(\frac{\xi_1+\eta_1}{2},\frac{\xi_2+\eta_2}{2},k_1,k_2\right)\exp\{-i[(\xi_1-\eta_1)k_1+(\xi_2-\eta_2)k_2]\}. \tag{5.22}$$

In conclusion, in momentum space, after taking the Fourier transform,

$$(2\pi)^4\delta(p_1+p_2-p_1'-p_2')\hat{I}(p_1,p_2;p_1',p_2')=-4\int d^4\xi_1 d^4\xi_2\int d^4\eta_1 d^4\eta_2\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}e^{-i(p_1-k_1)\xi_1-i(p_2-k_2)\xi_2}$$
$$\times R\left(\frac{\xi_1+\eta_1}{2},\frac{\xi_2+\eta_2}{2},k_1,k_2\right)e^{i(p_1'-k_1)\eta_1+i(p_2'-k_2)\eta_2}, \tag{5.23}$$

we obtain at the lowest order in α_s and in σa^2 the kernel given in Eqs. (1.3)–(1.7).

VI. HOMOGENEOUS EQUATION AND EFFECTIVE HAMILTONIAN

Let us now consider the usual decomposition of the $\hat{H}_4$ in a single bound state contribution $[(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\hat{H}_4$ denotes the Fourier trasform of $H_4]$:

$$\hat{H}_4(k,k',P)=\sum_B\frac{\Phi_B(k)\overline{\Phi}_B(k')}{P^2-m_B^2}+\text{regular terms}, \tag{6.1}$$

where we have explicitly introduced the total momentum $P=p_1+p_2$ and the relative momenta $k=\eta_2p_1-\eta_1p_2$ and $k'=\eta_2p_1'-\eta_1p_2'$ with $\eta_1=m_1/(m_1+m_2)$ and $\eta_2=m_2/(m_1+m_2)$ [$\Phi_B(k)$ being a kind of second order BS wave function]. Replacing Eq. (6.1) in the momentum space inhomogeneous BS equation corresponding to Eq. (5.21) and taking the limit $P^2\to m_B^2$ in the usual way, we obtain the homogeneous equation

$$\Phi_B(k)=-i\int\frac{d^4k'}{(2\pi)^4}\hat{H}_2^{(1)}(\eta_1P_B+k)\hat{H}_2^{(2)}(\eta_2P_B-k)$$
$$\times\hat{I}(k,k';P_B)\Phi_B(k'), \tag{6.2}$$

which is more appropriate for the bound state problem. Notice that $P_B=(m_B,0)$ and the center of mass frame are explicitly understood.

From Eq. (6.2) in the so-called instantaneous approximation we can also have an effective square mass operator or an effective Hamiltonian. The instantaneous approximation consists in replacing in Eq. (6.2) $\hat{H}_2^{(j)}(p)$ with the free propagator $i/(p^2-m^2)$ and the kernel $\hat{I}(k,k';P)$ with $\hat{I}_{\text{inst}}(\mathbf{k},\mathbf{k}')$ obtained from $\hat{I}(k,k',P)$ by setting $k_0=k_0'=\eta_2(w_1+w_1')/2-\eta_1(w_2+w_2')/2$ with $w_j=\sqrt{m_j^2+\mathbf{k}^2}$ and $w_j'=\sqrt{m_j^2+\mathbf{k}^{2\prime}}$. Then we put

$$\varphi_P(\mathbf{k})=\sqrt{\frac{2w_1(\mathbf{k})w_2(\mathbf{k})}{w_1(\mathbf{k})+w_2(\mathbf{k})}}\int_{-\infty}^{\infty}dk_0\Phi_P(k) \tag{6.3}$$

and integrate over k_0 and k_0', using

$$\int dk_0\frac{1}{(k_0+\eta_1m_B)^2-\mathbf{k}^2-m_1^2+i\varepsilon}$$
$$\times\frac{1}{(-k_0+\eta_2m_B)^2-\mathbf{k}^2-m_2^2+i\varepsilon}$$
$$=-\pi i\frac{w_1+w_2}{w_1w_2}\frac{1}{m_B^2-(w_1+w_2)^2}. \tag{6.4}$$

We obtain

$$[w_1(\mathbf{k})+w_2(\mathbf{k})]^2\varphi_{m_B}(\mathbf{k})$$
$$+\int\frac{d^3k'}{(2\pi)^3}\sqrt{\frac{w_1(\mathbf{k})+w_2(\mathbf{k})}{2w_1(\mathbf{k})w_2(\mathbf{k})}}\hat{I}_{\text{inst}}(\mathbf{k},\mathbf{k}')$$
$$\times\sqrt{\frac{w_1(\mathbf{k}')+w_2(\mathbf{k}')}{2w_1(\mathbf{k}')w_2(\mathbf{k}')}}\varphi_{m_B}(\mathbf{k}')$$
$$=m_B^2\varphi_{m_B}(\mathbf{k}), \tag{6.5}$$

which is the eigenvalue equation for the squared mass operator,

$$M^2=M_0^2+U, \tag{6.6}$$

with $M_0=\sqrt{m_1^2+\mathbf{k}^2}+\sqrt{m_2^2+\mathbf{k}^2}$ and

$$\langle\mathbf{k}|U|\mathbf{k}'\rangle=\frac{1}{(2\pi)^3}\sqrt{\frac{w_1+w_2}{2w_1w_2}}\hat{I}_{\rm inst}(\mathbf{k},\mathbf{k}')\sqrt{\frac{w_1'+w_2'}{2w_1'w_2'}}. \tag{6.7}$$

The quadratic form of Eq. (6.6) obviously derives from the second order character of the formalism we have used. It should be mentioned that for light mesons this form seems to be phenomenologically favoured with respect to the linear one.

In more usual terms one can also write

$$M=M_0+V, \tag{6.8}$$

with

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=\frac{1}{w_1+w_2+w_1'+w_2'}\langle\mathbf{k}|U|\mathbf{k}'\rangle+\cdots$$
$$=\frac{1}{(2\pi)^3}\frac{1}{4\sqrt{w_1w_2w_1'w_2'}}\hat{I}_{\rm inst}(\mathbf{k},\mathbf{k}')+\cdots, \tag{6.9}$$

where the ellipses stand for higher order terms in α_s and σa^2 and kinematical factors equal to 1 on the energy shell have been neglected. If in the potential V as given by Eq. (6.9) we neglect the spin-dependent terms, we reobtain the Hamiltonian of the relativistic flux tube model [13] with an appropriate ordering prescription [11,14]. Precisely, working in the Coulomb gauge the resulting potential is

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=-\frac{1}{2\pi^2}\frac{4}{3}\alpha_s\left\{\frac{1}{(\mathbf{k}'-\mathbf{k})^2}+\frac{1}{q_{10}q_{20}(\mathbf{k}'-\mathbf{k})^2}\left[\mathbf{q}^2+\frac{[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{q}]^2}{(\mathbf{k}'-\mathbf{k})^2}\right]\right\}$$
$$+\frac{1}{(2\pi)^3}\int d^3\mathbf{r}e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}\frac{\sigma r}{2}\frac{1}{q_{10}+q_{20}}\left\{\frac{q_{20}}{q_{10}}\sqrt{q_{10}^2-\mathbf{q}_T^2}+\frac{q_{10}}{q_{20}}\sqrt{q_{20}^2-\mathbf{q}_T^2}+\frac{q_{10}q_{20}}{|\mathbf{q}_T|}\left(\arcsin\frac{|\mathbf{q}_T|}{|q_{10}|}+\arcsin\frac{|\mathbf{q}_T|}{|q_{20}|}\right)\right\}+\cdots, \tag{6.10}$$

with $\mathbf{q}=(\mathbf{k}+\mathbf{k}')/2$, $q_{j0}=[w_j(\mathbf{k})+w_j(\mathbf{k}')]/2$, and where again kinematical factors equal to 1 on shell have been disregarded.

On the contrary, by performing a $1/m^2$ expansion in Eq. (6.9) we find the $q\bar{q}$ potential at $1/m^2$ order,

$$\langle\mathbf{k}|V|\mathbf{k}'\rangle=-\frac{4}{3}\alpha_s\frac{1}{2\pi^2\mathbf{Q}^2}-\frac{\sigma}{\pi^2}\frac{1}{\mathbf{Q}^4}-\frac{4}{3}\frac{\alpha_s}{2\pi^2}\frac{1}{m_1m_2\mathbf{Q}^2}\left[\mathbf{q}^2-\frac{(\mathbf{q}\cdot\mathbf{Q})^2}{\mathbf{Q}^2}\right]-\frac{4}{3}i\alpha_s\left\langle\mathbf{k}\left|\left(\frac{1}{2m_1}\frac{\boldsymbol{\alpha}_1\cdot\mathbf{r}}{r^3}-\frac{1}{2m_2}\frac{\boldsymbol{\alpha}_2\cdot\mathbf{r}}{r^3}\right)\right|\mathbf{k}'\right\rangle$$
$$+\frac{4}{3}\frac{\alpha_s}{2m_1m_2}\varepsilon_{hkl}\frac{k_l+k_l'}{2}(\sigma_1^h+\sigma_2^h)\left\langle\mathbf{k}\left|\frac{r_k}{r^3}\right|\mathbf{k}'\right\rangle+\frac{1}{3}\frac{\alpha_s}{m_1m_2}\left\langle\mathbf{k}\left|\left(3\frac{r^hr^k}{r^5}-\frac{\delta^{hk}}{r^3}\right)\right|\mathbf{k}'\right\rangle\sigma_1^h\sigma_2^k+\frac{4}{3}\frac{\alpha_s}{m_1m_2}\frac{2\pi}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$
$$-\frac{\sigma}{6}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}-\frac{1}{m_1m_2}\right)\langle\mathbf{k}|\mathbf{q}_T^2r|\mathbf{k}'\rangle-\frac{\sigma}{2}\varepsilon_{hkl}\frac{k_l+k_l'}{2}\left(\frac{\sigma_1^h}{m_1^2}+\frac{\sigma_2^h}{m_2^2}\right)\left\langle\mathbf{k}\left|\frac{r_k}{r}\right|\mathbf{k}'\right\rangle$$
$$+\frac{\sigma i}{2}\left\langle\mathbf{k}\left|\left(-\frac{1}{m_1}\frac{\boldsymbol{\alpha}_1\cdot\mathbf{r}}{r}+\frac{1}{m_2}\frac{\boldsymbol{\alpha}_2\cdot\mathbf{r}}{r}\right)\right|\mathbf{k}'\right\rangle+\cdots, \tag{6.11}$$

with $\mathbf{q}=(\mathbf{k}+\mathbf{k}')/2$ and $\mathbf{Q}=\mathbf{k}-\mathbf{k}'$ as usual.

In the coordinate representation we may also write

$$V=-\frac{4}{3}\frac{\alpha_s}{r}+\sigma r+\frac{4}{3}\frac{\alpha_s}{m_1m_2}\left\{\frac{1}{2r}\left(\delta^{hk}+\frac{r^hr^k}{r^2}\right)q^hq^k\right\}_W-\frac{4}{3}i\alpha_s\left(\frac{1}{2m_1}\frac{\boldsymbol{\alpha}_1\cdot\mathbf{r}}{r^3}-\frac{1}{2m_2}\frac{\boldsymbol{\alpha}_2\cdot\mathbf{r}}{r^3}\right)+\frac{4}{3}\frac{\alpha_s}{2m_1m_2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\frac{(\mathbf{r}\times\mathbf{q})}{\tau^3}$$
$$+\frac{1}{3}\frac{\alpha_s}{m_1m_2}\left[\frac{3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})}{r^5}-\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^3}\right]+\frac{4}{3}\frac{\alpha_s}{m_1m_2}\frac{2\pi}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\delta^3(\mathbf{r})-\frac{\sigma}{6}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}-\frac{1}{m_1m_2}\right)\{\mathbf{q}_T^2r\}_W$$
$$-\frac{\sigma}{2}\left(\frac{\boldsymbol{\sigma}_1}{m_1^2}+\frac{\boldsymbol{\sigma}_2}{m_2^2}\right)\cdot\left(\frac{\mathbf{r}}{r}\times\mathbf{q}\right)-\frac{\sigma i}{2}\left[\frac{1}{m_1}\frac{\boldsymbol{\alpha}_1\cdot\mathbf{r}}{r}-\frac{1}{m_2}\frac{\boldsymbol{\alpha}_2\cdot\mathbf{r}}{r}\right], \tag{6.12}$$

where now $\mathbf{q}$ stands for the momentum operator and and the symbol $\{\,\}_W$ stands for the Weyl ordering prescription. Notice the non-Hermitian terms[4] in the Dirac matrices α_1 and α_2. Such terms can be eliminated by performing a Foldy-Wouthuysen transformation with the non-Hermitian generator

$$S=\frac{i}{2m_1}\boldsymbol{\alpha}_1\cdot\mathbf{q}-\frac{i}{2m_2}\boldsymbol{\alpha}_2\cdot\mathbf{q}, \tag{6.13}$$

and we end up with the $1/m^2$ potential

$$V=-\frac{4}{3}\frac{\alpha_s}{r}+\sigma r+\frac{1}{2m_1m_2}\left\{\frac{4}{3}\frac{\alpha_s}{r}(\delta^{hk}+\hat{r}^h\hat{r}^k)p_1^hp_2^k\right\}_W-\sum_{j=1}^{2}\frac{1}{6m_j^2}\{\sigma r\mathbf{p}_{jT}^2\}_W-\frac{1}{6m_1m_2}\{\sigma r\mathbf{p}_{1T}\cdot\mathbf{p}_{2T}\}_W+\frac{1}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\nabla^2$$
$$\times\left(-\frac{4}{3}\frac{\alpha_s}{r}+\sigma r\right)+\frac{1}{2}\left(\frac{4}{3}\frac{\alpha_s}{r^3}-\frac{\sigma}{r}\right)\left[\frac{1}{m_1^2}\mathbf{S}_1\cdot(\mathbf{r}\times\mathbf{p}_1)-\frac{1}{m_2^2}\mathbf{S}_2\cdot(\mathbf{r}\times\mathbf{p}_2)\right]+\frac{1}{m_1m_2}\frac{4}{3}\frac{\alpha_s}{r^3}[\mathbf{S}_2\cdot(\mathbf{r}\times\mathbf{p}_1)-\mathbf{S}_1\cdot(\mathbf{r}\times\mathbf{p}_2)]$$
$$+\frac{1}{m_1m_2}\frac{4}{3}\alpha_s\left\{\frac{1}{r^3}\left[\frac{3}{r^2}(\mathbf{S}_1\cdot\mathbf{r})(\mathbf{S}_2\cdot\mathbf{r})-\mathbf{S}_1\cdot\mathbf{S}_2\right]+\frac{8\pi}{3}\delta^3(\mathbf{r})\mathbf{S}_1\cdot\mathbf{S}_2\right\}, \tag{6.14}$$

which coincides with the semirelativistic potential as obtained in [10,11].

VII. CONCLUSIONS

In conclusion, under the assumptions (1.2) and (4.1), (4.8) for the evaluation of the Wilson loop integral, we have derived the quark-antiquark Bethe-Salpeter (BS) equation from QCD. The assumptions are the same previously used for the derivation of a semirelativistic heavy quark potential and the technique is strictly similar. The kernel is constructed as an expansion in α_s and σa^2 and at the lowest order is given by Eqs. (1.3)–(1.7).

The BS equation that has been obtained is a second order one, analogous in some way to the iterated Dirac equation. Correspondingly, by instantaneous approximation, one can obtain from Eq. (6.2) an effective square mass operator which is given by Eqs. (6.6) and (6.7).

At the lowest order in α_s and σa^2 even a linear mass operator can be written with a potential V given by Eq. (6.9). Then, neglecting the spin-dependent terms in V, the Hamiltonian for the relativistic flux tube model comes out. On the contrary, by a $1/m$ expansion and an appropriate Foldy-Wouthuysen transformation the ordinary semirelativistic potential is reobtained.

In Eq. (5.21) or (6.4) a color-independent dressed quark propagator appears which is defined by Eqs. (5.7) and (5.18). Notice that only the perturbative expansion gives a contribution to this quantity.

A few additional remarks are in order.

First of all, notice that the result does not depend strictly on Eq. (4.8) but on the possibility of writing the interaction term as an integral on the world lines of the quark and the antiquark. Multiple integrations of the same type would be admissible, as occurs for the perturbative contribution or in cumulant expansion [16] (see footnote 1), but dependence of the integrand on higher derivatives in the parameters τ_1 and τ_2 would not enable one to carry on the argument.

A second point concerns the significance of the lowest order BS kernel we have derived. As the analysis in terms of potentials shows, the inclusion of terms in α_s^2 is essential for an understanding of the fine and hyperfine structures. For what concerns the importance of $(\sigma a^2)^2$ contributions an indication can be obtained considering the corresponding terms in the relativistic flux tube model. Neglecting the Coulomb terms, in the equal mass case the center of mass Hamiltonian for such a model at the $(\sigma a^2)^2$ order can be written [14]

$$\begin{aligned}H=&2\sqrt{m^2+\mathbf{q}^2}\\&+\frac{\sigma r}{2}\left[\frac{\sqrt{m^2+\mathbf{q}^2}}{|\mathbf{q}_T|}\arcsin\frac{|\mathbf{q}_T|}{\sqrt{m^2+\mathbf{q}^2}}+\sqrt{\frac{m^2+\mathbf{q}_r^2}{m^2+\mathbf{q}^2}}\right]\\&+\frac{\sigma^2r^2}{16\mathbf{q}_T^2}\frac{m^2+\mathbf{q}_r^2}{\sqrt{m^2+\mathbf{q}^2}}\left[\frac{\sqrt{m^2+\mathbf{q}^2}}{|\mathbf{q}_T|}\arcsin\frac{|\mathbf{q}_T|}{\sqrt{m^2+\mathbf{q}^2}}\right.\\&\left.-\sqrt{\frac{m^2+\mathbf{q}_r^2}{m^2+\mathbf{q}^2}}\right]^2+\cdots,\end{aligned} \tag{7.1}$$

where the appropriate ordering is understood. To better appreciate the relative magnitude of the two potential terms let us consider the case of small $\mathbf{q}_T$ (small angular momentum) in which the above equation becomes simply

$$\begin{aligned}H=&2\sqrt{m^2+\mathbf{q}^2}+\sigma r\left(1-\frac{1}{6}\frac{\mathbf{q}_T^2}{m^2+\mathbf{q}^2}+\cdots\right)\\&+\frac{\sigma^2r^2}{36}\frac{\mathbf{q}_T^2}{(m^2+\mathbf{q}^2)^{3/2}}+\cdots.\end{aligned} \tag{7.2}$$

[4]Such terms become, however, Hermitian with reference to the metric operator $\gamma_1^0\gamma_2^0$.

We notice that in s wave only the σr terms survive. Then, taking into account that $a\sim 1/(\sigma m)^{1/3}$, $q\sim 1/a$, and assuming typically $\sigma=0.17$ GeV^2, $m_u=0.35$ GeV, $m_c=1.7$ GeV, and $m_b=5$ GeV we find that for the p wave the last term in Eq. (7.2) is of the order of the 2%, 0.1%, and 0.006% of the preceding one for the $u\bar{u}$, $c\bar{c}$, and $b\bar{b}$ systems, respectively. This would correspond to contributions tothe mass of the meson of about 8, 0.2, and 0.01 MeV. The inclusion of the Coulomb term would reduce a and improve the result. In the $u\bar{u}$ case, e.g., it would amount to cutting the above contribution by a factor of 2. Therefore only in this last case would the σ^2 term be of any significance.

Finally let us come to the problem of the type of confinement, which has been largely discussed in the literature. By this terminology what is usually meant is the tentative assumption of a BS (first order) confining kernel K of the instantaneous form

$$\hat{K}_{\rm conf}=-(2\pi)^3\Gamma\frac{\sigma}{\pi^2}\frac{1}{\mathbf{Q}^4} \tag{7.3}$$

or even the covariant counterpart of it,

$$\hat{K}_{\rm conf}=-(2\pi)^3\Gamma\frac{\sigma}{\pi^2}\frac{1}{Q^4}, \tag{7.4}$$

where Γ is a combination of Dirac matrices. Typically the cases $\Gamma=1$ (scalar confinement), $\Gamma=\gamma_1^0\gamma_2^0$ (vectorial confinement), or a combination of them have been considered.

Equation (7.4) is immediately ruled out by the fact that, even if formally it corresponds to Eq. (7.3) (by instantaneous approximation), actually, due to the strong infrared singularity, it gives results very different from Eq. (7.3) [17]. As well known, Eq. (7.3) with $\Gamma=1$ was motivated by the fact that it reproduces the static potential σr and the spin-dependent potential as obtained in the Wilson loop context. As we already mentioned, however, this choice gets both into phenomenological and theoretical difficulties: (1) It gives a first order velocity-dependent relativistic correction to the potential which differs from the Wilson loop one [10,11]; (2) yields unstable variational solutions [6]; (3) it does not seem to agree with the heavy meson data [7,8,5,6], and does not reproduce straight line Regge trajectories [5,6]. The choice $\Gamma=\gamma_1^0\gamma_2^0$ does not meet difficulty (2), but it gives both wrong velocity and spin-dependent potentials.

On the contrary, even if we have not yet attempted calculations directly with the kernel established in this paper, very encouraging results have been obtained in the context of the relativistic flux tube model [13], of the dual QCD [18], and of the effective relativistic Hamiltonian [3], formalisms that are all strictly related to ours. Therefore the complicated momentum dependence appearing in Eqs. (1.3)–(1.7) seems essential to understand both the light and the heavy meson phenomenology.

APPENDIX A: ONE-DIMENSIONAL POTENTIAL THEORY

Here we illustrate the method used in the paper for the derivation of the BS equation on a potential-type model. Let us consider the model made by a nonrelativistic particle in one dimension with the Hamiltonian

$$H=\frac{p^2}{2m}+U(x,p)=H_0+U \tag{A1}$$

and the corresponding Schrödinger propagators

$$K(x,y,t)=\langle x|e^{-iHt}|y\rangle,\quad K_0(x,y,t)=\langle x|e^{-iH_0t}|y\rangle. \tag{A2}$$

From the operatorial identity

$$e^{-iHt}=e^{-iH_0t}-i\int_0^t dt'\,e^{-iH_0(t-t')}Ue^{-iHt'}, \tag{A3}$$

we obtain the equation

$$K(x,y,t)=K_0(x,y,t)-i\int_0^t dt'\int d\xi\int d\eta K_0(x,\xi,t-t')\times\langle\xi|U|\eta\rangle K(\eta,y,t'), \tag{A4}$$

which is somewhat analogous to the nonhomogeneous Bethe-Salpeter equation in configuration space.

We want to derive Eq. (1.4) by means of the path-integral formalism.

Let us take

$$U=V(x)+\{W(x)p^2\}_{\rm ord}, \tag{A5}$$

where $\{\,\}_{\rm ord}$ stands for some ordering prescription. In terms of the path integral we can write, in phase space,

$$K(x,y,t)=\int_y^x\mathcal{D}z\mathcal{D}p\exp\left\{i\int_0^t dt'\left[p'\dot{z}'-\frac{p'^2}{2m}-V(z')-W(z')p'^2\right]\right\}, \tag{A6}$$

with $z'=z(t')$, $p'=p(t')$, and $\dot{z}'=dz(t')/dt'$. In Eq. (A.6) the functional ''measures'' are supposed to be defined by

$$\mathcal{D}z=\left(\frac{m}{2\pi i\varepsilon}\right)^{N/2}dz_1\cdots dz_{N-1},$$
$$\mathcal{D}p=\left(\frac{i\varepsilon}{2\pi m}\right)^{N/2}dp_1\cdots dp_{N-1}dp_N, \tag{A7}$$

and again the end points x and y stand for the condition $z_0=y$, $z_N=x$. As is well known (see, e.g., [19]) the ordering prescription is concealed under the particular discretization adopted in the limit procedure implied in the definition of Eq. (A6).

Let us first consider the case $W=0$. In this case there is no ordering problem and it is possible to perform explicitly the p integration in Eq. (A6) obtaining the path-integral representation in configuration space,

$$K(x,y,t)=\int_y^x \mathcal{D}z \exp\left[i\int_0^t dt' \left(m\frac{\dot{z}'^2}{2}-V(z')\right)\right]. \quad \text{(A8)}$$

Then using the identity

$$\exp\left(-i\int_0^t dt' V(z')\right)=1-i\int_0^t dt' V(z') \times\exp\left(-i\int_0^{t'} dt'' V(z'')\right), \quad \text{(A9)}$$

one obtains

$$K(x,y,t)=K_0(x,y,t)-i\int_0^t dt' \int_y^x \mathcal{D}z V(z')\exp\left(i\int_{t'}^t dt'' m\frac{\dot{z}''^2}{2} + i\int_0^{t'} dt'' m\frac{\dot{z}''^2}{2} - i\int_0^{t'} dt'' V(z'')\right), \quad \text{(A10)}$$

which, taking into account that

$$\int_y^x \mathcal{D}z \cdots = \int d\xi \int_\xi^x \mathcal{D}z \int_y^\xi \mathcal{D}z \cdots \quad \text{(A11)}$$

[having identified $z(t')=\xi$], can be rewritten in the form (A4) with $\langle \xi | U | \eta\rangle = V(\xi)\delta(\xi-\eta)$.

In the general case $W(x)\neq 0$, it is convenient to work with the original path-integral representation in phase space, Eq. (A6), and it is necessary to use discretized expressions explicitly. For Weyl ordering in Eq. (A5) the correct discretization is the midpoint one. We can write therefore

$$K(x,y,t)=\frac{1}{(2\pi)^N}\int dp_N dz_{N-1} dp_{N-1}\cdots dz_1 dp_1 \exp\left(i\sum_{n=1}^N \left\{ p_n(z_n-z_{n-1})-\varepsilon\left[\frac{p_n^2}{2m}+V\left(\frac{z_n+z_{n-1}}{2}\right) + W\left(\frac{z_n+z_{n-1}}{2}\right)p_n^2\right]\right\}\right) \quad \text{(A12)}$$

and use the discrete counterpart of Eq. (A9),

$$\exp\left\{-i\varepsilon\sum_{n=1}^N \left[V\left(\frac{z_n+z_{n-1}}{2}\right)+W\left(\frac{z_n+z_{n-1}}{2}\right)p_n^2\right]\right\}=1-i\varepsilon\sum_{R=1}^N\left[V\left(\frac{z_R+z_{R-1}}{2}\right)+W\left(\frac{z_R+z_{R-1}}{2}\right)p_R^2\right] \times\exp\left\{-i\varepsilon\sum_{r=1}^{R-1}\left[V\left(\frac{z_r+z_{r-1}}{2}\right)+W\left(\frac{z_r+z_{r-1}}{2}\right)p_r^2\right]\right\}. \quad \text{(A13)}$$

Then we have

$$\begin{aligned}
K(x,y,t)={}&K_0(x,y,t)-i\varepsilon\frac{1}{(2\pi)^N}\sum_{R=1}^N \int dp_N dz_{N-1}\cdots dz_1 dp_1 \exp\left\{ i\sum_{n=R}^N\left[p_n(z_n-z_{n-1})-\varepsilon\frac{p_n^2}{2m}\right]\right\}\\
&\times\left[V\left(\frac{z_R+z_{R-1}}{2}\right)+W\left(\frac{z_R+z_{R-1}}{2}\right)p_R^2\right]\exp\left(i\sum_{n=1}^{R-1}\left\{p_n(z_n-z_{n-1})-\varepsilon\left[\frac{p_n^2}{2m}+V\left(\frac{z_n+z_{n-1}}{2}\right)+W\left(\frac{z_n+z_{n-1}}{2}\right)p_n^2\right]\right\}\right)\\
={}&K_0(x,y,t)-i\varepsilon\sum_{R=1}^N\int dz_R\int dz_{R-1}\left\{\frac{1}{(2\pi)^{N-R}}\int dp_N dz_N dp_{N-1}\cdots dz_{R+1}dp_{R+1}\right\}\\
&\times\exp\left\{ i\sum_{n=R+1}^N\left[p_n(z_n-z_{n-1})-\varepsilon\frac{p_n^2}{2m}\right]\right\}\frac{1}{2\pi}\int dp_R\left[V\left(\frac{z_R+z_{R-1}}{2}\right)+W\left(\frac{z_R+z_{R-1}}{2}\right)p_R^2\right]e^{ip_R(z_R-z_{R-1})}\frac{1}{(2\pi)^{R-1}}\\
&\times\int dp_{R-1}dz_{R-2}\cdots dp_1 e^{ip_{R-1}(z_{R-1}-z_{R-2})}\\
&\times\exp\left(i\sum_{n=1}^{R-2}\left\{p_n(z_n-z_{n-1})-\varepsilon\left[\frac{p_n^2}{2m}+V\left(\frac{z_n+z_{n-1}}{2}\right)+W\left(\frac{z_n+z_{n-1}}{2}\right)p_n^2\right]\right\}\right),
\end{aligned} \quad \text{(A14)}$$

which in the continuous limit reads

$$K(x,y,t)=K_0(x,y,t)-i\int_0^t dt'\int d\xi\int d\eta K_0(x,\xi,t-t')I_W(\xi,\eta)K(\eta,y,t'), \tag{A15}$$

with

$$I_W(\xi,\eta)=\frac{1}{2\pi}\int dp\left[V\left(\frac{\xi+\eta}{2}\right)+W\left(\frac{\xi+\eta}{2}\right)p^2\right]e^{ip(\xi-\eta)}. \tag{A16}$$

Passing to momentum space

$$\tilde{K}(k,q,t)=\int dx\int dy e^{-ikx}K(x,y,t)e^{iqy}, \tag{A17}$$

we can also write

$$\begin{aligned}\tilde{I}_W(k,q)&=\frac{1}{2\pi}\int d\xi dp d\eta e^{i(p-k)\xi}\left[V\left(\frac{\xi+\eta}{2}\right)+W\left(\frac{\xi+\eta}{2}\right)p^2\right]e^{-i(p-q)\eta}=\tilde{V}(k-q)+\tilde{W}(k-q)\left(\frac{k+q}{2}\right)^2\\&\equiv\left\langle k\left|V(x)+\frac{1}{4}\{p,\{p,W(x)\}\}\right|q\right\rangle.\end{aligned} \tag{A18}$$

Equation (A15) is equivalent to Eq. (A4) and the kernel reduces to the Fourier transformation of the case $W(x)=0$.

Had we considered the symmetric ordering in Eq. (A5) we should have replaced, in Eqs. (A13) and (A14),

$$V\left(\frac{z_n+z_{n-1}}{2}\right)+W\left(\frac{z_n+z_{n-1}}{2}\right)p_n^2$$

with

$$\frac{1}{2}[V(z_n)+V(z_{n-1})]+\frac{1}{2}[W(z_n)+W(z_{n-1})]p_n^2,$$

and the result would be

$$\begin{aligned}\tilde{I}_S(k,q)&=\frac{1}{2\pi}\int d\xi dp d\eta e^{i(p-k)\xi}\left[\frac{V(\xi)+V(\eta)}{2}+\frac{W(\xi)+W(\eta)}{2}p^2\right]e^{-i(p-q)\eta}=\tilde{V}(k-q)+\tilde{W}(k-q)\frac{k^2+q^2}{2}\\&\equiv\langle k|V(x)+\frac{1}{2}\{p^2,W(x)\}|q\rangle.\end{aligned} \tag{A19}$$

APPENDIX B: BEYOND THE LADDER APPROXIMATION

As an example let us derive the contribution in Eq. (2.14) corresponding to the crossed diagram in Fig. 1. Let us expand the last exponential (call it BL) in Eq. (2.11) (which was previously simply replaced with 1):

$$BL=\left\{1-\frac{ig_1^2}{4}\int_{\tau_1}^{s_1}d\tau_1'\int_0^{\tau_1}d\tau_1''D_F(z_1'-z_1'')-\frac{ig_2^2}{4}\int_{\tau_2}^{s_2}d\tau_2'\int_0^{\tau_2}d\tau_2''D_F(z_2'-z_2'')-\frac{ig_1g_2}{4}\int_0^{\tau_1}d\tau_1'\int_{\tau_2}^{s_2}d\tau_2'D_F(z_1'-z_2')+\cdots\right\}. \tag{B1}$$

The contribution corresponding to the crossed diagram (CD) is the one coming from the last term inside the curly brackets in Eq. (B1). We can write

$$\mathrm{CD}=-\frac{g_1g_2}{4}\frac{g_1g_2}{4}\left(\frac{1}{2}\right)^2\int_0^\infty d\tau_1\int_{\tau_1}^\infty ds_1\int_0^\infty d\tau_2\int_{\tau_2}^\infty ds_2\int d^4\xi_1\int d^4\xi_2\int_{\xi_1}^{x_1}\mathcal{D}z_1\int_{\xi_2}^{x_2}\mathcal{D}z_2\int_{y_1}^{\xi_1}\mathcal{D}z_1\int_{y_2}^{\xi_2}\mathcal{D}z_2\int_0^{\tau_1}d\tau_1'\int_{\tau_2}^{s_2}d\tau_2'D_F \tag{B2}$$

$$\times(z_1'-z_2')D_F(\xi_1-\xi_2)\exp\left\{-\frac{i}{2}\int_{\tau_1}^{s_1}d\tau_1''\left[\dot z_1''^2+m_1^2+\frac{g_1^2}{2}\int_{\tau_1}^{\tau_1''}d\tau_1'''D_F(z_1''-z_1''')\right]\right.$$

$$\left.-\frac{i}{2}\int_{\tau_2}^{s_2}d\tau_2''\left[\dot z_1''^2+m_2^2+\frac{g_2^2}{2}\int_{\tau_2}^{\tau_2''}d\tau_2'''D_F(z_2''-z_2''')\right]\right\}$$

$$\times\exp\left\{-\frac{i}{2}\int_0^{\tau_1}d\tau_1''\left[\dot z_1''^2+m_1^2+\frac{g_1^2}{2}\int_0^{\tau_1''}d\tau_1'''D_F(z_1''-z_1''')\right]-\frac{i}{2}\int_0^{\tau_2}d\tau_2''\left[\dot z_2''^2+m_2^2+\frac{g_2^2}{2}\int_0^{\tau_2''}d\tau_2'''D_F(z_2''-z_2''')\right]\right.$$

$$\left.-\frac{ig_1g_2}{4}\int_0^{\tau_1}d\tau_1''\int_0^{\tau_2}d\tau_2''D_F(z_1''-z_2'')\right\}$$

$$=-\frac{g_1^2g_2^2}{16}\left(\frac{1}{2}\right)^2\int d^4\eta_1\int d^4\eta_2\int d^4\xi_1\int d^4\xi_2D_F(\xi_1-\xi_2)D_F(\eta_1-\eta_2)\int_0^\infty d\tau_1'\int_{\tau_1'}^\infty d\tau_1\int_{\tau_1}^\infty ds_1\int_0^\infty d\tau_2\int_{\tau_2}^\infty d\tau_2'\int_{\tau_2'}^\infty ds_2$$

$$\times\int_{\xi_1}^{x_1}\mathcal{D}z_1\int_{\eta_1}^{\xi_1}\mathcal{D}z_1\int_{y_1}^{\eta_1}\mathcal{D}z_1\int_{\eta_2}^{x_2}\mathcal{D}z_2\int_{\xi_2}^{\eta_2}\mathcal{D}z_2\int_{y_2}^{\xi_2}\mathcal{D}z_2\exp\left\{-\frac{i}{2}\int_{\tau_1}^{s_1}d\tau_1''\left[\dot z_1'''^2+m_1^2+\frac{g_1^2}{2}\int_{\tau_1}^{\tau_1'''}d\tau_1'''D(z_2'''-z_2''')\right]\right.$$

$$\left.-\frac{i}{2}\int_{\tau_2'}^{\tau_1}d\tau_1''\left[\dot z_1''^2+m_1^2+\frac{g_1^2}{2}\int_{\tau_1'}^{\tau_1''}d\tau_1'''D_F(z_1''-z_1''')\right]\right\}\exp\left\{-\frac{i}{2}\int_{\tau_2'}^{s_2}d\tau_2''\left[\dot z_2''^2+m_2^2+\frac{g_2^2}{2}\int_{\tau_{2'}}^{\tau 2''}d\tau_2'''D(z_2''-z_2''')\right]\right.$$

$$\left.-\frac{i}{2}\int_{\tau_2}^{\tau_2'}d\tau_2''\left[\dot z_2''^2+m_2^2+\frac{g_2^2}{2}\int_{\tau_2}^{\tau_2''}d\tau_2'''D(z_2''-z_2''')\right]\right\}\exp\left\{-\frac{i}{2}\int_0^{\tau_1'}d\tau_1''\left[\dot z_1''^2+m_1^2+\frac{g_1^2}{2}\int_0^{\tau_1''}d\tau_1'''D_F(z_1''-z_1''')\right]\right.$$

$$\left.-\frac{i}{2}\int_0^{\tau_2}d\tau_2''\left[\dot z_2''^2+m_2^2+\frac{g_2^2}{2}\int_0^{\tau_2''}d\tau_2'''D_F(z_2''-z_2''')\right]-i\frac{g_1g_2}{4}\int_0^{\tau_1'}d\tau_1''\int_0^{\tau_2}d\tau_2''D_F(z_1''-z_2'')\right\}$$

$$\times\exp\left\{-\frac{ig_1^2}{4}\int_{\tau_1'}^{\tau_1}d\tau_1''\int_0^{\tau_1'}d\tau_1'''D_F(z_1''-z_1''')-\frac{ig_2^2}{4}\int_{\tau_2}^{\tau_2'}d\tau_2''\int_0^{\tau_2}d\tau_2'''D_F(z_2''-z_2''')-\frac{ig_1g_2}{4}\int_{\tau_1'}^{\tau_1}d\tau_1''\int_0^{\tau_2}d\tau_2''D_F(z_1''-z_2'')\right\}.$$

In conclusion, replacing again the last exponential in Eq. (2.2) with 1, we have

$$\mathrm{CD}=-g_1^2g_2^2\int d^4\xi_1\int d^4\xi_2\int d^4\eta_1\int d^4\eta_2D_F(\xi_1-\eta_2)D_F(\xi_2-\eta_1)G_2(x_1-\xi_1)G_2(\xi_1-\eta_1)$$

$$\times G_2(x_2-\xi_2)G_2(\eta_2-\xi_2)G_4(\eta_1,\eta_2;y_1,y_2), \tag{B3}$$

which apart from a reordering of the various factors and a redenomination of the variables corresponds to the last term in Eq. (2.14). The method can be immediately extended to higher order terms in BL or iterated on Eq. (B2).

APPENDIX C: FUNCTIONAL DERIVATIVES OF THE AREA TERM

We prove Eq. (5.10).

Let us first consider the confinement part and go back to the original Eq. (4.2). We write

$$S_{\min}=\int_{t_i}^{t_f}dt\int_0^1 d\lambda\,\mathcal{S}(u), \tag{C1}$$

with

$$\mathcal{S}(u)=\left[-\left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial t}\right)\left(\frac{\partial u^\nu}{\partial\lambda}\frac{\partial u_\nu}{\partial\lambda}\right)+\left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial\lambda}\right)^2\right]^{1/2}, \tag{C2}$$

$u^\mu=u^\mu(\lambda,t)$ being the equation of the minimal surface enclosed by the loop. Obviously u^μ must be the solution of the Euler equations

$$\frac{\partial}{\partial\lambda}\frac{\partial\mathcal{S}}{\partial(\partial u^\mu/\partial\lambda)}+\frac{\partial}{\partial t}\frac{\partial\mathcal{S}}{\partial(\partial u^\mu/\partial t)}=0, \tag{C3}$$

satisfying the boundary conditions $u^\mu(1,t)=z_1^\mu(t)$ and $u^\mu(0,t)=z_2^\mu(t)$. Then, considering an infinitesimal variation of the world line of the quark 1, $z_1^\mu(t)\to z_1^\mu(t)+\delta z_1^\mu(t)$, even $u^\mu(\lambda,t)$ must change, $u^\mu(\lambda,t)\to u^\mu(\lambda,t)+\delta u^\mu(\lambda,t)$, and one has

$$\delta S_{\min}=\int_{t_i}^{t_f}dt\int_0^1 d\lambda\left[\frac{\partial\mathcal{S}}{\partial(\partial u^\mu/\partial\lambda)}\frac{\partial}{\partial\lambda}\delta u^\mu+\frac{\partial\mathcal{S}}{\partial(\partial u^\mu/\partial t)}\frac{\partial}{\partial t}\delta u^\mu\right]=\int_{t_i}^{t_f}dt\left[\frac{\partial\mathcal{S}}{\partial(\partial u^\mu/\partial\lambda)}\delta u^\mu\right]_{\lambda=1}, \tag{C4}$$

where $\delta z_1^\mu(t)$ is assumed to vanish out of a small neighborhood of a specific value of t. Finally taking into account that

$$\delta u^\mu(1,t)=\delta z_1^\mu(t),\qquad \frac{\partial u^\mu(1,t)}{\partial t}=\dot z_1^\mu(t), \tag{C5}$$

one obtains

$$\begin{aligned}\delta S_{\min}&=\int_{t_i}^{t_f}dt\frac{1}{[\mathcal{S}]_{\lambda=1}}\left[-\dot z_1^2\left(\frac{\partial u_\nu}{\partial\lambda}\right)_1+\left(\frac{\partial u_\mu}{\partial\lambda}\right)_1\dot z_1^\mu\dot z_{1\nu}\right]\delta z_1^\nu\\&=\frac12\int_{t_i}^{t_f}(dz_1^\mu\delta z_1^\nu-dz_1^\nu\delta z_1^\mu)\left[\left(\frac{\partial u_\mu}{\partial\lambda}\right)_1\dot z_{1\nu}-\left(\frac{\partial u_\nu}{\partial\lambda}\right)_1\dot z_{1\mu}\right]\\&\quad\times\left\{-\dot z_1^2\left(\frac{\partial u}{\partial\lambda}\right)_1^2+\left[\dot z_1\cdot\left(\frac{\partial u}{\partial\lambda}\right)_1\right]^2\right\}^{-1/2}\end{aligned} \tag{C6}$$

and, more explicitly,

$$\frac{\delta S_{\min}}{\delta S^{\mu\nu}(z_1)}=\frac{(\partial u_\mu/\partial\lambda)_1\dot z_{1\nu}-(\partial u_\nu/\partial\lambda)_1\dot z_{1\mu}}{[-\dot z_1^2(\partial u/\partial\lambda)_1^2+(\dot z_1\cdot(\partial u/\partial\lambda)_1)^2]^{1/2}}. \tag{C7}$$

Then, passing to the straight line approximation we have

$$\frac{\partial u_\mu}{\partial\lambda}=z_{1\mu}-z_{2\mu}=r_\mu \tag{C8}$$

and

$$\frac{\delta S_{\min}}{\delta S^{\mu\nu}(z_1)}=\frac{r_\mu\dot z_{1\nu}-r_\nu\dot z_{1\mu}}{[-\dot z_1^2r^2+(\dot z_1\cdot r)^2]^{1/2}}. \tag{C9}$$

Using Eq. (C9) (having replaced the velocities with the momenta) and Eqs. (5.2) and (4.5), we obtain the second term in Eq. (5.10).

Let us come to the perturbative part. Consider a variation $z_1\to z_1+\delta z_1$; then,

$$\begin{aligned}&\delta_1\int_0^{s_2}d\tau_1\int+0^{s_2}d\tau_2\dot z_1^\rho D_{\rho\sigma}(z_1-z_2)\dot z_2^\sigma\\&=\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2\Big[\delta\dot z_1^\rho D_{\rho\sigma}(z_1-z_2)\\&\quad+\dot z_1^\rho\delta z_1^\nu\partial_\nu D_{\rho\sigma}(z_1-z_2)\dot z_2^\sigma\\&=\int\delta S^{\rho\nu}\int_0^{s_2}d\tau_2[\partial_\nu D_{\rho\sigma}(z_1-z_2)-\partial_\rho D_{\nu\sigma}(z_1-z_2)]\dot z_2^\sigma,\end{aligned} \tag{C10}$$

whence

$$\begin{aligned}&\frac{\delta}{\delta S^{\mu\nu}(z_1)}\int_0^{s_1}d\tau_1\int_0^{s_2}d\tau_2p_1^\rho D_{\rho\sigma}(z_1-z_2)p_2^\sigma\\&=\int_0^{s_2}d\tau_2(\delta_\mu^\rho\partial_{1\nu}-\delta_\nu^\rho\partial_{1\mu})D_{\rho\sigma}(z_1-z_2)p_2^\sigma,\end{aligned} \tag{C11}$$

and we recover the first term in Eq. (5.10).

From Eqs. (C9) and (C10) even Eqs. (5.11) and (5.12) can be directly obtained [notice that $z_1'\neq z_1$ or $z_1'\neq z_2$ have to be assumed in Eq. (5.11) in the case of a second derivative].

APPENDIX D: DISCRETE FORM FOR THE PATH-INTEGRAL REPRESENTATION OF THE $Q\bar q$ PROPAGATOR

Taking explicitly into account the discrete form of Eq. (5.19) and using Eq. (5.20) we have

$$\begin{aligned}H_4(x_1,x_2;y_1,y_2)&=H_2(x_1-y_1)H_2(x_2-y_2)-\frac{i}{4}\varepsilon^2\sum_{N_1=0}^\infty\sum_{N_2=0}^\infty\frac{1}{(2\pi)^{N_1+N_2}}\int d^4p_{11}d^4z_{11}\cdots d^4z_{1N_1-1}d^4p_{1N_1}\\&\times\int d^4p_{21}d^4z_{21}\cdots d^4z_{2N_2-1}d^4p_{2N_2}T_{x_1y_1}T_{x_2y_2}\sum_{R_1=1}^{N_1}\sum_{R_2=1}^{N_2}R\left(\frac{z_{1R_1}+z_{1R_1-1}}{2},\frac{z_{2R_2}+z_{2R_2-1}}{2},p_{1R},p_{2R}\right)\\&\times\mathcal{S}_0^{s_1}\mathcal{S}_0^{s_2}\exp\Bigg\{i\sum_{j=1}^2\sum_{n=1}^{N_j}\Bigg[-p_{jn}(z_{jn}-z_{jn-1})+\frac{\varepsilon}{2}(p_{jn}^2-m_j^2)\\&+\frac23g^2\varepsilon^2\sum_{n'=1}^{n-1}D_{\mu\nu}\left(\frac{z_{jn'}+z_{jn'-1}}{2}-\frac{z_{jn'}+z_{jn'-1}}{2}\right)p_{jn'}^\mu p_{jn'}^\nu\Bigg]\\&-i\varepsilon^2\sum_{n_1=1}^{R_1-1}\sum_{n_2=1}^{N_2}E\left(\frac{z_{1n_1}+z_{1n_1-1}}{2},\frac{z_{2n_2}+z_{2n_2-1}}{2},p_{1n_1},p_{2n_2},\ldots\right)\Bigg\}.\end{aligned} \tag{D1}$$

If we neglect in the exponent the terms $\Sigma_{n=R_j+1}^{N_j}\Sigma_{n=1}^{R_j}D_{\mu\nu}p_{jn_j}^{\mu}p_{jn_j'}^{\nu}$ and $\Sigma_{n_1=1}^{R_1}\Sigma_{n_2=R_2+1}^{N_2}E((z_{1n_1}+z_{1n_1-1})/2,(z_{2n_2}+z_{2n_2-1})/2,p_{1n_1},p_{2n_2},\dots)$, corresponding in the continuous to the quantity [cf. Eq. (2.11)],

$$\int_{\tau_j}^{s_j}d\tau_j'\int_0^{\tau_j}d\tau_j''D_{\mu\nu}(z_j'-z_j'')p_j^{\mu\prime}p_j^{\nu\prime\prime} \tag{D2}$$

and

$$\int_0^{\tau_1}d\tau_1'\int_{\tau_2}^{s_2}d\tau_2'E(z_1',z_2',p_1',p_2',\dots). \tag{D3}$$

Equation (D1) can be written

$$\begin{aligned}
H_4(x_1,x_2,y_1,y_2)=&H_2(x_1-y_1)H_2(x_2-y_2)-\frac{i}{4}\varepsilon^4\sum_{R_1=1}^{\infty}\sum_{R_2=1}^{\infty}\sum_{N_1=R_1}^{\infty}\sum_{N_2=R_2}^{\infty}\frac{1}{(2\pi)^4}\int d^4z_{1R_1}d^4p_{1R_1}d^4z_{1R_1-1}\frac{1}{(2\pi)^4}\\
&\times\int d^4z_{2R_2}d^4p_{2R_2}d^4z_{R_2-1}\int_{z_{1R_1+1}}^{x_1}\mathcal{D}z_1\mathcal{D}p_1\int_{z_{2R_2+1}}^{x_2}\mathcal{D}z_2\mathcal{D}p_2\int_{y_1}^{z_{1R_1-1}}\mathcal{D}z_1\mathcal{D}p_1\\
&\times\int_{y_2}^{z_{2R_2-1}}\mathcal{D}z_2\mathcal{D}p_2T_{x_1z_{1R}}T_{x_2z_{2R}}\mathcal{S}_{\tau_1}^{s_1}\mathcal{S}_{\tau_2}^{s_2}\exp\left\{i\sum_{j=1}^{2}\sum_{n=R_j+1}^{N_1}\left[-p_{jn}(z_{jn}-z_{jn-1})+\frac{\varepsilon}{2}(p_{jn}^2-m_j^2)\right.\right.\\
&\left.\left.+\frac{2}{3}g^2\varepsilon^2\sum_{n'=R_j+1}^{N_1}D_{\mu\nu}p_{jn}^{\mu}p_{jn'}^{\nu}\right]\right\}\exp\left\{-i\sum_{j=1}^{2}p_{jR}(z_{jR_j}-z_{jR_{j-1}})\right\}\\
&\times R\left(\frac{z_{1R_1}+z_{1R_1-1}}{2},\frac{z_{2R_2}+z_{2R_2-1}}{2},p_{1R},p_{2R}\right)T_{z_{1R}y_1}T_{z_{2R}y_2}\mathcal{S}_0^{\tau_1}\mathcal{S}_0^{\tau_2}\exp\left\{i\sum_{j=1}^{2}\sum_{n=1}^{R_j-1}[-p_{jn}(z_{jn}-z_{jn-1})\right.\\
&\left.+\frac{\varepsilon}{2}(p_{jn}^2-m_j^2)+\frac{2}{3}g^2\varepsilon^2\sum_{n'=1}^{R_j-1}D_{\mu\nu}p_{jn}^{\mu}p_{jn'}^{\nu}]-\varepsilon^2\sum_{n_1=1}^{R_1-1}\sum_{n_2=1}^{R_2-1}E\left(\frac{z_{1n_1}+z_{1n_1-1}}{2},\frac{z_{2n_2}+z_{2n_2-1}}{2},p_{1n_1},p_{2n_2}\right)\right\},
\end{aligned} \tag{D4}$$

which, going back to the continuous, corresponds to Eq. (5.21) with the kernel given by Eq. (5.22).

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[1] See, e.g., E. Eichten, in *Lattice '90*, Proceedings of the International Symposium, Tallahassee, Florida, edited by V. A. Heller, A. D. Kennedy, and S. Sanielevici [Nucl. Phys. B (Proc. Suppl.) **20**, 475 (1991)].

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