Fixing the renormalization scheme in NNLO perturbative QCD using conformal limit arguments

J. Rathsman*
Department of Radiation Sciences, Uppsala University, Box 535, S-751 21 Uppsala, Sweden
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We discuss how the renormalization scheme ambiguities in QCD can be fixed, when two observables are related, by requiring the coefficients in the perturbative expansion relating the two observables to have their conformal limit values, i.e., to be independent of the β function of the renormalized coupling. We show how the next-to-leading order BLM automatic scale-fixing method can be extended to next-to-next-to-leading order to fix both the renormalization scale and β_2 in a unique way. As an example we apply the method to the relation between Bjorken's sum rule and $R_{e^+e^-}$ and compare with experimental data as well as other scheme-fixing methods. [S0556-2821(96)02817-2]

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I. INTRODUCTION

In perturbative QCD, observables are given by expansions in the strong coupling α_s:

$$R=\left(\frac{\alpha_s}{\pi}\right)^N\left[R_0+R_1\frac{\alpha_s}{\pi}+R_2\left(\frac{\alpha_s}{\pi}\right)^2+\cdots\right], \qquad (1)$$

where the coefficients R_i can be calculated from the appropriate Feynman diagrams. The individual terms in the series depend on the renormalization scheme one is using but the sum of the entire series is independent of the scheme according to the renormalization group equation. However, when the series is truncated the result becomes renormalization scheme dependent. This dependence is formally of higher order than the terms calculated in the series but numerically the difference between different schemes can be large. These differences give a theoretical uncertainty which in principle makes it impossible to make any absolute predictions since any result can be obtained by a finite renormalization. By going to higher order in perturbation theory the renormalization scheme dependence becomes smaller but in principle the problem remains. One can argue that it is only bad scheme choices that give ''crazy'' results and that as long as one uses a ''sensible'' scheme the result will also be ''sensible.'' The question then arises, what is a sensible scheme?

The question of how to choose an appropriate renormalization scheme in QCD has been discussed many times. Three well-known methods for choosing the renormalization scheme are the ''effective charge scheme'' by Grunberg [1], the ''principle of minimum sensitivity'' by Stevenson [2], and ''automatic scale fixing'' by Brodsky, Lepage, and Mackenzie (BLM) [3]. All these methods are based on some more or less intuitive principle or set of arguments for how a perturbative series should behave.

Of special interest here is the BLM method which fixes the scale in next-to-leading order (NLO) using conformal limit arguments. In a conformally invariant theory the coupling $a=\alpha(\mu)/\pi$ is scale invariant: i.e.,

$$\frac{da}{d\ln\mu}=\beta(a)=-\beta_0a^2-\beta_1a^3-\beta_2a^4-\ \dots=0. \qquad (2)$$

It is therefore natural to define the conformal limit of perturbative QCD as the limit $\beta_i\to 0$ [3,4]. This means that the coefficients R_i in the perturbative series have their conformal limit values if they do not contain any explicit dependence on the β function. For example, in NLO the perturbative coefficients should have no explicit β_0 dependence. In the BLM method this is achieved by absorbing all β_0-dependent NLO terms ($\beta_0=\frac{11}{2}-\frac{1}{3}N_f$ where $N_f=$ number of active quark flavors) into the running of α_s by a suitable redefinition of the renormalization scale. It should be noted that the renormalization scale obtained by the BLM method can also be interpreted as the mean value of the virtualities in the gluon propagators [3,5,4,6,7].

A useful concept when discussing renormalization scheme uncertainties is the effective charge [1] of an observable which contains all QCD corrections. For example, the effective charge $\hat{a}_R$ of $R_{e^+e^-}$ is defined by

$$R_{e^+e^-}(Q_R)=\frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}=3\sum_{i=1}^{N_f}e_i^2[1+\tfrac{3}{4}C_F\hat{a}_R(Q_R)]. \qquad (3)$$

Each effective charge has its own β function [1] connected to it:

$$\frac{d\hat{a}_R}{d\ln Q_R}=\hat{\beta}_R(\hat{a}_R)=-\beta_0\hat{a}_R^2-\beta_1\hat{a}_R^3-\hat{\beta}_{2,R}\hat{a}_R^4-\cdots, \qquad (4)$$

where β_0 and β_1 are renormalization scheme independent and $\hat{\beta}_{i,R}$, $i\geqslant 2$ are renormalization scheme invariants. Thus, for each physical observable A there is a specific $\hat{\beta}_{2,A}$ connected to it which is an inherent property of the effective charge. The perturbative series for an effective charge depends on the renormalization scheme even in the conformally invariant theory, but when two effective charges are related, one gets a relation that is independent of the intermediate scheme that was used.

In this paper we present a new generalization of the BLM method to next-to-next-to-leading order (NNLO) using the

*Electronic address: rathsman@tsl.uu.se

conformal limit arguments as a starting point which fixes both the renormalization scale and β_2 when two physical observables are related. The value for β_2 that is obtained is an intermediate value between the $\hat{\beta}_2$'s of the two effective charges. (A generalization to the factorization scheme problem will be considered in a separate paper [8].) This is a variation of an approach by Grunberg and Kataev [9], but whereas they claimed that the prescription for making the coefficients N_f independent is ambiguous, we will show that once the initial renormalization scheme is fixed by relating two physical observables, the conformal limit arguments fixes the scheme in a unique way. We also compare with the single-scale [10] and multiscale extensions [4,9,11] of the BLM method to NNLO which fixes the renormalization scale when two effective charges A and B are related, $\hat{a}_A = \hat{a}_B(1 + r_{1,A/B}\hat{a}_B + \cdots)$, using $\hat{\beta}_{2,B}$. As an example the conformal-limit scheme-fixing method is applied to the relation between Bjorken's sum rule in polarized deep-inelastic scattering and R_{e+e-}. The result is compared with a recently reported experimental determination of Bjorken's sum rule and the general renormalization scheme dependence.

II. CONFORMAL-LIMIT SCHEME-FIXING METHOD

Consider an observable in NNLO depending on one energy scale Q such as $R_{e^+e^-}(Q_R)$ defined by Eq. (3). The effective charge $\hat{a}_R$ contains all QCD corrections:

$$\hat{a}_R(Q_R) = a(\mu,\beta_2,\dots)[1 + r_1(Q_R,\mu)a(\mu,\beta_2,\dots) + r_2(Q_R,\mu,\beta_2)a^2(\mu,\beta_2,\dots)], \tag{5}$$

where the coefficients r_i can be calculated using perturbative QCD. The renormalization scheme dependence can be parametrized through the renormalization scale μ and the coefficients in the β function, β_i for $i \geq 2$ [2]. Strictly speaking it is the ratio μ/Λ of the renormalization scale and the QCD scale parameter Λ that is the relevant parameter but in the following we will often make the implicit assumption that Λ is held fixed when μ is varied. This can be done by choosing a measurement of an effective charge to define Λ as will be shown later.

The first two terms in the renormalization group equation for the coupling $a = \alpha_s(\mu)/\pi$,

$$\frac{da}{d\ln\mu} = \beta(a) = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 - \cdots, \tag{6}$$

are renormalization scheme independent,

$$\beta_0 = \tfrac{11}{6}N_C - \tfrac{1}{3}N_f, \tag{7}$$

$$\beta_1 = \tfrac{17}{12}N_C^2 - \tfrac{5}{12}N_C N_f - \tfrac{1}{4}C_F N_f, \tag{8}$$

whereas the higher order terms depend on the renormalization scheme.

Applying self-consistency for the perturbative expansion of the effective charge with respect to the renormalization scheme parameters,

$$\frac{d\hat{a}_R}{d\ln\mu}, \frac{d\hat{a}_R}{d\beta_2} = O(a^4),$$

gives [2] the renormalization scheme invariants

$$\hat{r}_1 = r_1 - \beta_0 \ln\frac{\mu}{\Lambda}, \tag{9}$$

$$\hat{\beta}_{2,R} = \beta_2 - \beta_1 r_1 - \beta_0 r_1^2 + \beta_0 r_2, \tag{10}$$

where $\hat{\beta}_{2,R}$ is the coefficient in the renormalization group equation for the effective charge given by Eq. (4). In passing we also note that the expression for the renormalization scheme invariant $\hat{r}_1$ shows explicitly that it is μ/Λ that is the relevant parameter for parametrizing the renormalization scheme dependence.

From the self-consistency requirements we also get the explicit μ and β_j dependence of the coefficients r_i:

$$r_1 = r_1^* + \beta_0\left[d^* + \ln\frac{\mu}{Q}\right], \tag{11}$$

$$r_2 = r_2^* - \frac{\beta_2 - \beta_{2,\overline{\rm MS}}}{\beta_0}\beta_1\left[d^* + \ln\frac{\mu}{Q}\right] + \beta_0\left[e^* + 2r_1\ln\frac{\mu}{Q}\right] + \beta_0^2\left[f^* - \ln^2\frac{\mu}{Q}\right], \tag{12}$$

where we have assumed that the coefficients have been calculated in the modified minimal subtraction ($\overline{\rm MS}$) scheme with $\mu = Q$ to fix the integration constants. (The asterisk is used to indicate terms that are independent of β_0 and β_1.) We also assume that r_1 and r_2 only contain β_0- and β_1-dependent terms from loop insertions which is why the β_0 term in r_1 and the β_1 term in r_2 are the same; i.e., they are both given by d^*. This way we also fix the redundancy in how to divide r_2 into β_0- and β_1-dependent parts.

We are now in the position to apply the conformal limit arguments to the effective charge $\hat{a}_R$ to fix the renormalization scheme parameters μ and β_2. First the renormalization scale is fixed by requiring r_1 to be β_0 independent. From Eq. (11) we see that this can be obtained by choosing the renormalization scale as

$$\mu^* = \mu_{\rm BLM} = Q\exp(-d^*). \tag{13}$$

We also note that the renormalization scale obtained in this way is the same as in the original BLM method.

Next β_2 is fixed by requiring r_2 to be β_0 and β_1 independent, i.e., $r_2 = r_2^*$. Using the renormalization scheme invariant $\hat{\beta}_{2,R}$ we get the following expression for r_2:

$$r_2 = (r_1^*)^2 + \frac{\beta_1}{\beta_0}r_1^* + \frac{\hat{\beta}_{2,R} - \beta_2}{\beta_0}. \tag{14}$$

From this we see that by choosing a renormalization scheme where β_2 is given by

$$\beta_2^* = \hat{\beta}_{2,R} + \beta_1 r_1^* + \beta_0 (r_1^*)^2 - \beta_0 r_2^*, \tag{15}$$

we get $r_2 = r_2^*$. Note that this value of β_2 in general is different both from the effective charge value $\hat{\beta}_{2,R}$ and from $\beta_{2,\overline{\rm MS}}$ which was used in the calculation. However, if $r_i^* = 0$, then $\beta_2^* = \hat{\beta}_{2,R}$, and if $r_i^* = r_i$, then $\beta_2^* = \beta_{2,\overline{\rm MS}}$.

This fixes the renormalization scheme in NNLO up to the question of initial scheme, which is resolved when two physical observables are related as shown below. This does not introduce any new uncertainties since only relations between observables can be predicted in a renormalized theory and for each pair of observables we get a unique relation. The situation here is not different from what happens in the BLM method and its earlier extensions where it is also necessary to fix the initial renormalization scheme to get a unique result. In [9] it was argued that ''in QCD, setting $r_i = r_i^*$ is always possible, but leaves us with an ambiguous prescription.'' However, as we have shown above, there are no ambiguities once the initial renormalization scheme has been fixed and this can be done using a physical observable as shown below.

The perturbative series for the effective charge $\hat{a}_R$ in NNLO thus becomes

$$\hat{a}_R = a^*[1 + r_1^* a^* + r_2^*(a^*)^2] \tag{16}$$

where the r_i^*'s contain no explicit β_j terms. In this way we obtain the required feature that all signs of scale breaking, i.e., $\beta \neq 0$, are confined into the running of the coupling and the coefficients in the perturbative series have their conformal limit values. Finally a^* can be obtained by solving the renormalization group equation (6) with $\beta_2 = \beta_2^*$.

Before ending this section we note that the method for fixing β_2 can be generalized to arbitrary order, $n \geqslant 2$. For this we need the renormalization scheme invariants $\hat{\beta}_{n,R}$ in the renormalization group equation for the effective charge, Eq. (4). The general form for $\hat{\beta}_{n,R}$ is given in [1] and can be rewritten as

$$r_n = \sum_{j=0}^{n} c_{j,n} r_1^n + \frac{1}{n-1}\frac{\hat{\beta}_{n,R} - \beta_n}{\beta_0}, \tag{17}$$

where $c_{j,n}$ only depends on $\{\hat{\beta}_{i,R}, \beta_i\}$ with $i \leqslant n-1$. In previous steps of applying the conformal limit arguments, the renormalization scale has been fixed so that $r_1 = r_1^*$ and the β_i's $(2 \leqslant i \leqslant n-1)$ have been fixed to $\beta_i = \beta_i^*$ So by requiring $r_n = r_n^*$ to contain no explicit β_i terms for $i \leqslant n-1$ the value of β_n is fixed to be

$$\beta_n^* = \hat{\beta}_{n,R} - (n-1)\beta_0 r_n^* + (n-1)\beta_0 \sum_{j=0}^{n} c_{j,n}^* (r_1^*)^n, \tag{18}$$

which is a generalization of Eq. (15) to arbitrary $n \geqslant 2$.

III. COMPARISON WITH OTHER SCALE-FIXING METHODS

In previous multiscale extensions of the BLM method (denoted MBLM in the following) [4,9,11] one has different scales for each α_s term: i.e.,

$$\hat{a}(Q) = a(\mu_1) + r_1(\mu_1)a^2(\mu_2) + r_2(\mu_1,\mu_2)a^3(\mu_3), \tag{19}$$

where μ_1 is parametrized as $\mu_1 = \mu_0 \exp[\theta a(\mu)]$ and μ as well as μ_3 is arbitrary (they will be fixed in higher order approximations but here we simply set them to be the same as μ_2).

The MBLM scale-fixing method is constructed to have β_2 unchanged and instead θ and μ_2 are introduced which gives three ($\mu_1 = \mu_0 \exp[\theta a(\mu)]$, μ_2, and β_2) unphysical[1] parameters instead of the minimal two (μ and β_2). Requiring that the effective charge does not depend on these parameters, to the present order of perturbation theory, gives the explicit μ, β_j, and θ dependence of the coefficients r_i:

$$r_1 = r_1^* + \beta_0\left[d^* + \ln\frac{\mu_0}{Q}\right], \tag{20}$$

$$r_2 = r_2^* - \frac{\beta_2 - \beta_{2,\overline{\mathrm{MS}}}}{\beta_0} + \beta_1\left[d^* + \ln\frac{\mu_0}{Q}\right] + \beta_0\theta + \beta_0\left[e^* + 2r_1\ln\frac{\mu_2}{Q}\right] + \beta_0^2\left[f^* - \ln^2\frac{\mu_0}{Q}\right], \tag{21}$$

where again the integration constants are fixed by assuming that the calculation was made in the $\overline{\mathrm{MS}}$ scheme with $\mu = Q$. Comparing with Eqs. (11) and (12) we see the effects of having different renormalization scales and also how the θ dependence enters. In the MBLM scale fixing all N_f-dependent terms should be absorbed so that $r_1 = r_1^*$ and $r_2 = r_2^*$ just as in the conformal limit scheme. Keeping in mind that β_2 should be unchanged we see that this can be achieved by choosing

$$\beta_2 = \beta_{2,\overline{\mathrm{MS}}},$$

$$\mu_0 = Q\exp(-d^*),$$

$$\theta = \beta_0[-f^* + (d^*)^2],$$

$$\mu_2 = Q\exp[-e^*/(2r_1^*)],$$

so that $\mu_1 = Q\exp\{-d^* - \beta_0[f^* - (d^*)^2]a(\mu_2)\}$. From Eqs. (20) and (21) it is also easy to see that one only needs a single renormalization scale if θ is chosen appropriately. In this single-scale extension [10] of the BLM scale-fixing method (denoted SBLM in the following) one chooses $\mu_2 = \mu_1 = \mu_0\exp[\theta a(\mu)]$ where $\mu_0 = Q\exp(-d^*)$ and $\theta = \beta_0[-f^* + (d^*)^2] - e^* + 2r_1^* d^*$.

IV. FIXING THE INITIAL SCHEME WITH A PHYSICAL OBSERVABLE

Up to now we have assumed that the initial renormalization scheme (and thereby Λ) is fixed. Now we will show how this can be accomplished using a physical observable so that a unique prediction of another physical observable can be made. As an example we will relate R defined in Eq. (3), to K, Bjorken's sum rule for polarized deep-inelastic electroproduction [12].

The effective charge for R is in NNLO given by (in the $\overline{\mathrm{MS}}$ scheme),

[1]When two physical observables are related the MBLM (and SBLM) method uses the effective charge $\hat{\beta}_2$ of one of the observables which in principle is a measurable quantity.

$$\hat{a}_R = a_{\overline{\rm MS}}(1 + r_1 a_{\overline{\rm MS}} + r_2 a_{\overline{\rm MS}}^2), \tag{22}$$

where r_1 and r_2 can be obtained from [13,14]. For Bjorken's sum rule, one can also define an effective charge $\hat{a}_K$ (using the same normalization as in [15]):

$$K = \int_0^1 dx[g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]$$
$$= \frac{1}{6}\left|\frac{g_A}{g_V}\right|\left(1 - \frac{3}{4}C_F\hat{a}_K(Q)\right). \tag{23}$$

In NNLO, $\hat{a}_K$ is given by

$$\hat{a}_K = a_{\overline{\rm MS}}(1 + k_1 a_{\overline{\rm MS}} + k_2 a_{\overline{\rm MS}}^2), \tag{24}$$

where k_1 and k_2 have been calculated in [16].

Recognizing that the $\overline{\rm MS}$ scheme is only an intermediary scheme suited for calculations we can find a unique relation between the two observables $\hat{a}_R$ and $\hat{a}_K$. Inverting Eq. (22) for $a_{\overline{\rm MS}}$ and inserting into Eq. (24) gives

$$\hat{a}_K(Q_K) = \hat{a}_R(Q_R)[1 + c_1\hat{a}_R(Q_R) + c_2\hat{a}_R^2(Q_R)] \tag{25}$$

where now Q_R is the renormalization scale. The coefficients c_i, which are independent of the intermediate scheme, are given by

$$c_1 = -\frac{3}{4}C_F - \beta_0\left(\frac{7}{4} - 2\zeta_3 - \ln\frac{Q_R}{Q_K}\right), \tag{26}$$

$$\begin{aligned} c_2 = {} & \frac{9}{16}C_F^2 - \ell - \frac{\beta_2 - \hat{\beta}_{2,R}}{\beta_0} - \beta_1\left(\frac{7}{4} - 2\zeta_3 - \ln\frac{Q_R}{Q_K}\right) \\ & + \beta_0 C_F\left(\frac{523}{144} + \frac{14}{3}\zeta_3 - 10\zeta_5\right) - \beta_0 N_C\left(\frac{13}{36} - \frac{1}{3}\zeta_3\right) \\ & + \beta_0 2c_1\ln\frac{Q_R}{Q_K} - \beta_0^2\left[-\frac{17}{6} + \left(\frac{35}{3} - 8\zeta_3\right)\zeta_3 \right. \\ & \left. - \frac{\pi^2}{12} + \ln^2\frac{Q_R}{Q_K}\right], \end{aligned} \tag{27}$$

where ℓ is the light-by-light term:[2]

$$\ell = \frac{d^{abc}d^{abc}(\Sigma_{i=1}^{N_f} e_i)^2}{N_C C_F \Sigma_{i=1}^{N_f} e_i^2}\left(\frac{11}{144} - \frac{1}{6}\zeta_3\right). \tag{28}$$

Hereby all dependence on the $\overline{\rm MS}$ scheme has disappeared. Effectively what we have done is to go from the $\overline{\rm MS}$ scheme to the R scheme, the effective charge scheme for R which is the renormalization scheme where $\hat{a}_R(Q_R)$ has no perturbative corrections, i.e., $\beta_2 = \hat{\beta}_{2,R}$ and $\ln(\mu/\Lambda_R) = -\hat{r}_1/\beta_0$ as seen from Eqs. (9) and (10).

Applying the conformal limit criteria so that the coefficients c_1 and c_2 take their conformal limit values, $c_1^* = -\frac{3}{4}C_F = -1$ and $c_2^* = \frac{9}{16}C_F^2 - \ell = 1 - \ell$, we get the conformal limit renormalization scheme parameters Q_R^* and β_2^*

$$Q_R^* = Q_K \exp(\tfrac{7}{4} - 2\zeta_3), \tag{29}$$

$$\begin{aligned} \beta_2^* &= \hat{\beta}_{2,K} + c_1^*\beta_1 + (c_1^*)^2\beta_0 - c_2^*\beta_0 \\ &= \hat{\beta}_{2,K} - \beta_1 + \ell\beta_0, \end{aligned} \tag{30}$$

where β_2^* is obtained from the invariant $\hat{\beta}_{2,K} = \beta_2^* - c_1^*\beta_1 - (c_1^*)^2\beta_0 + c_2^*\beta_0$. Equation (29) has been called a commensurate scale relation [4] since it gives the relation between the renormalization scales when two physical observables are related. In the same sense one can call Eq. (30) a commensurate β_2 relation since it gives β_2 when two observables are related. The resulting value for β_2^* is an intermediate value between the two effective charge values $\hat{\beta}_{2,K}$ and $\hat{\beta}_{2,R}$. For a general relation between two effective charges, $\hat{a}_A = \hat{a}_B(1 + r_{1,A/B}\hat{a}_B + r_{2,A/B}\hat{a}_B^2)$, β_2^* depends on the conformal limit values of the coefficients $r_{i,A/B}^*$. If $r_{i,A/B}^* = 0$, then $\beta_2^* = \hat{\beta}_{2,A}$, and if $r_{i,A/B}^* = r_{i,A/B}$, then $\beta_2^* = \hat{\beta}_{2,B}$. In other words the conformal limit scheme value β_2^* "interpolates" between the two effective charge values $\hat{\beta}_{2,A}$ and $\hat{\beta}_{2,B}$, depending on the conformal limit values of the coefficients.

Finally we have the conformal limit result in NNLO,

$$\hat{a}_K(Q_K) = a_R^*[1 - a_R^* + (1 - \ell)(a_R^*)^2], \tag{31}$$

which relates one effective charge ($\hat{a}_K$) to another one (a_R^*) which has been modified in a unique way. This relation resembles the nonperturbative Crewther relation [17], $3S = KR'$, which is derived using conformal and chiral invariance. It relates Adler's anomalous constant (S), Bjorken's sum rule for polarized deep-inelastic electroproduction (K), and the isovector part of R (R'). According to the no-renormalization theorem for the axial anomaly [18] one might think [19] that the perturbative corrections to K and R cancel. This is not the case, but instead one finds [19] that the combined corrections are proportional to the β function, $(1 + \hat{a}_R)(1 - \hat{a}_K) - 1 \propto \beta(a)$, if the light-by-light term is neglected. The generalized Crewther relation has been studied in more detail in [10].

The modified effective charge $a_R^*(Q_R^*, \beta_2^*)$ can be obtained from the third order standard solution to Eq. (6):

$$a_R^*(Q_R^*, \beta_2^*) = \frac{1}{\beta_0\ln(Q_R^*/\Lambda_R)} - \frac{\beta_1\ln\ln(Q_R^*/\Lambda_R)}{\beta_0^3\ln^2(Q_R^*/\Lambda_R)} + \frac{\beta_1^2\ln^2\ln(Q_R^*/\Lambda_R) - \beta_1^2\ln\ln(Q_R^*/\Lambda_R) + \beta_2^*\beta_0 - \beta_1^2}{\beta_0^5\ln^3(Q_R^*/\Lambda_R)}, \tag{32}$$

[2]Numerically the light-by-light term is small: $\ell = -0.0376$ for $N_f = 5$, $\ell = -0.1653$ for $N_f = 4$, and $\ell = 0$ for $N_f = 3$. The light-by-light term is not affected by the conformal limit arguments since it is not proportional to β_0.

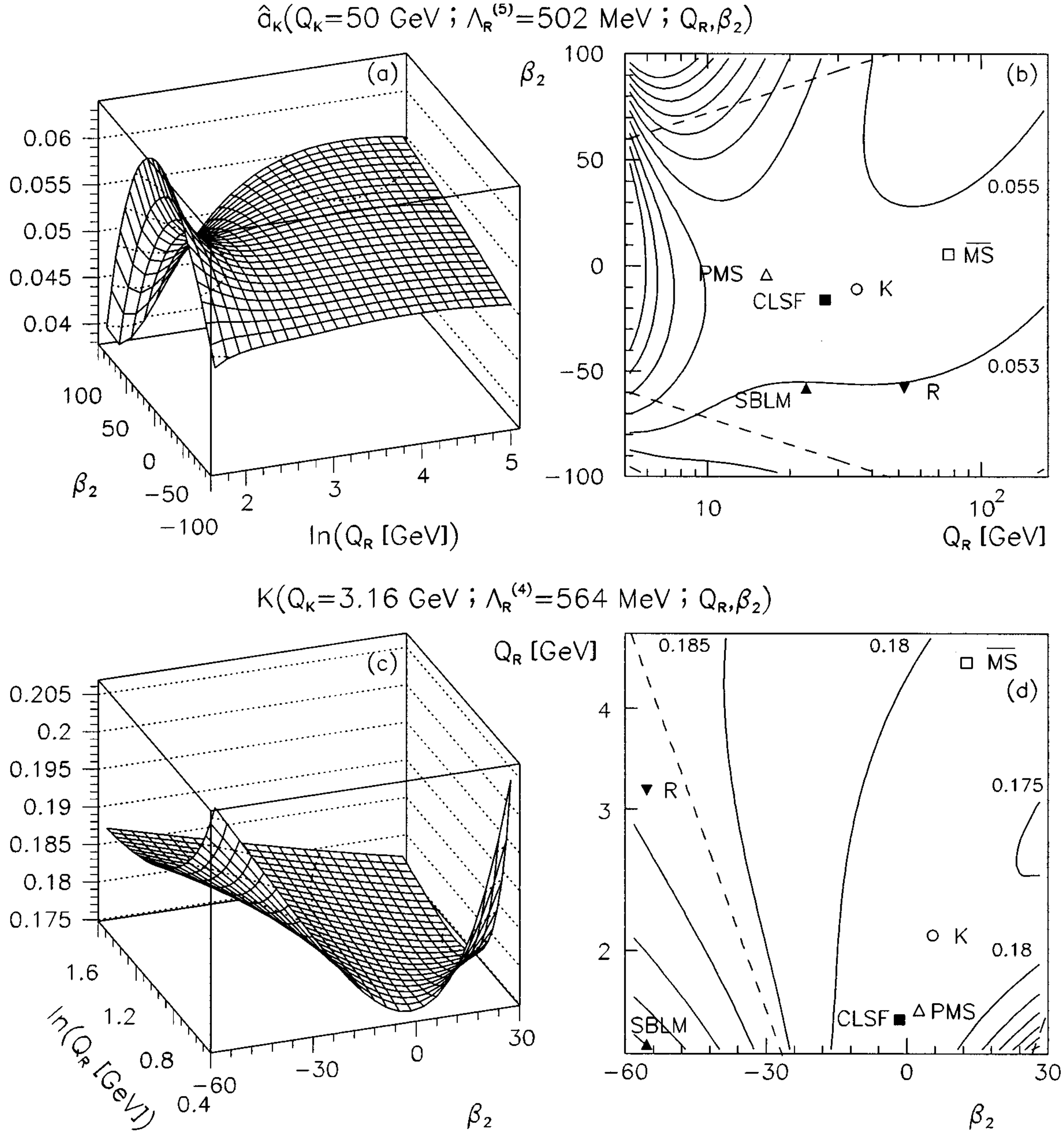

FIG. 1. Renormalization scheme dependence of $\hat{a}_K(Q_K=50\text{ GeV})$ in (a), (b) and Bjorken's sum rule $K(Q_K=3.16\text{ GeV})$ in (c), (d), shown as surface plots (a), (c) and contour plots (b), (d). Note that Λ has been kept fixed so that all the renormalization scheme dependence is given by the renormalization scale Q_R and β_2. Some well-known scheme and scale choices are marked in (b) and (d) for reference and the corresponding numerical values are given in Tables I and II, respectively. The dashed lines indicate the limit of the perturbative regime, $|\beta_2|\leqslant\beta_1/a_R\simeq\beta_1\beta_0\ln(Q_R/\Lambda_R)$, as explained in the text. In addition to the scheme dependence there is also an experimental uncertainty from the value of Λ_R.

which is valid for $\ln(Q_R^*/\Lambda_R)\gg 1$. The value of Λ_R should be determined by experiment from $\hat{a}_R(Q_R)$ with $\beta_2=\hat{\beta}_{2,R}$ and $Q_R=\sqrt{s}$ using the same solution for α_s. (The definition of Λ depends on which solution that is used but sticking to one definition or solution presents no problem.) In other words the effective charge $\hat{a}_R(Q_R)$ gives an experimentally measurable Λ_R and a well-defined starting renormalization scheme which is then modified into the conformal limit scheme where the scheme parameters are given by Q_R^* and β_2^*.

V. DISCUSSION

Figures 1(a) and 1(b) illustrate the renormalization scheme dependence of $\hat{a}_K(Q_K=50\text{ GeV})$ as given by Eq. (26) using the standard solution, Eq. (32), for a_R with $\Lambda_R^{(5)}=502$ MeV. We see that for not too small renormalization scales the β_2 dependence dominates whereas for smaller scales both the scale dependence and β_2 dependence are quite strong. Since the renormalization scheme dependence is parametrized by the renormalization scale Q_R and β_2

TABLE I. Numerical values of β_2, Q_R, $\hat{a}_K$, and Bjorken's sum rule K in different schemes for $Q_K=50$ GeV and $\Lambda_R^{(5)}=502$ MeV (Λ is kept fixed so that the scheme dependence is given by β_2 and Q_R). Note that there are two scales for the MBLM method given as $Q_{R,1}(Q_{R,2})$.

Scheme	β_2	Q_R [GeV]	$\hat{a}_K$	K
CLSF	-15.98	26.00	0.5404	0.1982
SBLM	-57.86	22.18	0.5271	0.1985
MBLM	-57.86	22.17(25.02)	0.5290	0.1985
R	-57.86	50.00	0.5291	0.1985
K	-11.00	33.74	0.5421	0.1982
$\overline{\rm MS}$	5.65	72.22	0.5444	0.1981
PMS	-3.98	15.88	0.5403	0.1982

TABLE II. Numerical values of β_2, Q_R, $\hat{a}_K$, and Bjorken's sum rule K in different schemes for $Q_K=3.16$ GeV and $\Lambda_R^{(4)}=564$ MeV (Λ is kept fixed so that the scheme dependence is given by β_2 and Q_R). Note that there are two scales for the MBLM method given as $Q_{R,1}(Q_{R,2})$.

Scheme	β_2	Q_R [GeV]	$\hat{a}_K$	K
CLSF	-1.56	1.64	0.159	0.176
SBLM	-55.46	1.53	0.0229	0.205
MBLM	-55.46	1.49(1.58)	0.0126	0.207
R	-55.46	3.16	0.100	0.189
K	5.54	2.09	0.160	0.176
$\overline{\rm MS}$	12.70	4.56	0.147	0.179
PMS	2.62	1.69	0.160	0.176

when Λ_R is kept fixed, the whole space of schemes should in principle be obtained by varying Q_R and β_2. However, this does not take into account the region of validity for Eq. (32). For too small renormalization scales Q_R or too large β_2 this solution is no longer valid.

To be self-consistent, one should also take into account that Eq. (6) has to make sense perturbatively. In other words, if $|\beta_2|a_R \geqslant \beta_1$, then we are no longer in the perturbative regime where the contributions from consecutive terms are smaller than the preceding ones and therefore the perturbative expansion breaks down. The lines $\beta_2 = \pm\beta_1/a_R \simeq \pm\beta_1\beta_0\ln(Q_R/\Lambda_R)$ which indicate where this happens are shown in Fig. 1(b). These lines also indicate where the solution given by Eq. (32) is no longer valid and the numerical results should therefore not be trusted in that region. The conformal-limit scheme-fixing (CLSF) method and the SBLM scale-fixing method are indicated in Fig. 1(b) together with some other well-known schemes like the $\overline{\rm MS}$ scheme, the ''principle of minimum sensitivity'' (PMS), and the effective charge (ECH) schemes for R and K.

Conceptually the PMS and ECH schemes are different from the conformal limit schemes in that they prescribe a unique scheme for each observable instead of giving schemes for relations between observables. It should also be noted that the PMS and ECH schemes sometimes give renormalization scales which are difficult to interpret physically. For instance, in jet production, both in e^+e^- [20] and deep-inelastic scattering (DIS) [21], the resulting renormalization scales grow when the typical jet mass ($y_{\rm cut}W^2$) is decreased, which is counter intuitive. In addition the PMS prescription depends on the intermediate scheme. For example, applying the PMS prescription to two observables given in the $\overline{\rm MS}$ scheme separately and then relating them gives a different result compared to if the observables are first related so that the dependence on the $\overline{\rm MS}$ scheme is removed and then the PMS prescription is applied.

For reference, the numerical values of β_2, Q_R, $\hat{a}_K$, and Bjorken's sum rule K in the different schemes are given in Table I (together with the MBLM method which has two renormalization scales). Comparing the conformal-limit scheme-fixing with the SBLM and MBLM scale-fixing methods in Table I we see that even though the coefficients c_i are the same, the scales, β_2 and the resulting effective charge $\hat{a}_K$ are different. This shows the importance of not only taking the commensurate scale relation into account as in the SBLM and MBLM methods but also the commensurate β_2 relation as in the CLSF method.

From Fig. 1(b) we also see that the CLSF point is closer to the saddle point (PMS) than the SBLM point, which means that the scheme dependence is smaller in the CLSF point. One might also worry that the SBLM and MBLM scale-fixing methods are too close to the line $\beta_2 = \pm\beta_1\beta_0\ln(Q_R/\Lambda_R)$ where the perturbative expansion for the effective charge $\hat{a}_R$ breaks down. Note that both in Fig. 1 and in Table I (and Table II) the renormalization scales are related to Λ_R. This means that, for example, in the $\overline{\rm MS}$ scheme where one normally would use $\mu=Q_K$ as the renormalization scale for $\Lambda=\Lambda_{\overline{\rm MS}}$, the scale becomes $Q_K\exp(r_{1,\overline{\rm MS}}/\beta_0)$ for $\Lambda=\Lambda_R$ [compare with Eq. (9)].

As a concrete example of the conformal-limit-scheme fixing method we will use the global analysis of R_{e+e-} data in the range $2.64 < Q_R < 52$ GeV by Marshall [22] to calculate the Bjorken sum rule K at $Q_K=3.16=\sqrt{10}$ GeV, which can be compared with the SMC measurement, $K=\Gamma_1^p-\Gamma_1^n=0.199\pm0.038$ [15]. The result of the analysis in [22] is a global fit taking both electroweak and QCD corrections into account, $R_{e+e-}=R_{\rm Born}^{\gamma,Z}(1+\hat{a}_R)$, and numerical values for $R_Q=1+\hat{a}_R$ are given for some distinct energies. In the following we have used the value $R_Q=1.0463\pm0.0044$ for $Q_R=59.2$ GeV. The reasons for picking this particular energy are that we want to have a large scale Q_R where the standard solution, Eq. (32), is a good approximation (especially since $\hat{\beta}_{2,R}$ is so large) and at the same time we do not want to extrapolate the experimental result too much outside the measured range.

Following the prescription given above for determining $\Lambda_R^{(5)}$ we get

$$\Lambda_R^{(5)} = 502^{+326}_{-225} \text{ MeV},$$

from $\hat{a}_R=0.0463\pm0.0044$ using Eq. (32) with $\hat{\beta}_{2,R}=-57.9$ and $Q_R=59.2$ GeV. To be able to compare with the Spin Muon Collaboration (SMC) measurement we also need $\Lambda_R^{(4)}$. This is obtained by matching a_R^* numerically at the flavor threshold, $Q_R=m_b=5$ GeV, for $N_f=4$ and 5 with $\beta_2=\beta_2^*$, which gives,

$$\Lambda_R^{(4)} = 564^{+282}_{-224} \text{ MeV}.$$

The conformal limit renormalization scheme parameters are then obtained from Eqs. (29) and (30), $Q_R^* = 1.64$ GeV and $\beta_2^* = -1.56$, and together with the conformal limit coefficients $c_1^* = -1$, $c_2^* = 1.165$, and $\Lambda_R^{(4)}$ this gives $\hat{a}_K = 0.159^{+0.139}_{-0.048}$. Finally the conformal limit result for Bjorken's sum rule given by Eq. (24) becomes

$$K = 0.176^{-0.029}_{+0.010},$$

where $|g_A/g_V| = 1.2573 \pm 0.0028$ [23] has been used and the error comes from the uncertainty in R_Q. This is in good agreement with the experimental value $K = 0.199 \pm 0.038$ measured by SMC. To be able to make a more challenging test of the conformal limit scheme arguments one would need much more precise measurements of both R_{e+e-} and K.

For illustration we also show the renormalization scheme dependence of Bjorken's sum rule in Figs. 1(c) and 1(d) and the numerical values of β_2 and Q_R for the specific schemes indicated in Fig. 1(d) are given in Table II together with the resulting values for $\hat{a}_K$ and Bjorken's sum rule K. Comparing with Figs. 1(a) and 1(b) we see that the scheme dependence is much stronger which is also expected since we are at a much smaller Q_K. We also see that the perturbative regime indicated by the dashed line in Fig. 1(d) is narrower than in Fig. 1(b) and in fact both the SBLM and MBLM methods as well as the R scheme are outside the perturbative regime. Therefore the numerical results given for these schemes should not be trusted. However, one must keep in mind that the SBLM and MBLM scale-fixing methods advocate the use of a physical measurement of $\hat{a}_R$ at this scale and in this way the problem with the validity of the solution used for α_s never enters. Even so, Figs. 1(c) and 1(d) illustrate clearly that there is a strong renormalization scheme dependence for Bjorken's sum rule at this scale which should be taken into account when comparing the experimental result with theoretical expectations.

VI. SUMMARY AND CONCLUSIONS

In summary we have shown that it is possible to generalize the conformal limit arguments of the BLM method to NNLO to fix the renormalization scheme, i.e., both the renormalization scale and β_2, when two observables are related. In this way all signs of scale breaking, i.e., $\beta \neq 0$, are confined into the running of the coupling and the coefficients in the perturbative series have their conformal limit values. We have also shown (contrary to [9]) that this prescription for making the coefficients have their conformal limit values is unique. Comparing with earlier extensions of the BLM method to NNLO they only fix the scale using β_2 from the effective charge. This means that the conformal-limit scheme-fixing method gives both a so-called commensurate scale relation as well as a commensurate β_2 relation which, in a unique way, specifies how β_2 should be modified when two physical observables are related to each other.

Applying the conformal-limit scheme-fixing method to the relation between Bjorken's sum rule K and R_{e+e-} gives a simple relation between the two. Using the effective charge value $\hat{a}_R = 0.0463 \pm 0.0044$ for $Q_R = 59.2$ GeV from a global analysis of R_{e+e-} data gives $K = 0.176^{-0.029}_{+0.010}$ for $Q_K^2 = 10$ GeV^2 where the error comes from the experimental uncertainty in $\hat{a}_R$. Assessing a theoretical error is much more complicated. The theoretical uncertainty is illustrated by the renormalization scheme dependence which is shown to be quite large even though it can be reduced by requiring the scheme to belong to the perturbative regime. Still, the problem of quantifying the theoretical error remains to be solved. However, comparing with the experimentally measured $K = 0.199 \pm 0.038$ the agreement is good and theoretical uncertainties are still smaller than the experimental ones.

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