Relativistic and binding energy corrections to direct photon production in Υ decay

Mohammad Ali Yusuf

Department of Physics, Quaid-e-Azam University, Islamabad, Pakistan

Pervez Hoodbhoy

Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
and Department of Physics, Quaid-e-Azam University, Islamabad, Pakistan

A systematic gauge-invariant method, which starts directly from QCD, is used to calculate the rate for an Υ meson to decay inclusively into a prompt photon. An expansion is made in the quark relative velocity v, which is a small natural parameter for heavy quark systems. Inclusion of these $O(v^2)$ corrections tends to increase the photon rate in the middle z range and to lower it for larger z, a feature supported by the data. [S0556-2821(96)01117-4]

INTRODUCTION

The hadronic decays of the Υ family of $b\bar{b}$ mesons proceeds mainly via an intermediate state consisting of three gluons. By replacing one of the outgoing gluons with a photon one obtains the leading-order contribution to the production of direct photons, i.e., those photons which do not result from π^0 decay, etc. The spectrum of such photons provides, in principle, an excellent test of perturbative quantum chromodynamics (QCD) because in this case one has a large number of data points against which theoretical predictions can be compared. This is in contrast with the prediction of a decay rate, which is a single number. However it is well known [1] that the photon spectrum $^3S_1 \rightarrow \gamma + X$, calculated at leading order [2], is too hard when compared against experiment, both in J/ψ and Υ decays. Such calculations yield an almost linearly rising spectrum in $z = 2E_\gamma/M$ with a sudden decrease at $z = 1$. A next-to-leading-order calculation by Photiadis [3] sums leading logs of the type $\ln(1-z)$ and yields some softening. However, the peak is still too sharp and close to $z = 1$. An earlier calculation by Field [4] predicts a much softer spectrum which fits the relatively recent data [5] quite well. This uses a parton-shower Monte Carlo approximation wherein the two gluons recoiling against the direct photon acquire a nonzero invariant mass by radiating further bremsstrahlung gluons. This does not, therefore, qualify it as an *ab initio* perturbative QCD calculation. We note that in [2–4] the non-perturbative dynamics of the decaying hadron is described by a single parameter $\phi(0)$, the quark wave function at the origin. This leads to the assertion that the ratio of widths for decay into prompt photons and l^+l^- pairs is independent of quark dynamics.

In this paper we compute the rate for $^3S_1 \rightarrow \gamma + X$ taking into account the bound-state structure of the decaying quarkonium state. We note that the description of hadron dynamics in this decay process by just $\phi(0)$ is correct *only* if one assumes that Q and $\bar{Q}$ are exactly on shell and at rest relative to each other. This assumption is only approximately true, heavy quarkonia are weakly bound $Q\bar{Q}$ composites and v^2/c^2 is a small parameter. Improvement requires introduction of additional hadronic quantities, which we identify

within the context of a systematically improvable gauge-invariant theory for quarkonium decays. This formalism has been recently applied to one- and two-particle decays [6,7]. Here we apply the method of [6] to the more complicated three particle case and obtain the photon spectrum for the process $\Upsilon \rightarrow \gamma + 2g$. We find that inclusion of binding and relativistic effects via the two additional parameters, ϵ_B/M and $\nabla^2\phi(0)/M^2\phi(0)$, makes the computed spectrum softer for large z ($z < 0.9$). For still larger z, $0.9 < z < 1$, there are nonperturbative effects due to final-state gluon interactions which cannot be reliably computed and which, therefore, we shall not address.

FORMALISM

The starting point of our approach is that the decay amplitude for $^3S_1 \rightarrow \gamma + X$ is given by the sum of all distinct Feynman diagrams leading from the initial to the final state. The first step is to write a given diagram in the form of a (multiple) loop integral. Consider, for example, one of the six leading-order diagrams [Fig. 1(a)]. Omitting color matrices and coupling constants for brevity, its contribution can be expressed as
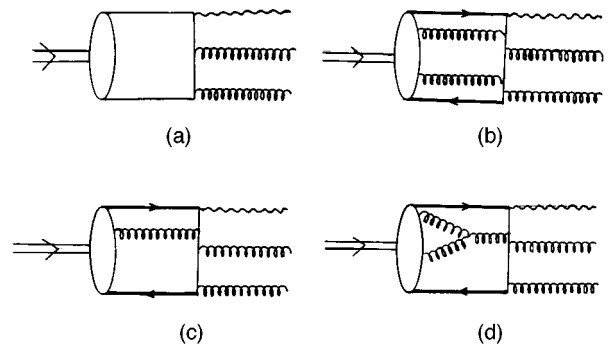

(a)   (b)

(c)   (d)

FIG. 1. (a) One of the six leading order diagrams. (b) One of the 12 one-gluon diagrams. (c) One of the 24 two-gluon diagrams. (d) One of the 12 gluon self-coupling diagrams.

$$T_{0[1(b)]}^{\mu_1\mu_2\mu_3} = \int \frac{d^4k}{(2\pi)^4} \, \text{Tr}[\gamma^{\mu_2} S_F(k+s_2) \gamma^{\mu_1}$$
$$\times S_F(k-s_3) \gamma^{\mu_3} M(k)]. \tag{1}$$

$M(k)$ is the usual, but obviously non-gauge-invariant zero-gluon, Bethe-Salpeter (BS) amplitude,

$$M(k) = \int d^4x e^{ik\cdot x} \langle 0| T[\bar{\psi}(-x/2)\psi(x/2)] |P,\epsilon\rangle. \tag{2}$$

In Eqs. (1) and (2) x^μ is the relative distance between quarks, k_μ is the relative momentum, $P^2 = M^2$, and $s_i = q_i - (1/2)P$. We define the binding energy as $\epsilon_B = 2m - M$. Provided all propagators are far off shell, they may be expanded in the two small quantities ϵ_B/M and k/M. This yields the expression

$$T_0^{\mu_1\mu_2\mu_3} = \text{Tr}[\langle 0|\bar{\psi}\psi|P,\epsilon\rangle h^{\mu_1\mu_2\mu_3}$$
$$+ \langle 0|\bar{\psi}i\overleftrightarrow{\partial}_\alpha\psi|P,\epsilon\rangle \partial^\alpha h^{\mu_1\mu_2\mu_3}$$
$$+ \langle 0|\bar{\psi}i\overleftrightarrow{\partial}_\alpha i\overleftrightarrow{\partial}_\beta\psi|P,\epsilon\rangle \tfrac{1}{2}\partial^\alpha\partial^\beta h^{\mu_1\mu_2\mu_3} + \cdots]. \tag{3}$$

We have defined $\overleftrightarrow{\partial}^\alpha = (1/2)(\overrightarrow{\partial}^\alpha - \overleftarrow{\partial}^\alpha)$, and $h^{\mu_1\mu_2\mu_3}$ is the "hard part" which combines terms from all six leading diagrams.[1] One can readily see that it is the sum of terms of the type in Eq. (1) corresponding to different permutations of indices and momenta. There are 12 one-gluon diagrams one of which is illustrated in Fig. 1(b), which must be added as corrections to the no-gluon amplitude. These all have the general form

$$T_1^{\mu_1\mu_2\mu_3} = \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \, \text{Tr} \, M^\rho(k,k') H_\rho^{\mu_1\mu_2\mu_3}(k,k'), \tag{4}$$

where $M_\rho(k,k')$ is a generalized BS amplitude,

$$M^\rho(k,k') = \int d^4x d^4z e^{ik\cdot x} e^{ik'\cdot z}$$
$$\times \langle 0|T[\bar{\psi}(-x/2)A^\rho(z)\psi(x/2)]|P,\epsilon\rangle. \tag{5}$$

The gluon which originates from the blob is part of the $Q\bar{Q}g$ Fock state component of the meson and has its momentum k' bounded by $R^{-1} \lesssim k' \ll M$, where R is the meson's spatial size. Hence it is to be considered soft on the scale of quark mass. Again, one may expand the propagators in $H_\rho^{\mu_1\mu_2\mu_3}(k,k')$ about $k = k' = 0$ to get

$$T_1^{\mu_1\mu_2\mu_3} = \text{Tr}[M^\rho H_\rho^{\mu_1\mu_2\mu_3} + M^{\rho,\alpha}\partial_\alpha H_\rho^{\mu_1\mu_2\mu_3}$$
$$+ M'^{\rho,\alpha}\partial'_\alpha H_\rho^{\mu_1\mu_2\mu_3} + \cdots], \tag{6}$$

where

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[1] In actual fact, only three of the diagrams need to be evaluated because of time-reversal symmetry. This simplification halves the number of diagrams in the one-gluon and two-gluon cases as well.

$$M^\rho = \langle 0|\bar{\psi}\psi A^\rho|P,\epsilon\rangle,$$

$$M^{\rho,\alpha} = \langle 0|\bar{\psi}i\overleftrightarrow{\partial}^\alpha\psi A^\rho|P,\epsilon\rangle,$$

$$M'^{\rho,\alpha} = \langle 0|\bar{\psi}\psi i\overleftrightarrow{\partial}^\alpha A^\rho|P,\epsilon\rangle. \tag{7}$$

The derivative $i\overleftrightarrow{\partial}^\alpha$ acts only upon the quark operators.

The two soft-gluon contribution [Figs. 1(c) and 1(d)] to the amplitude is handled similarly but is more complicated. The matrix elements which enter into $T_2^{\mu_1\mu_2\mu_3}(k)$ are of the type $\langle 0|\bar{\psi}\psi A^\rho A^\lambda|P,\epsilon\rangle$ and so on; we shall not list these explicitly here since the principle is rather clear. It is straightforward to show that in the sum $T_0 + T_1 + T_2$, all ∂'s combine with A's to yield covariant derivatives and/or field strength tensors:

$$(T_0 + T_1 + T_2)^{\mu_1\mu_2\mu_3} = \text{Tr}[\langle 0|\bar{\psi}\psi|P,\epsilon\rangle h^{\mu_1\mu_2\mu_3}$$
$$+ \langle 0|\bar{\psi}i\overleftrightarrow{D}_\alpha\psi|P,\epsilon\rangle \partial^\alpha h^{\mu_1\mu_2\mu_3}$$
$$+ \langle 0|\bar{\psi}i\overleftrightarrow{D}_\alpha i\overleftrightarrow{D}_\beta\psi|P,\epsilon\rangle$$
$$\times \tfrac{1}{2}\partial^\alpha\partial^\beta h^{\mu_1\mu_2\mu_3} + \langle 0|\bar{\psi}F^{\alpha\beta}\psi|P,\epsilon\rangle$$
$$\times \tfrac{1}{2}\partial'_\alpha H_\beta^{\mu_1\mu_2\mu_3} + \cdots]. \tag{8}$$

The above is a sum of terms, each of which is the product of a soft hadronic matrix element and a hard perturbative part.

To proceed, one can perform a Lorentz and CPT-invariant decomposition of each of the hadronic matrix elements in Eq. (10). This is somewhat complicated [8] and involves a large number of constants which characterize the hadron. Considerable simplification results from choosing the Coulomb gauge, together with the counting rules of Lepage et al. [9]. The upshot of using this analysis is that, in this particular gauge, the gluons contribute at $O(v^3)$ to the reaction $^3S_1 \rightarrow \gamma + X$, and hence can be ignored. Even this leaves us with too many parameters, and forces us to search for a dynamical theory for the 1^{--} quarkonium state. We shall assume, in common with many other authors, that the Bethe-Salpeter equation with an instantaneous kernel does provide an adequate description. This has been conveniently reviewed by Keung and Muzinich [10] and we adopt their notation.[2]

The momentum space BS amplitude $\chi(p)$ satisfies the homogeneous equation

$$\chi(p) = iG_0(P,p) \int \frac{d^4p'}{(2\pi)^4} K(P,p,p')\chi(p'), \tag{9}$$

which, after making the instantaneous approximation $K(P,p,p') = V(\vec{p},\vec{p}')$ and reduction to the nonrelativistic limit yields

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[2] We find the analysis of [10] to be wanting because it does not properly deal with the issue of gauge invariance of the meson state. Further, while the binding energy is taken into account, the wave-function corrections, which are essentially short-distance or relativistic effects, are not.

$$\chi(p) = \frac{M^{1/2}(M-2E)(E+m-\vec{p}\cdot\vec{\gamma})\not{\epsilon}(1-\gamma_0)(E+m-\vec{p}\cdot\vec{\gamma})\phi(|\vec{p}|)}{4E(E+m)(p^0+M/2-E+i\epsilon)(p^0-M/2+E-i\epsilon)}. \tag{10}$$

The scalar wave function $\phi(|\vec{p}|)$ is normalized to unity:

$$\int \frac{d^3p}{(2\pi)^3}\,|\phi(|\vec{p}|)|^2 = 1, \tag{11}$$

and

$$E = \sqrt{\vec{p}^2+m^2}. \tag{12}$$

Fourier transforming $\chi(p)$ to position space yields

$$\langle 0|\bar{\psi}(-x/2)\psi(x/2)|P\rangle$$

from which, by tracing with appropriate γ matrices, the coefficients below can be extracted. So finally, to $O(v^2)$, one has a rather simple result

$$\langle 0|\bar{\psi}\psi|P,\epsilon\rangle = \frac{1}{2}M^{1/2}\left(1+\frac{\nabla^2}{M^2}\right)\phi\left(1+\frac{\not{P}}{M}\right)\not{\epsilon}$$
$$-\frac{1}{2}M^{1/2}\frac{\nabla^2\phi}{3M^2}\left(1-\frac{\not{P}}{M}\right)\not{\epsilon},$$

$$\langle 0|\bar{\psi}i\overleftrightarrow{\partial}_\alpha\psi|P,\epsilon\rangle = -\frac{1}{3}M^{3/2}\frac{\nabla^2\phi}{M^2}\,\epsilon^\beta$$
$$\times\left[-g_{\alpha\beta}+i\epsilon_{\mu\nu\alpha\beta}\frac{P^\nu}{M}\gamma^\mu\gamma^5\right],$$

$$\langle 0|\bar{\psi}i\overleftrightarrow{\partial}_\alpha i\overleftrightarrow{\partial}_\beta\psi|P,\epsilon\rangle = \frac{1}{6}M^{5/2}\frac{\nabla^2\phi}{M^2}\left(g_{\alpha\beta}-\frac{P_\alpha P_\beta}{M^2}\right)$$
$$\times\left(1+\frac{\not{P}}{M}\right)\not{\epsilon}. \tag{13}$$

DECAY RATE

All the ingredients are now in place to calculate the decay $\Upsilon\to\gamma+2g$. In squaring the amplitude obtained by substituting Eqs. (13) into Eq. (8), terms involving the product of ϵ_B and $\nabla^2\phi$ may be neglected. We assume the emitted gluons to be massless and transverse, and to decay with unit probability into hadrons. Polarizations of the final-state particles are summed over, and the spin states of the initial meson are averaged over. Summing over final-state colors yields 2/3, and one must include a factor of 1/2 for identical gluons. The Lorentz-invariant phase-space factor for 3 massless particles has a standard expression [13] which is best expressed in terms of the dimensionless energy fractions $x_i=2E_i/M$ which satisfy $x_1+x_2+x_3=2$. The variables s,t,u are symmetric functions of x_i:

$$s = (P-q_1)^2 = M^2(1-x_1). \tag{14}$$

Ignoring radiative corrections for the moment, a tedious calculation[3] yields

$$\frac{d^2\Gamma}{dx_1 dx_2} = \frac{256}{9}e_q^2\alpha_s^2\alpha_e\frac{|\phi(0)|^2}{M^2}[\eta_0 f_0(s,t,u)+\eta_B f_B(s,t,u)$$
$$+\eta_W f_W(s,t,u)]. \tag{15}$$

e_q is the quark charge and

$$\eta_0 = 1, \quad \eta_B = \frac{\epsilon_B}{M}, \quad \eta_W = \frac{\nabla^2\phi}{M^2\phi}. \tag{16}$$

The function f_0 provides the standard, leading-order result:

$$f_0(s,t,u) = \frac{M^4(s^2t^2+t^2u^2+u^2s^2+M^2stu)}{(s-M^2)^2(t-M^2)^2(u-M^2)^2}. \tag{17}$$

The binding energy and wave-function corrections, f_B and f_W, respectively, are more complicated:

$$f_B(s,t,u) = \frac{M^4}{2D}[-7stu(s^4+t^4+u^4)+7M^2(s^3t^3+t^3u^3+u^3s^3)+(s^2t^2+t^2u^2+u^2s^2)(s^3+t^3+u^3+15stu)$$
$$+M^2stu(s^3+t^3+u^3)+29M^2s^2t^2u^2],$$

$$f_W(s,t,u) = \frac{M^4}{3D}\left[141stu(s^4+t^4+u^4)-85M^2(s^3t^3+t^3u^3+u^3s^3)-27(s^2t^2+t^2u^2+u^2s^2)\left(s^3+t^3+u^3+\frac{205}{27}stu\right)\right.$$
$$\left.-139M^2stu(s^3+t^3+u^3)-463M^2s^2t^2u^2\right]. \tag{18}$$

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[3]We used MATHEMATICA [12], supplemented by the HIP package [14], for computation of traces and simplification of algebra.

The denominator D is

$$D=(s-M^2)^3(t-M^2)^3(u-M^2)^3. \tag{19}$$

Integrating over the energies of the outgoing gluons for a fixed photon energy yields

$$\frac{d\Gamma}{dz}=\frac{256}{9}e_q^2\alpha_e\alpha_s^2\frac{|\phi(0)|^2}{M^2}[\eta_0 F_0(z)+\eta_B F_B(z)+F_W(z)], \tag{20}$$

where $z=2E_\gamma/M$ and

$$F_0=[1+4\xi-2\xi^3-\xi^4-2\xi^5+2\xi(1+2\xi+5\xi^2)\ln\xi]/(1-\xi)^2(1+\xi)^3,$$

$$F_B=[2-16\xi+10\xi^2-48\xi^3-10\xi^4+64\xi^5-2\xi^6+(1-3\xi+14\xi^2-106\xi^3+17\xi^4-51\xi^5)\ln\xi]/2(1-\xi)^3(1+\xi)^4,$$

$$F_W=[-26+14\xi-210\xi^2+134\xi^3+274\xi^4-150\xi^5-38\xi^6+2\xi^7-(27+50\xi+257\xi^2-292\xi^3+205\xi^4-78\xi^5-41\xi^6)$$
$$\times\ln\xi]/3(1-\xi)^3(1+\xi)^5. \tag{21}$$

In the above, $\xi=1-z$. The integrated decay width is[4]

$$\Gamma_{1^{--}\to\gamma+2g}=\frac{128}{9}(\pi^2-9)e_q^2\alpha_e\alpha_s^2\frac{|\phi(0)|^2}{M^2}$$
$$\times\left(1+a\frac{\alpha_s}{\pi}-1.03\eta_B+19.34\eta_W\right). \tag{22}$$

We have included the radiative corrections of $O(\alpha_s)$ which are of the same order in v^2/c^2 as the other corrections, and were calculated [11] many years ago:

$$a=\beta_0\ln(\mu/m_Q)-4.37-0.77n_f, \tag{23}$$

where $\beta_0=11-2n_f/3$. The parameters η_W and η_B are independent of each other in the present treatment. We note, however, that if we impose the condition $\eta_W=(1/2)\eta_B$ then the result, Eq. (3.5) of Keung and Muzinich [10], is precisely recovered. This latter condition is equivalent to $(1/M)\nabla^2\phi(0)=(1/2)\epsilon_B\phi(0)$, which is the Schrödinger equation for quark relative motion in a potential which vanishes at zero relative separation. It is also worthy of note that the same condition emerges as a renormalization condition in the treatment of positronium by Labelle *et al.* [15] [see their equations (11) and (12)]. However in our treatment there is

no principle which *a priori* constrains η_B to bear a fixed relation to η_W and therefore both will be considered adjustable parameters.

The application of Eq. (22) must be done with caution because extraction of the direct photon decay rate from the data requires an extrapolation down to small photon energies. But in this energy range the prompt photons are heavily contaminated by photons from π^0 decays. A numerical estimate of the correction factors requires the value of η_B and η_W. We have chosen $m_b=4.5$ which gives $\eta_B=-0.048$. If we take $\alpha_S=0.20$ then η_W can be fixed by using the experimentally known numbers [17]:

$$\Gamma(\Upsilon\to 2g+\gamma)=1.28\pm0.10\ \text{keV},$$

$$\Gamma(\Upsilon\to l\bar{l})=1.34\pm0.04\ \text{keV}. \tag{24}$$

This gives a range of values for η_W. We have plotted the graphs in Fig. 2 at $\eta_W=-0.0059$. The binding $F_B(z)$ and wave function $F_W(z)$ correction terms tend to cancel each other over part of the z region. The effect of final-state interaction corrections can be reasonably well estimated [3] provided one stays away from the end point $z=1$. In Fig. 2 we compare the data, taken from [5], with the prediction of our model appropriately folded with the experimental photon energy resolution (assumed to be Gaussian). The effect of the binding and wave function corrections calculated in this work is sizable, and tends to increase the photon rate in the middle z range and to lower it for larger z. While this appears to be in the right direction, it would be highly desirable to have more precise data.

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[4]Note that Eq. (22) does not take into account nonperturbative effects [16] which are significant in the part of the phase space where one of the quark propagators becomes soft, and where the gluon vacuum condensate plays a role.
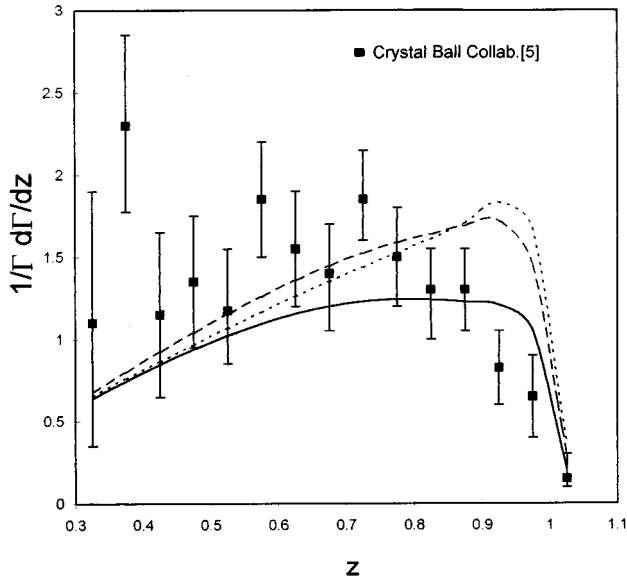
FIG. 2. The photon spectrum as a function of z, folded with the experimental photon energy resolution. The dotted line is the zeroth-order QCD result, the dashed line incorporates the binding and wave-function corrections, with $\eta_B = -0.048$ and $\eta_W = +0.0059$. The solid line is the final result including final-state interaction of [3].

SUMMARY

The approach taken in this paper for calculating the amplitude for $\Upsilon \rightarrow \gamma + X$ is to take the sum of all distinct Feynman diagrams leading from the initial quarkonium state to the final state. Each diagram is put into the form of a (multiple) loop integral with a kernel which is a product of a hard part and a soft part. The hard part is treated with perturbative QCD, and the soft part is analyzed into its different components with the use of Lorentz, $\mathscr{C}$, and $\mathscr{P}$ symmetries. Use of the QCD equations of motion enables separation of these components according to their importance in powers of v. At the last step, a specific commitment to dynamics is made and the BS equation is used to express the components in the form of wave functions. However, the unregularized value of $\nabla^2 \phi(0)$ is singular at the origin $\nabla^2 \phi(0) \sim M \phi(0)/r$. As is clear from the uncertainty principle, the local kinetic energy becomes very large at short distances and the expansion in powers of v breaks down. This difficulty was circumvented by imagining that annihilation takes place in a diffused region of size $O(1/m)$, i.e., that $\phi(0)$ and $\nabla^2 \phi(0)$ are quantities renormalized at this scale, and to be considered as adjustable parameters. The numerical investigation we undertook showed that varying these within reasonable limits led to substantial improvement in the intermediate z region but was insufficient to reproduce the data near $z=1$, once again underscoring the importance of final-state interactions between collinear gluons.

[1] For a review, see Gerhard A. Schuler, ''Quarkonium Production And Decays,'' CERN Report No. CERN-TH. 7170/94 (unpublished).

[2] S. Brodsky, G. P. Lepage, and P. B. Mackenzie, Phys. Rev. D **28**, 228 (1983).

[3] Douglas M. Photiadis, Phys. Lett. **164B**, 160 (1985).

[4] R. D. Field, Phys. Lett. **133B**, 248 (1983).

[5] Crystal Ball Collaboration, A. Bizzeti *et al.*, Phys. Lett. B **267**, 286 (1991).

[6] Hafsa Khan and Pervez Hoodbhoy, Phys. Rev. D **53**, 2534 (1996).

[7] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995).

[8] M. A. Yusuf, Ph.D. thesis, Quaid-i-Azam University, Islamabad, Pakistan (in preparation).

[9] G. P. Lepage, L. Magnea, C. Nakleh, U. Magnea, and K. Hornbostel, Phys. Rev. D **46**, 4052 (1992).

[10] W. Y. Keung and I. J. Muzinich, Phys. Rev. D **27**, 1518 (1983).

[11] P. B. Mackenzie and G. P. Lepage, Phys. Rev. Lett. **47**, 1244 (1981); W. Kwong, P. B. Mackenzie, R. Rosenfield, and J. L. Rosner, Phys. Rev. D **37**, 3210 (1988).

[12] S. Wolfram, ''MATHEMATICA-A System for Doing Mathematics by Computer,'' 2nd ed. (Addison-Wesley, Reading, MA, 1991).

[13] C. Itzykson and J. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 237.

[14] A. Hsieh and E. Yehudai, Comput. Phys. **6**, 253 (1992).

[15] P. Labelle, G. P. Lepage, and U. Magnea, Phys. Rev. Lett. **72**, 2006 (1994).

[16] M. B. Voloshin, Sov. J. Nucl. Phys. **40**, 662 (1984).

[17] Particle Data Group, L. Montanet *et al.*, Phys. Rev. D **50**, 1173 (1994).