Lowest order hadronic contribution to the muon $g-2$ value with systematic error correlations

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We have performed a new evaluation of the hadronic contribution to $a_\mu=(g-2)/2$ of the muon with explicit correlations of systematic errors among the experimental data on $\sigma(e^+e^-\to\text{hadrons})$. Our result for the lowest order hadronic vacuum polarization contribution is $a_\mu^{\text{had}}=702.6(7.8)(14.0)\times10^{-10}$ where the first error is statistical and the second is systematic. The total systematic error contributions from below and above $\sqrt{s}=1.4$ GeV are $(13.1)\times10^{-10}$ and $(5.1)\times10^{-10}$, respectively, and are hence dominated by the low energy region. Therefore, new measurements on $\sigma(e^+e^-\to\text{hadrons})$ below 1.4 GeV can significantly reduce the total error on a_μ^{had}. In particular, the effect on the total errors of new hypothetical data with 3% statistical and 0.5–1.0 % systematic errors is presented. [S0556-2821(96)03117-7]

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I. ROLE OF HADRONIC CONTRIBUTION IN $g-2$

A new measurement of the anomalous magnetic moment of the muon, $a_\mu\equiv(g-2)/2$, to an absolute accuracy of $\sigma_{\text{expt}}^{a_\mu}\sim\pm4.0\times10^{-10}$ is proposed by the Brookhaven National Laboratory (BNL) E821 Collaboration [1,2]. The theoretical value of the muon $g-2$ value consists of at least the three standard model contributions: quantum electrodynamics (QED), electroweak (EW), and hadronic. The latter arise from hadronic vacuum polarization effects caused by effective photon couplings to hadrons via charged quarks and consequent quantum chromodynamics (QCD) interactions with gluons (see Fig. 1).

Any residual difference between the sum of the standard model contributions and the new experimental value a_μ^{expt} will be indicative of new physics:

$$a_\mu^{\text{residual}}=a_\mu^{\text{expt}}-a_\mu^{\text{QED}}-a_\mu^{\text{EW}}-a_\mu^{\text{had}}.$$

The new experimental value can only be sensitive to electroweak and possibly supergravity [3,4] and muon substructure effects [5] provided the errors on the standard model contributions are known better than the experimental accuracy. The QED and EW contributions have been calculated from theory and are known an order of magnitude better than the expected experimental accuracy ($a_\mu^{\text{QED}}=11658470.6\pm0.2\times10^{-10}$ [6] and $a_\mu^{\text{EW}}=15.1\pm0.4\times10^{-10}$ [7]). At the same time, the lowest order hadronic contribution cannot be calculated accurately enough by QCD and hence a phenomenological procedure must be used for its calculation. While the results of the published a_μ^{had} calculations have errors which vary from just above E821 error to greater than the electroweak contribution (see Table I) and even differ in principle of approach, it is of interest for the interpretation of a new experimental measurement of the muon $g-2$ value to investigate in more detail the precise procedure used for calculation of the hadronic contribution total error.

Fortunately the hadronic contribution to the muon $g-2$ value can be related to a dispersion integral over the experimental cross section $\sigma(e^+e^-\to\text{hadrons})$ (see Sec. II). The error on the hadronic contribution to a_μ is hence determined by the error on the experimental cross section and is by all acounts larger than the expected E821 accuracy. Therefore, the hadronic contribution error will largely determine the sensitivity level of E821 to new physics:

$$\sigma_{\text{residual}}=\sqrt{\sigma_{\text{expt}}^2+\sigma_{\text{QED}}^2+\sigma_{\text{EW}}^2+\sigma_{\text{had}}^2}\sim\sqrt{\sigma_{\text{expt}}^2+\sigma_{\text{had}}^2}.$$

This is the motivation for several recent evaluations of the lowest order hadronic contribution to a_μ which are presented in Table I. These calculations may roughly be placed into two categories of approach whether they are based primarily upon (aggressive) model dependent techniques which yield relatively smaller errors [8–10] or on (conservative) model independent techniques (trapezoidal integration) which yield relatively larger errors [11,12]. A typical method for evalu-

TABLE I. History of $a_\mu^{\text{Fig. 1}}$ calculations. The values 4 and 15×10^{-10} are the BNL E821 error goal and electroweak contribution to a_μ, respectively.

Calculations based on experimental data			
Author(s)	Year	$10^{10}a_\mu^{\text{Fig. 1}}$(stat)(syst)	Ref.
Budker Institute	1985	684(11)	[8]
Kinoshita, Nizic, Okamoto	1985	707(6)(17)	[11]
Dubnickova, Dubnicka, Stricinec	1992	699(4)(2)	[9]
Eidelman, Jegerlehner	1995	702(6)(14)	[12]
Apel, Yndurain	1995	710(11)	[10]
WFSA evaluation	1996	703(8)(14)	

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ating systematic error is comparison of the model-dependent calculation with one based on trapezoidal integration.

While the model dependent calculations are based on theoretical innovations to represent various components of the cross section $\sigma(e^+e^- \to \text{hadrons})$ they lack the merit of a clear prescription for correlating the systematic errors of the experimental input. These errors are in some cases as large as 20% in overall normalization of the cross section. Therefore, the χ^2 criterion used to select a set of fit parameters, although indicative of the best overall fit to the data, may perhaps not indicate the true error on the integral over experimental data points which is the hadronic contribution to muon $g-2$. In particular, a shift of the whole curve up or down may fall within the experimental data point errors and yet yield a variation in a_μ^{had} larger than the stated errors.

The model-independent techniques for calculating a_μ^{had} are based upon trapezoidal integration over the experimental data points. As will be described below, this approach allows for a definite procedure for determining both the central value for a_μ^{had} and its errors both of which make explicit use of the statistical and systematic errors for each experimental data point input. In particular, for each experiment an error weighted fraction will be defined and used in averaging over different experiments which make measurements of the same cross sections over common energy regions.

The most recent model independent analysis [12] was the first to account for systematic error correlations among the experimental data used in the a_μ^{had} calculation. The systematic errors were correlated at each energy point separately in the process of determining an error-weighted average cross section ratio $R(s)$ (see Sec. II) *before* performing the $g-2$ integral. Special care was taken to calculate the ρ meson contribution separately from the rest of the hadronic production cross sections and there are many significant contributions to the discussion of a_μ^{had} calculations in this important communication.

The main new aspect of the present a_μ^{had} calculation is that the systematic errors are correlated by a different error combination formula *after* performing the $g-2$ integral. Furthermore, correlations are accounted for across energy regions and between experimental measurements of the ρ meson and of the additional hadronic cross sections which contribute to $\sigma(e^+e^- \to \text{hadrons})$. (It appears that by treating the ρ meson separately the previous calculation did not correlate some errors completely.)

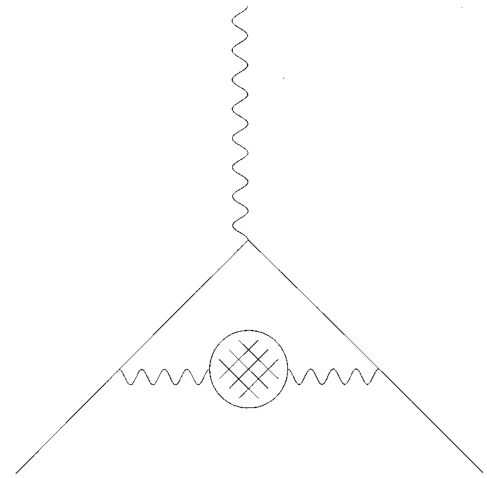

FIG. 1. Lowest order hadronic contribution to a_μ^{had}.

In short, the $g-2$ integration has been performed over each hadronic cross section separately without combining them into their equivalent $R(s)$ value for the present calculation. The focus is therefore upon the error-weighted average of the a_μ^{had} integral value itself (and its errors). These innovations have yielded a result with total error remarkably equal to that obtained previously [12]. However, there is a difference obtained as to how much of the error derives from the energy region above and below 1.4 GeV which calls attention to the precise procedure of error combination employed in the calculation.

II. BACKGROUND ON a_μ^{had} CALCULATIONS

The hadronic contribution to a_μ consists of a dominant lowest order term, shown in Fig. 1, several higher order terms in Fig. 2 (the number above each diagram indicates how many contributing diagrams are in its class), and finally a group of hadronic light-by-light scattering terms (not shown). Detailed calculations are given elsewhere of the Fig. 2 and hadronic light-by-light contributions. Since their errors are well below the expected E821 error of $\sigma_{\text{expt}}^{a_\mu}$

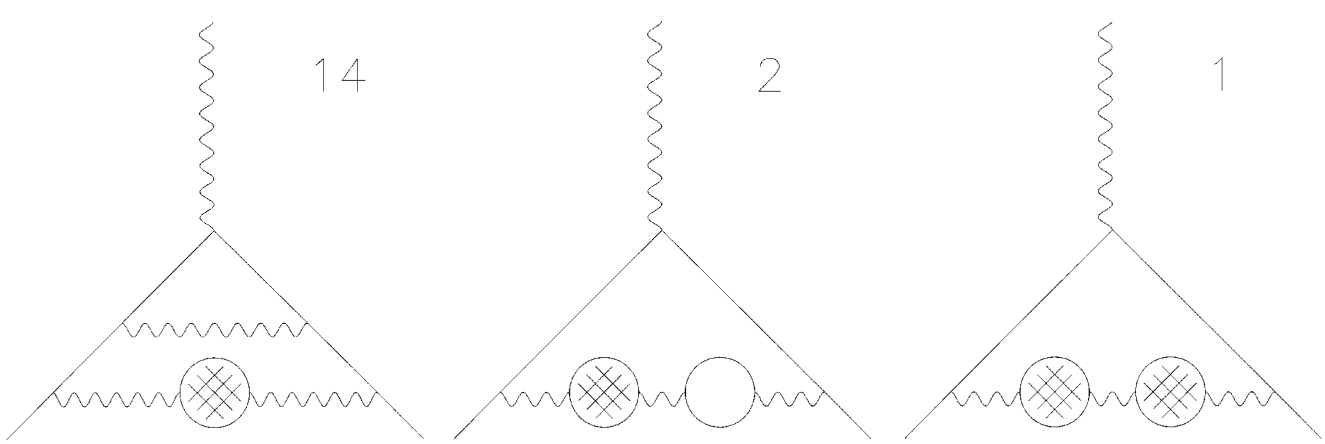

FIG. 2. Higher order hadronic contributions to a_μ^{had}.

$\sim \pm 4.0\times 10^{-10}$ ($a_\mu^{\text{Fig. 2}} = -9.0\pm 0.5\times 10^{-10}$ [11] and $a_\mu^{\text{light}} = -5.2\pm 1.8\times 10^{-10}$ [13]) they are not discussed here. There is another hadronic light-by-light calculation with a larger error $a_\mu^{\text{light}} = -11\pm 5\times 10^{-10}$ [14] but it does not depend on the cross section $\sigma(e^+e^-\to\text{hadrons})$ and therefore is not of concern to the present calculation. In view of the difference between the two latest hadronic light-by-light contributions, and the consequent ambiguity over defining the *total* hadronic contribution, this paper is concerned only with the lowest order hadronic contribution to the muon $g-2$ value shown in Fig. 1.

Formalism of the hadronic contribution a_μ^{had}

The largest contribution to a_μ^{had} shown in Fig. 1, can be related to the total Born cross section (lowest order in QED) for hadron production in electron-positron annihilations, $\sigma^0_{\text{had}} = \sigma^0(e^+e^-\to\text{hadrons})$, by means of dispersion theory and the optical theorem [15]. Defining $\xi \equiv s/m_\mu^2$ and $\beta \equiv \sqrt{1-4/\xi}$ the result is

$$a_\mu^{\text{had}} = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty} ds\,\sigma^0_{\text{had}} K(s) \tag{1}$$

$$= \frac{m_\mu^2}{9}\left(\frac{\alpha}{\pi}\right)^2 \int_{4m_\pi^2}^{\infty} ds\,R(s) K_2(s)\frac{1}{s^2},$$

where the kernel function $K(s)$ in general arises from a massive photon propagator in the $g-2$ Schwinger calculation:

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)\xi} \cong \frac{m_\mu^2}{3s}, \quad s\to\infty,$$

$$K(s) = \frac{1}{2\beta}\left[\frac{(1-\beta)^2}{1+\beta}\left(1-\frac{2}{1+\beta}\ln\frac{2}{1-\beta}\right) - \frac{(1+\beta)^2}{1-\beta}\left(1-\frac{2}{1-\beta}\ln\frac{2}{1+\beta}\right)\right] - \frac{1}{2}. \tag{2}$$

The cross section ratio $R(s) \equiv \sigma^0_{\text{had}}/\sigma^0_{\mu\mu}$ with $\sigma^0_{\mu\mu} = 4\pi\alpha^2/3s$ and the kernel $K_2(s)$ is used in the $R(s)$ formulation of the a_μ^{had} dispersion integral:

$$K_2(s) \equiv \frac{3s}{m_\mu^2}K(s).$$

The σ_{had} formulation is useful for low energy data which are usually published as individual exclusive hadronic cross sections; the $R(s)$ form of the a_μ^{had} dispersion integral is useful for higher energies where experimental data are usually published as the inclusive ratio R.

From the a_μ^{had} dispersion integrals it is apparent that the error on the hadronic cross sections determines the error on a_μ^{had} Therefore, the AGS E821 experimental error goal just larger than 3.5×10^{-10} on the hadronic contribution of roughly 700×10^{-10} requires a 0.5% accuracy on the hadronic contribution calculation.

III. THE WFSA EVALUATION PROCEDURE

A. Correlation postulates

As is implied by the $g-2$ dispersion integral [Eq. (1)] the a_μ^{had} calculation procedure consists of some combination of the following steps: (1) integration over energy, (2) weighted average over detectors, and (3) sum over exclusive hadronic modes which contribute to $\sigma^0(e^+e^-\to\text{hadrons})$. The sequence of these three sums may be interchanged as for example performing the energy integration last [12]. Most calculations, however, employ the sequence (1,2,3)—the most natural one—where in the last step, the quadrature sum of errors over modes implies the assumption of zero correlation of systematic errors among experiments (a reasonable first approximation).

To go beyond the first approximation, it is necessary to survey the existing experimental data on hadronic production cross sections. The published data and recent preprints on exclusive hadron production in electron-positron annihilations used in the present calculation are listed in Table XIII where it is evident that eight detectors have measured more than one mode. Since a given detector uses the same luminosity and similar correction factor calculations (e.g., radiative corrections, efficiencies which use some of the same subroutines) it is reasonable to suppose that the cross section determinations of different exclusive hadronic modes by a single detector may in fact be correlated. The following correlation postulates are therefore intended to address this situation.

(1) A single detector measuring more than one exclusive hadronic mode has 100% correlations among systematic errors due to common luminosity and correction factor calculations.

(2) Different detectors have uncorrelated errors since they do not share luminosity and correction factor calculations.

The accomodation of these correlation postulates requires a particular sequence of the three sums (1,2,3) for combination of the a_μ^{had} central values and an alternative sequence (1,3,2) for the a_μ^{had} errors. In both cases, the energy is integrated over first and separately for each detector and exclusive hadronic mode measured over energy subregions, where these subregions are defined by common energy coverage among detectors.

The former sequence (1,2,3) facilitates the need to first calculate the error weighted fractions (defined in detail below) while the latter sequence (1,3,2) is necessary for, according to postulate 1, correlating individual detector systematic errors over the modes measured by that detector. This must be done before the final uncorrelated combination of errors is made, according to postulate 2, across detectors. As this method is based on *weighted fraction* averaging with S factor application [see Eq. (5)] *after* a_μ^{had} integration over the energy, it is here referred to as the WFSA method.

Lastly, in the present calculation it is noted that exclusive hadronic cross sections (modes) up to 2.0 GeV have been used because this can reveal the propagation of errors from each detector and exclusive mode separately. An additional consideration, although less important, is that the exclusive hadronic mode spectra may contain interference effects (when more than one vector meson contributes to a given exclusive hadronic mode) which would otherwise require special care if individual vector meson contributions were calculated separately. Further, the uncertainty over the generation mechanism, between the e^+e^- annihilation photon and the hadronic final state, is avoided by focusing on exclusive hadronic modes themselves.

For the energy region 2.0 to 3.1 GeV, the inclusive cross section ratio $R(s)$ has been used (in the absence of exclusive data in this energy region) with the WFSA procedure for the contributing experiments. In the region above 3.1 GeV, the QCD expression has been used without the WFSA procedure since perturbative QCD is expected to be valid (see Sec. IV D).

B. Trapezoidal integration procedure

The usual trapezoidal integration technique takes the experimental data points pairwise: the cross sections, systematic and squared statistical errors are averaged per pair, then multiplied by the energy width of the pair, and finally they are summed over all pairs. However, it is convenient for treatment of statistical errors to expand the sum in order to remove terms which cancel. Denoting by s_k, K_k, c_{ijk}, $\sigma_{ijk}^{\rm stat}$ and $\sigma_{ijk}^{\rm syst}$ the energy, kernel function, the cross section and its statistical and systematic error from the ith detector, jth exclusive mode, at the kth energy point, the integration of Eq. (1) can be represented by

$$a_{ij}=\frac{1}{4\pi^3}\sum_{k=1}^{n-1}\left\{\frac{1}{2}(c_{ijk}K_k+c_{ij,k+1}K_{k+1})\right\}(s_{k+1}-s_k)$$

$$=\frac{1}{4\pi^3}\frac{1}{2}\left(A_1+\sum_{k=2}^{n-1}A_k+A_n\right),$$

$$A_1=c_{ij,1}K_1(s_2-s_1),$$

$$A_k=c_{ij,k}K_k(s_{k+1}-s_{k-1}),$$

$$A_n=c_{ij,n}K_n(s_n-s_{n-1}),$$

where the first and last terms in the sum are handled separately and the middle terms have an energy width across both upper and lower neighboring data points instead of across the points in pairs. This latter form is necessary for the proper treatment of the statistical errors:

$$\sigma_{ij}^{\rm stat}=\frac{1}{4\pi^3}\frac{1}{2}\sqrt{\sigma_1{}^2+\sum_{k=2}^{n-1}\sigma_k{}^2+\sigma_n{}^2},$$

$$\sigma_1^2=(\sigma_{ij,1}^{\rm stat}K_1)^2(s_2-s_1)^2,$$

$$\sigma_k^2=(\sigma_{ij,k}^{\rm stat}K_k)^2(s_{k+1}-s_{k-1})^2,$$

$$\sigma_n^2=(\sigma_{ij,n}^{\rm stat}K_n)^2(s_n-s_{n-1})^2.$$

The systematic errors are specified by an array of values $p_{ijk}^{\rm syst}$ which are given as a percentage of the total cross section. Since the $a_\mu^{\rm had}$ contribution terms above are linear in the cross section this implies that the systematic errors should be

TABLE II. Systematic errors used in the WFSA evaluation of the $\pi^+\pi^-$ contribution to $a_\mu^{\rm had}$.

Detector	Systematic error
NA7	0.01,0.02,0.025,0.05
CMD	0.02
TOF	0.035[a]
DM1	0.022
Olya	0.04 – 0.15
M2N	0.0[b]
BCF	0.1[c]
$\mu\pi$	0.1[c]
MEA	0.1[c]
DM2	0.12

[a]Evaluated from errors on radiative correction or efficiency factors.
[b]Systematic error is included with statistical in quoted errors.
[c]Arbitrary value—inconsequential since statistical errors are large.

$$\sigma_{ij}^{\rm syst}=p_{ij,1}^{\rm syst}A_1+\sum_{k=2}^{n-1}p_{ij,k}^{\rm syst}A_k+p_{ij,n}^{\rm syst}A_n.$$

Usually the systematic error is the same for all energy points; however, it is different for each point of the NA7 and Olya $\pi^+\pi^-$ measurements. In the cases where no systematic errors were given or readily located in the literature, values from comparable measurements with the same detector have been taken wherever possible, or as 10% if the statistical errors dominate. In addition, a calculation with 20% for these ambiguous values has shown that the WFSA results do not depend on the selection of 10% [16]. All of the systematic errors that were used are listed in Table II for $\pi^+\pi^-$, Table III for $\pi^+\pi^-\pi^0$, Table IV for the higher multiplicity modes, and Table V for the energy region 2.0 – 3.1 GeV.

If the desired limits of integration are inside (outside) the given data energy range, the cross section and error are linearly interpolated (extrapolated) to give the relevant pair of points. Thus our energy ranges are variable and have been set to match each of the previous evaluations for detailed comparisons [16].

C. Weighted fraction averaging

In order to arrive at an $a_\mu^{\rm had}$ contribution per exclusive mode, $a_\mu^{\rm mode}\equiv a_j$, error-weighted averages across detectors

TABLE III. Systematic errors used in the WFSA evaluation of the $\pi^+\pi^-\pi^0$ contribution to $a_\mu^{\rm had}$.

Detector	Energy range [GeV]	Systematic error
CMD	0.76 – 0.81	0.0[a]
CMD	0.84 – 1.013	0.07
ND	0.414 – 0.765	0.1
ND	0.805 – 1.003	0.1
ND	1.036 – 1.379	0.2
DM1	0.414 – 1.098	0.032
DM2	1.34 – 2.0	0.0866
$\gamma\gamma 2$	1.437 – 2.0	0.15

[a]Systematic errors are less than quoted statistical errors.

TABLE IV. Systematic errors used in the WFSA evaluation of higher multiplicity exclusive hadronic mode $a_\mu^{\rm had}$ contribution.

Mode	Olya	CMD	ND	ARGUS	DM1	DM2	$\gamma\gamma2$
K^+K^-	0.1[a]	0.038	0.1			0.1[a]	
K_LK_S	0.3	0.1	0.1		0.1[a]		
$\pi^+\pi^-\pi^+\pi^-$	0.15	0.1	0.1		0.1	0.1[a]	
$\pi^+\pi^-\pi^0\pi^0$	0.15		0.15			0.1	0.15
$\pi^+\pi^-\pi^+\pi^-\pi^0$		0.20			0.12		0.15
$K^+K^-\pi^+\pi^-$					0.13	0.1[a]	
$K_SK^\pm\pi^\mp$					0.0	0.1[a]	
$K^*K^\pm\pi^\mp$						0.1[a]	
$\pi^+\pi^-\pi^+\pi^-\pi^+\pi^-$					0.28		
$\pi^+\pi^-\pi^+\pi^-\pi^0\pi^0$							0.15
$p\overline{p}$					0.1[a]		
$\omega\pi$			0.1	0.1			
$\omega\pi\pi$					0.12	0.082	
$\eta\pi\pi$						0.1[a]	

[a]Systematic error not discussed in reference.

have been performed according to the Particle Data Group (PDG) [17] procedure (for the ith detector and jth exclusive hadronic mode)

$$a_j \pm \sigma_j = \frac{\Sigma_i w_{ij} a_{ij}}{\Sigma_i w_{ij}} \pm \left(\frac{1}{\Sigma_i w_{ij}}\right)^{1/2}, \tag{3}$$

where

$$w_{ij} = \frac{1}{\sigma_{ij}^2}, \quad \sigma_{ij}^2 = (\sigma_{ij}^{\rm stat})^2 + (\sigma_{ij}^{\rm syst})^2. \tag{4}$$

In addition, the quality of the error weighted combinations were assessed by calculating the χ^2 and PDG scale factor (i.e. χ^2 per degree of freedom) [17]:

$$\chi_j^2 = \sum_i^N w_{ij}(a_j - a_{ij})^2,$$

$$S_j \equiv \sqrt{\frac{\chi_j^2}{N-1}}, \tag{5}$$

where N is the number of detectors included in the average. If $S>1$ then the errors were scaled up by this factor.

At this point the prescription for determining the error-weighted fractional contribution to the total error from a given statistical or systematic error of the ith detector and jth exclusive mode is needed. For this purpose the PDG expression above for total squared error on a weighted average [see Eq. (3)] can be expanded as a sum over squared component errors where in the first step the trivial sum $\Sigma_{i=1}^N = N$ is used to multiply by unity:

$$\sigma_j{}^2 = \frac{1}{\Sigma_i w_{ij}} = \frac{1}{\Sigma_i w_{ij}}\left(\frac{1}{N}\sum_{i=1}^N \frac{\sigma_{ij}^2}{\sigma_{ij}^2}\right) = \sum_{i=1}^N \frac{1}{N}\frac{w_{ij}}{\Sigma_i w_{ij}}\sigma_{ij}^2 .$$

The last step makes use of the definition $w_{ij} = 1/\sigma_{ij}^2$ while the definition for the remaining σ_{ij}^2 in the numerator [see Eq. (4)] leads to the separation of squared statistical from systematic terms in the sum

$$\sigma_j{}^2 = \sum_{i=1}^N \left[\left(\overline{\sigma_{ij}^{\rm stat}}\right)^2 + \left(\overline{\sigma_{ij}^{\rm syst}}\right)^2\right],$$

$$\overline{\sigma_{ij}^{\rm stat}} = \sqrt{\frac{1}{N}\frac{w_{ij}}{\Sigma_i w_{ij}}} S \sigma_{ij}^{\rm stat}, \tag{6}$$

$$\overline{\sigma_{ij}^{\rm syst}} = \sqrt{\frac{1}{N}\frac{w_{ij}}{\Sigma_i w_{ij}}} S \sigma_{ij}^{\rm syst}. \tag{7}$$

Note that the PDG scale factor S has been inserted to emphasize the fact that the errors are to be increased if and only if the scale factor $S>1$. (These expressions differ from those in the previous $a_\mu^{\rm had}$ calculation [12] by the factors $1/N$ and S.)

Armed with these expressions it is possible to implement the correlation postulates for the ith detector by summing the *averaged* systematic errors [i.e., the weighted fractional systematic error contributions to the total errors in Eq. (7)] linearly over contributing exclusive modes, while leaving weighted fractional statistical errors [Eq. (6)] uncorrelated. The correlation postulates are simply executed by the sums

$$\overline{\sigma_i^{\rm syst}} = \sum_j \overline{\sigma_{ij}^{\rm syst}}, \tag{8}$$

$$\overline{\sigma_i^{\rm stat}} = \sqrt{\sum_j \left(\overline{\sigma_{ij}^{\rm stat}}\right)^2}. \tag{9}$$

Note that σ_j itself is not directly used to determine the total WFSA error; to sum σ_j over j would yield a total error which ignores correlations. This is what most previous $a_\mu^{\rm had}$ calculations have done.

IV. APPLICATION OF THE WFSA METHOD

A. Correlations over energies

In general, if different detectors measure hadronic cross sections over different energy regions, there must be different error-weighted fractions and S factors for each common energy region (ith detector, jth energy region). Such a WFSA application over energies has been applied to the dominant $a_\mu^{\rm had}$ contributions from the hadronic modes: $\pi^+\pi^-$, $\pi^+\pi^-\pi^0$, K^+K^-, and K_LK_S. An example of WFSA correlations over energy is shown in Table VI for the $R(s)$ $a_\mu^{\rm had}$ contribution in the energy range from 2.0 to 3.1 GeV. The $a_\mu^{\rm had}$ central values are error-weighted averaged across detectors (horizontally in the table) and summed over energies (vertically) while the detector specific statistical

TABLE V. Systematic errors used in the WFSA evaluation of the energy region 2.0 – 3.1 GeV $a_\mu^{\rm had}$ contribution.

Detector	Systematic error
BCF	0.02
$\gamma\gamma2$	0.21
Mark I	0.25

TABLE VI. The WFSA $10^{10}a_\mu^{\rm had}$ results for $R(s)$ from 2.0 – 3.1 GeV. Error weighted averages with S factor in ''Total'' column. Numbers in upper (lower) parentheses are WFSA contributions to the statistical (systematic) errors [Eqs. (6) and (7)] which are combined in quadrature (linearly) in each column separately. Lower right corner errors are quadrature sums of errors in the ''Total'' row.

Energy range [GeV]	BCF	$\gamma\gamma2$	Mark I	Total	S factor
2.0 – 2.6	18.778 (2.149) (0.207)	22.441 (0.527) (2.093)		20.480	0.637
2.6 – 2.87	3.718 (0.469) (0.032)	6.535 (0.130) (0.451)	4.781 (0.236) (0.406)	4.898	1.210
2.87 – 3.0	1.721 (0.202) (0.019)		1.777 (0.061) (0.193)	1.742	0.096
3.0 – 3.1			1.254 (0.062) (0.313)	1.253	0.0
Total	(2.208) (0.258)	(0.543) (2.544)	(0.251) (0.913)	28.374 (2.288) (2.700)	

(systematic) errors are combined in quadrature (linearly) across the energies (vertically). The correlation of systematic errors over energies does not appear to be taken into account by the WFB method discussed in the previous $a_\mu^{\rm had}$ calculation [12]: *weighted fraction* averaging [of $R(s)$ values] *before* energy integration.

In this energy range, published inclusive $R(s)$ values have been used from the $\gamma\gamma2$ detector [18,19] and Mark I [20]. It is noted that the $\gamma\gamma2$ Collaboration has published values for $R_2(s)$ (two hadrons exclusively) [18] and $R_{\geqslant 3}(s)$ (three or more hadrons) [19] separately which are added for the present calculation. This does not appear to have been done in the previous $a_\mu^{\rm had}$ calculations. In addition, data published as $\sigma(e^+e^-\to\text{hadrons})$ by the BCF Collaboration [21] (not apparently included previously) has been divided by $\sigma_{\mu\mu}^0$ to make them also $R(s)$ values.

B. Correlations over modes

If the hadronic cross sections are measured over similar energy regions, as in the case of the >2 hadrons multiplicity cross sections shown in Table VII, then the WFSA method can simply be applied over the modes (ith detector, jth hadronic mode). In this case, a single error-weighted fraction and S factor suffices for evaluation of $a_\mu^{\rm had}$ central value and errors. As before, the $a_\mu^{\rm had}$ central values are error-weighted averaged across detectors (horizontally in the table) and summed over modes (vertically), while the detector specific statistical (systematic) errors are combined in quadrature (linearly) across, in this case, the modes (vertically). The errors are not to be combined across detectors until all modes have been treated separately and then taken together. It is this correlation over all modes, in particular correlating the $\pi^+\pi^-$ detector total systematic errors with all the other two body and higher multiplicity modes, over all energies, which appears not to have been done before. (The previous $a_\mu^{\rm had}$ calculation [12] appears to have accounted for these correlations only above 0.81 GeV, after the peak of the ρ meson. Hence it appears that the $\pi^+\pi^-$ systematic errors below 0.81 GeV were not correlated with systematic errors of measurements by the same detectors of the other modes over all energies and of the $\pi^+\pi^-$ mode above 0.81 GeV.)

C. Kaons, narrow resonances, and higher VMD modes

Some additional hadronic modes which contribute to the total hadronic production cross section $\sigma^0(e^+e^-\to\text{hadrons})$ require comment. In particular, experimental data in the form of total cross sections neither exists on the radiative decays of the ω and ϕ mesons ($\pi^0\gamma,\eta\gamma$) nor on the kaon pair production of the ϕ meson below certain energies. Further, there are additional contributions to hadronic vacuum polarization than from just the lowest order single vector meson dominated (VMD) amplitudes represented by the hadronic (decay) modes previously discussed.

1. Kaons and narrow resonances

Experimental data on the total cross sections for production of kaon pairs (charged and neutral) are limited by the fact that nuclear interactions of low momentum kaons are not well measured. Hence kaon detection efficiencies are difficult to calculate precisely, and although existing data does span both sides of the ϕ meson it is usually a *relative* cross section useful for measurements of ϕ meson parameters. Hence a Breit-Wigner (BW) line shape with PDG 1994 parameters has been used, below the (most recent) lowest data point for kaon pair production and for the whole energy range for the radiative decays, where the errors (considered totally systematic) are evaluated by differentiating the BW formula.

TABLE VII. The WFSA $10^{10}a_\mu^{\rm had}$ results for >2 hadrons exclusive modes below 2.0 GeV. Upper (lower) contributions listed in the total column are for below (above) 1.4 GeV. Upper (lower) numbers in parentheses are contributions to the statistical (systematic) errors [Eqs. (6) and (7)]. Systematic (statistical) errors are combined linearly (in quadrature) in each column separately.

Mode	CMD	Olya	ND	DM1D	DM2	Other	Total	S factor
$\pi^+\pi^-\pi^0\pi^0$		(0.325)	(0.193)		(0.256)	(0.335)	10.783	1.614
		(1.491)	(1.514)		(0.868)	(0.841)	8.543	1.692
$\pi^+\pi^-\pi^+\pi^-$	(0.225)	(0.174)	(0.066)	(0.104)	(0.070)		5.161	0.486
	(0.527)	(0.914)	(0.526)	(0.623)	(0.628)		10.215	1.221
$\pi^+\pi^-\pi^+\pi^-\pi^0\pi^0$						(0.494)		
						(0.763)	5.089	0.0
$A_1(\pi^+\pi^-\pi^+\pi^-\pi^0)$	(0.072)			(0.124)		(0.187)	0.305	0.0
	(0.062)			(0.192)		(0.131)	2.533	1.010
$K_SK^\pm\pi^\mp$				(0.158)	(0.104)			
				(0.0)	(0.119)		0.951	3.199
$K^+K^-\pi^+\pi^-$				(0.072)	(0.091)			
				(0.135)	(0.123)		0.815	2.709
$K^*K^\pm\pi^\mp$					(0.057)			
					(0.069)		0.692	0.0
$B_1(\omega\pi^0)$			(0.017)			(0.079)	0.533	1.132
			(0.030)			(0.021)	0.210	0.0
$B_2(\eta\pi^+\pi^-)$					(0.039)			
					(0.044)		0.444	0.0
$(\pi^+\pi^-\pi^+\pi^-\pi^+\pi^-)$				(0.040)				
				(0.117)			0.419	0.0
$B_3(\omega\pi^+\pi^-)$				(0.005)	(0.003)			
				(0.003)	(0.004)		0.084	0.982
							16.782	
Total	(0.236)	(0.369)	(0.205)	(0.241)	(0.307)	(0.631)	29.995	
	(0.589)	(2.405)	(2.070)	(1.071)	(1.857)	(1.735)		

In the charm and bottom threshold regions the six states each of the J/ψ and Υ resonance families have been calculated separately using PDG 1994 values and the peak approximation formula for $a_\mu^{\rm had}$ [22,11]. In view of the small contribution to $a_\mu^{\rm had}$ from the b quark [due to kernel function supression in Eq. (1)], the top quark contribution is neglected.

2. Higher order VMD contributions

The vector meson dominance (VMD) model approximation of QCD expresses the fact that vector mesons, instead of quarks and gluons, are the relevant degrees of freedom in low energy (near threshold) QCD interactions. The VMD model approximation for hadronic vacuum polarization is depicted in Fig. 3. Most of the $a_\mu^{\rm had}$ contributions are contained in the first term on the right of the figure. The decay modes of the vector mesons ρ, ω, ϕ, and ω' account for $\sim 597\times 10^{-10}$ (85%), while ρ' accounts for $\sim 34.7\times 10^{-10}$ (5%), and ϕ' for $\sim 1.6\times 10^{-10}$ by rough accounting. The remaining part ($\sim 10\%$) is derived from a non- or very broadly resonant background for which it is instructive to consider some additional generation mechanisms not already included in the previous exclusive mode approach.

Namely, some portion of $a_\mu^{\rm had}$ derives from higher order VMD interactions $V\pi$ and $V\pi\pi$ (the next terms on the right in Fig. 3) where V is a vector meson. Although these amounts are individually less than the expected E821 error ($\sim 4.0\times 10^{-10}$) the fact that they may combine with other small contributions (whose sum may be greater than the E821 error) implies that they all should indeed be carefully considered.

The cross sections $\sigma(K^*K^\pm\pi^\mp)$, $\sigma(\omega\pi^0)$, $\sigma(\omega\ \pi^+\pi^-)$, $\sigma(\eta\ \pi^+\pi^-)$ have recently (1991,1992) been measured (ND,DM2). In particular, it has been pointed out

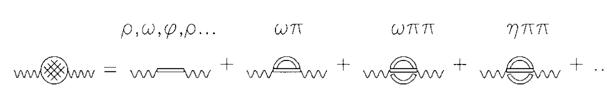

FIG. 3. Vector meson dominance representation of hadronic vacuum polarization.

[12] that the following exclusive hadronic modes marked with the comment ''not'' need to be included (the branching fraction for the decay modes in parentheses are $B=0.888$, 0.085, 0.0221 for $\omega\to\pi^+\pi^-\pi^0$, $\pi^0\gamma$, $\pi^+\pi^-$, respectively):

$$\begin{aligned}\omega\pi^0 &\to(\pi^+\pi^-\pi^0)\pi^0, \quad \text{is in } \pi^+\pi^-\pi^0\pi^0\\ &\to(\pi^0\gamma)\pi^0, \quad \textit{not} \text{ counted}\\ &\to(\pi^+\pi^-)\pi^0, \quad \text{is in } \pi^+\pi^-\pi^0,\end{aligned}$$

$$\begin{aligned}\omega\pi^+\pi^- &\to(\pi^+\pi^-\pi^0)\pi^+\pi^-, \quad \text{is in } \pi^+\pi^-\pi^+\pi^-\pi^0\\ &\to(\pi^0\gamma)\pi^+\pi^-, \quad \textit{not} \text{ in } \pi^+\pi^-\pi^0\\ &\to(\pi^+\pi^-)\pi^+\pi^-, \quad \text{is in } \pi^+\pi^-\pi^+\pi^-,\end{aligned}$$

$$\begin{aligned}\omega\pi^0\pi^0 &\to(\pi^+\pi^-\pi^0)\pi^0\pi^0, \quad \text{is in } \pi^+\pi^-\pi^0\pi^0\pi^0\\ &\to(\pi^0\gamma)\pi^0\pi^0, \quad \textit{not} \text{ counted}\\ &\to(\pi^+\pi^-)\pi^0\pi^0, \quad \text{is in } \pi^+\pi^-\pi^0\pi^0.\end{aligned}$$

They further point out that isospin considerations [23] imply

$$2\sigma(\pi^+\pi^-\pi^0\pi^0\pi^0)=\sigma(\pi^+\pi^-\pi^+\pi^-\pi^0),$$

$$2\sigma(\omega\pi^0\pi^0)=\sigma(\omega\,\pi^+\pi^-)$$

and that this must be used since $\sigma(\omega\pi^0\pi^0)$ and $\sigma(\pi^+\pi^-\pi^0\pi^0\pi^0)$ have not been measured. Therefore the $\sigma(\pi^+\pi^-\pi^+\pi^-\pi^0)$ contribution has been augmented by a factor $A_1=1.5$, and the contributions from $\sigma(\omega\pi^0)$ and $\sigma(\omega\,\pi^+\pi^-)$ will contribute with factors $B_1=0.085$ and $B_3=1.5B_1$, respectively, as noted in Table IV. However, the same logic applied to $\sigma(\eta\,\pi^+\pi^-)$ implies *not* including 100% of it as done previously [12] (the branching fractions for the decay modes in parentheses are $B=0.388$, 0.319, 0.236, 0.0488 for $\eta\to\gamma\gamma$, $\pi^0\pi^0\pi^0$, $\pi^+\pi^-\pi^0$, $\pi^+\pi^-\gamma$, respectively):

$$\begin{aligned}\eta\pi^+\pi^- &\to(\gamma\gamma)\pi^+\pi^-, \quad \textit{not} \text{ in } \pi^+\pi^-\\ &\to(\pi^0\pi^0\pi^0)\pi^+\pi^-, \quad \text{is in } \pi^+\pi^-\pi^0\pi^0\pi^0\\ &\to(\pi^+\pi^-\pi^0)\pi^+\pi^-, \quad \text{is in } \pi^+\pi^-\pi^+\pi^-\pi^0\\ &\to(\pi^+\pi^-\gamma)\pi^+\pi^-, \quad \textit{not} \text{ in } \pi^+\pi^-\pi^+\pi^-\pi^0.\end{aligned}$$

To avoid double counting in the $\pi^+\pi^-\pi^+\pi^-\pi^0$ and $\pi^+\pi^-\pi^0\pi^0\pi^0$ channels only the fraction $B_2=0.388+0.0488$ of the $\sigma(\eta\,\pi^+\pi^-)$ contribution has been included. [There is no further factor of 1/2 for inclusion of the cross section $\sigma(e^+e^-\to\eta\pi^0\pi^0)$ since it is forbidden.] The augmentation and branching factors (A_1, B_1, B_2, B_3) are noted in Table IV where all contributions are listed in descending order.

D. The perturbative QCD energy region

To test the validity of QCD for determination of the contribution to $a_\mu^{\rm had}$ in the energy region above 3.1 GeV, a QCD parameterization [24] (including second order terms) was compared with a data-based evaluation [12]. An asymptotic kernel function [12] was used in the $a_\mu^{\rm had}$ integral since Eq. (2) is numerically stable only up to ~20 GeV. The errors on the QCD contribution to $a_\mu^{\rm had}$ were determined by its variation with $\Lambda_{\overline{\rm MS}}\pm\Delta\Lambda_{\overline{\rm MS}}$, where $\overline{\rm MS}$ denotes the modified minimal subtraction scheme.

Marshall combined the results from 15 different e^+e^- annihilation experiments, fitting them to a third order QCD model with a single parameter, $\Lambda_{\overline{\rm MS}}$. Because the fit was overconstrained, he was able to evaluate whether these experiments had overestimated their systematic errors. He then went a step further, and fitted for the absolute normalization of each of the 15 experiments independently, bounded by double their stated systematic errors ($\pm2\sigma$). He found fitted normalizations for most experiments within their stated limits, with the exception of two of the earliest experiments: Mark II and $\gamma\gamma2$.

By using the error in his normalization constants (from the fit) rather than the stated systematic errors for each experiment, Marshall was able to significantly reduce his error on $\Lambda_{\overline{\rm MS}}$ and hence on his overall normalization for $R(s)$. Dubnicka followed Marshall in his 1992 preprint [9], and in the present calculation both Marshall's fitted parameterization (with his errors) and trapezoidal integration [12] have been compared for the higher-energy contributions to $a_\mu^{\rm had}$. As the QCD and data-based results are in good agreement it is clear that the second order QCD expression is sufficient.

V. WFSA RESULTS AND PROJECTIONS

A. Summary results

As the preceding discussion of the WFSA evaluation shows, the core $a_\mu^{\rm had}$ problem is how to combine the correlations among the systematic errors. In the present calculation, partial $a_\mu^{\rm had}$ integrations have been performed first so relative errors and error-weighted averaged systematic and statistical errors could be defined at an early stage of the error combination calculation. The averaged systematic errors have been correlated according to the postulates discussed in Sec. III A for all experiments measuring more than one hadronic mode; all other errors have been combined in quadrature only after such correlations have been taken into account.

All of the WFSA calculations have been collected in Table IX. The first two lines of the table are summary results of the $a_\mu^{\rm had}$ contributions and errors presented in Table VIII. In the >2 hadrons line of Table VIII the results from Table VII are presented. The third line in Table IX for $R(s)$ from 2.0–3.1 GeV is the summary result from Table VI. All of the $a_\mu^{\rm had}$ central values are averaged over detectors first and then summed over modes, while the errors are combined over modes first (to correlate systematics) and only combined over detectors in the end. The errors in Table IX may be combined in quadrature since there are no correlations remaining among the categories chosen. (This feature is not completely present in the previous calculations of $a_\mu^{\rm had}$.)

A subtotal for $a_\mu^{\rm had}$ contributions below 3.1 GeV is presented in Table IX in order to show the total errors before the use of QCD. It is there seen that the QCD results do not have much influence on the final results since the error from the lower energy (< 3.1 GeV) region dominates. For comparison, if in place of the QCD calculation the data based evaluation for the region above 3.1 GeV [12] is used the final

TABLE VIII. Our WFSA $10^{10}a_{\mu}^{\rm had}$ results below 2.0 GeV. Upper (lower) contributions listed in the total column are for below (above) 1.4 GeV. Upper (lower) numbers in parentheses are contributions to the statistical (systematic) errors. The ‘‘>2 hadrons’’ entries are taken from the total line of Table VII while two hadron entries are taken from similar tables not presented here for brevity. Systematic (statistical) errors are combined linearly (in quadrature) in each column separately. E_{θ} refers to the particular thresholds of the exclusive hadronic modes. DM1A=DM1 at ACO; DM1D=DM1 at DCI, Orsay.

Mode	CMD	Olya	ND	DM1A	DM1D	DM2	Other	Total
								502.184
$\pi^+\pi^-$	(4.806)	(1.306)		(5.064)		(0.058)	(0.814)	0.781
	(4.789)	(5.807)		(4.021)		(0.110)	(0.732)	
								50.886
$\pi^+\pi^-\pi^0$	(0.509)		(0.856)	(1.209)		(0.041)	(0.049)	0.677
	(0.104)		(1.303)	(0.715)		(0.053)	(0.009)	
								16.782
>2 hadrons	(0.236)	(0.369)	(0.205)		(0.241)	(0.307)	(0.631)	29.995
	(0.589)	(2.405)	(2.070)		(1.071)	(1.857)	(1.735)	
								20.623
K^+K^-	(0.119)	(0.953)				(0.046)		4.45[a]
	(0.058)	(1.900)				(0.090)	(0.230)[b]	0.759
								0.755
K_LK_S	(0.173)	(0.156)			(0.033)			14.07[a]
	(0.040)	(0.101)			(0.017)		(0.586)[b]	0.154
$p\overline{p}$					(0.020)			0.100
					(0.010)			
$\omega\to\pi^0\gamma,\eta\gamma$							(0.040)[b]	0.980[a]
$\phi\to\pi^0\gamma,\eta\gamma$							(0.024)[b]	0.602[a]
								611.33
Total	(4.842)	(1.666)	(0.880)	(5.206)			(0.807)	(7.399)
E_{θ}–1.4	(5.580)	(10.213)	(3.373)	(4.736)			(0.962)	(13.045)
								32.47
Total					(0.244)	(0.318)	(0.642)	(0.756)
1.4–2.0					(1.098)	(2.110)	(1.747)[c]	(2.379)

[a]Integration of energy dependent width Breit-Wigner resonance in absence of data.
[b]Errors determined by m_V, Γ_V, $B_{V\to ee}$ derivatives of the Breit-Wigner resonance formula.
[c]The $\gamma\gamma2$ systematic errors from energy regions 1.4–2.0 GeV (1.7) and 2.0–3.1 GeV (2.7) will be added linearly and presented in Table IX.

errors are only slightly higher: $a_{\mu}^{\rm had}=702.718(7.787)(14.147)$. Hence the WFSA result is thus shown to be largely insensitive to the use of QCD or the experimental data after 3.1 GeV. (This justifies both the use of QCD down to the energy 3.1 GeV and the neglect of third order QCD terms; as new data bring the errors from the lower energy regions downward, the QCD error will become more significant however.)

It is further apparent from Table IX that the error in the energy region below 1.4 GeV dominates the total error in $a_{\mu}^{\rm had}$, and this error comes mostly from the Olya $\pi^+\pi^-$ data. [One advantage of the WFSA method is that in presenting averaged errors (both statistical and systematic separately) for each detector and exclusive mode, it remains possible to identify which detector and exclusive mode is contributing the most to the overall total error.] The WFSA method most heavily weights the data with the smallest errors (see Table II), and hence most of the stated systematic errors are indeed being added linearly for the experiments with the smallest errors.

B. Energy region < 1.4 GeV

The overall WFSA results indicate that improved measurements below 1.4 GeV, can significantly improve upon the overall lowest order hadronic vacuum polarization uncertainty in $a_{\mu}^{\rm had}$. To show this, the WFSA procedure has been performed with an additional two experiments in view, the CMD2 experiment at VEPP-2M in Novosibirsk, Russia and a hypothetical detector at DAFNE at Frascati, Italy. The $a_{\mu}^{\rm had}$ results for the energy region below 1.4 GeV are shown in Table X for three cases: (a) without new data, (b) with

TABLE IX. The grand total WFSA $10^{10}a_\mu^{\rm had}$ results. E_θ refers to the particular thresholds of the exclusive hadronic modes.

Energy region [GeV]	WFSA $10^{10}a_\mu^{\rm had}$ (stat) (syst)
$\sigma(e^+e^-\to\text{hadrons})$ E_θ–1.4	611.332 (7.399) (13.045)
$\sigma(e^+e^-\to\text{hadrons})$ 1.4–2.0	32.466 (0.756) (2.379)
$R(s)$ 2.0–3.1	28.374 (2.288) (4.400)[a]
J/Ψ (6 states)	9.047 (-) (0.969)
Υ (6 states)	0.109 (-) (0.013)
QCD 3.1–∞	21.301 (-) (0.371)[b]
Subtotal $<$3.1+J/Ψ,Υ	681.328 (7.782) (14.005)
Subtotal $<$1.4	611.332 (7.399) (13.045)
Subtotal $>$1.4	91.297 (2.410) (5.108)
Total Fig. 1	702.629 (7.782) (14.009)

[a]Represents systematic errors mainly from $\gamma\gamma2$ detector added linearly for energy regions 1.4–2.0 and 2.0–3.1 GeV (1.7+2.7).
[b]Errors determined by $a_\mu^{\rm QCD}(\Lambda_{\overline{\rm MS}}\pm\Delta\Lambda_{\overline{\rm MS}.})$

expected errors (stat)(syst)=(3%)(0.5%) from the CMD2 experiment, and (c) CMD2 plus a second experiment with errors (stat)(syst)=(3%)(0.5%). In particular, the central values for hadronic cross sections have been chosen equal to CMD or Olya values (and hence the S factors will generally be less than one and so not applied), while the errors above have been determined for only the following exclusive hadronic modes: $\pi^+\pi^-$ (2π), $\pi^+\pi^-\pi^0$ (3π), $\pi^+\pi^-\pi^+\pi^-$ $(4\pi$ charged), $\pi^+\pi^-\pi^0\pi^0$ $(4\pi$ neutral), $\pi^+\pi^-\pi^+\pi^-\pi^0$ (5π), K^+K^- and K_LK_S.

The choice of 3% statistical error is based on the CMD2 data taking assumption of 1000 pion pairs per energy point since $\pi^+\pi^-$ is the dominant contribution to $a_\mu^{\rm had}$. Since the luminosity collected by CMD2 has been determined by this requirement on the $\pi^+\pi^-$ cross section (and by competing requirements for other vector meson physics goals), the actual statistical errors on the non-$\pi^+\pi^-$ higher multiplicity modes will in fact be somewhat different, but this is a higher order effect here neglected.

The choice of 0.5% systematic error is based on the fact that the limiting error on the new CMD2 cross section measurements appears to be the error on higher order corrections to the QED Bhabha cross sections used in calculations of the luminosity. While radiative corrections have been calculated to good accuracy (0.11% [25]) for the t-channel contributions to Bhabha scattering useful for the forward region luminosity monitors in use at LEP, they are not useful for CMD2; the principle luminosity is determined there by large angle Bhabha events in the barrel calorimeter where significant t- and s-channel interference terms are present. The task of calculating these interference terms to 0.5% accuracy is well underway [26] and it is assumed that this will be the limiting error on the new CMD2 cross section measurements.

The results in Table X show that without new data the errors on the contributions to $10^{10}a_\mu^{\rm had}$ below 1.4 GeV will be 10^{10} (stat)(syst)=(7.4)(13.1) and with particular new data they can become 10^{10} (stat)(syst)=(2.1)(2.9) which is equivalent to a total error of 3.6×10^{-10}. Comparing this with the AGS E821 experimental error goal of $\sigma_{\rm expt}^{a_\mu}\sim\pm4.0\times10^{-10}$ it is clear that the new data with near or better the assumed errors on the stated modes is needed.

C. Energy region >1.4 GeV

However, the new data from BINP Novosibirsk and INFN Frascati will not reduce the errors on $a_\mu^{\rm had}$ in the energy range above 1.4 GeV. The current error on $a_\mu^{\rm had}$ obtained by use of the WFSA procedure in the region above 1.4 GeV is 10^{10} (stat)(syst)=(2.4)(5.3). In this region, most of the error comes from the $\gamma\gamma2$ detector at the former 3 GeV Adone storage ring in INFN Frascati, Italy. As this error is larger than the BNL E821 experimental error goal of $\sigma_{\rm expt}^{a_\mu}\sim\pm4.0\times10^{-10}$, clearly more experiments are needed for interpretation of the new measurement of a_μ. Fortunately, it is possible to use new high statistics data on τ decays for new measurements of multipion production in the energy region 1.4–2.0 GeV, and above 2.2 GeV there are plans to make further measurements of the cross section ratio $R(s)$ at the Beijing Electron-Positron Collider (BEPC), China [27].

1. *τ decay data for energy region 1.4–2.0 GeV*

The new high statistics data on τ decays has already been included in the present calculation by use of the $\omega\pi^0$ cross section obtained from the ARGUS Collaboration [28,29]. The idea is based on the fact that the coupling of $W^\pm$ bosons

TABLE X. WFSA $10^{10}a_\mu^{\rm had}$ detector total statistical (upper) and systematic (lower) errors below 1.4 GeV; row (1) with no new data, row (2) one new experiment (CMD2) with 3% stat., 0.5% syst. errors, and row (3) two new experiments with 3% stat. and 0.5% syst. errors.

WFSA $10^{10}a_\mu^{\rm had}$ errors below 1.4 GeV							
New 2	New 1	CMD	Olya	ND	DM1A	Other	Total
		(4.842)	(1.666)	(0.880)	(5.206)	(0.807)	(7.399)
		(5.580)	(10.213)	(3.373)	(4.736)	(0.962)	(13.045)
	(2.426)	(1.266)	(1.148)	(0.136)	(1.239)	(0.451)	(3.250)
	(2.224)	(1.292)	(3.608)	(0.405)	(1.241)	(0.760)	(4.681)
(1.183)	(1.183)	(0.789)	(0.528)	(0.078)	(0.766)	(0.361)	(2.103)
(1.256)	(1.256)	(0.773)	(1.921)	(0.244)	(0.769)	(0.718)	(2.934)

TABLE XI. The WFSA $10^{10}a_\mu^{\rm had}$ detector total statistical (upper) and systematic (lower) errors in region 1.4–2.0 GeV. Row (1) with no new data, row (2) new hypothetical data (possibly from τ decay spectral functions) on modes $\pi^+\pi^-\pi^0\pi^0$, $\pi^+\pi^-\pi^+\pi^-\pi^0$ and $\pi^+\pi^-\pi^+\pi^-\pi^0\pi^0$ with 3% stat., 1.0 % syst. errors, and row (3) including as well the mode $\pi^+\pi^-\pi^+\pi^-$.

WFSA $10^{10}a_\mu^{\rm had}$ errors in 1.4–2.0 GeV region				
New 1	DM1D	DM2	$\gamma\gamma2$	Total
	(0.244)	(0.318)	(0.628)	(0.745)
	(1.098)	(2.110)	(1.744)	(2.950)
(0.068)	(0.210)	(0.192)	(0.067)	(0.300)
(0.115)	(0.922)	(1.320)	(0.141)	(1.620)
(0.082)	(0.179)	(0.157)	(0.067)	(0.261)
(0.175)	(0.346)	(0.514)	(0.141)	(0.659)

TABLE XII. The WFSA $10^{10}a_\mu^{\rm had}$ detector total statistical (upper) and systematic (lower) errors in region 2.0–3.1 GeV. Row (1) with no new data, and row (2) one new experiment above 2.2 GeV with 3% stat., 0.5% syst. errors

WFSA $10^{10}a_\mu^{\rm had}$ errors in 2.0–3.1 GeV region				
New 1	BCF	$\gamma\gamma2$	Mark I	Total
	(2.208)	(0.543)	(0.251)	(2.288)
	(0.258)	(2.544)	(0.913)	(2.715)
(0.109)	(0.485)	(0.140)	(0.025)	(0.517)
(0.058)	(0.109)	(0.601)	(0.084)	(0.619)

to quarks is related to the photon coupling to quarks by an isospin rotation [conserved vector current (CVC) relation] [30]. The effect on the total errors in the energy region 1.4–2.0 GeV are presented in Table XI where the assumed errors are (stat)(syst)=(3%)(1%) for the following hadronic modes: $\pi^+\pi^-\pi^+\pi^-$ (4π charged), $\pi^+\pi^-\pi^0\pi^0$ (4π neutral), $\pi^+\pi^-\pi^+\pi^-\pi^0$ (5π) and $\pi^+\pi^-\pi^+\pi^-\pi^0\pi^0$ (6π).

New data on the 4π neutral, 5π and 6π modes can reduce the $\gamma\gamma2$ contribution to the total errors (as shown in row 2) while new data on the 4π charged mode is required to reduce the DM1 and DM2 contribution to the total errors (as shown in row 3). Therefore, if many data points ($\sim 10-20$) for each of the specified modes across the energy region 1.4 to 2.0 GeV with the assumed errors can be extracted from τ decay data, then the contribution to the total errors from this energy region will be reduced significantly as shown in Table XI.

2. New $R(s)$ measurements for energy region 2.0–3.1 GeV

The effect of new data on the error in the 2.0–3.1 GeV energy region is presented in Table XII. The WFSA procedure has been performed with a new hypothetical detector measuring $R(s)$ from 2.2–3.1 GeV [central value of $R(s)=3.0$ for all points so no S factors exceed 1] with assumed errors of (stat)(syst)=(3%)(0.5%). If new $R(s)$ data

TABLE XIII. The experimental data used for the WFSA evaluation of $a_\mu^{\rm had}$ is listed by hadronic exclusive mode. There are eight detectors which have measured more than one mode.

Mode	OLYA	CMD	ND	DM1	DM2	$\gamma\gamma2$	BCF	MEA	Others
$\pi^+\pi^-$	[31]	[8]		[32]	[33]		[35]	[37]	[34,36,38,39]
$\pi^+\pi^-\pi^0$		[40,41]	[29]	[42]	[43]	[19]			
K^+K^-	[44]	[45]			[46]			[37]	
K_LK_S	[44]	[45]		[47]					
$\pi^+\pi^-\pi^+\pi^-$	[48]	[49]	[29]	[50]	[51]				
$\pi^+\pi^-\pi^0\pi^0$	[52]		[29]		[46]	[19]			
$\pi^+\pi^-\pi^+\pi^-\pi^0$		[45]		[53]		[19]			
$\pi^+\pi^-\pi^+\pi^-\pi^+\pi^-$				[54]					
$\pi^+\pi^-\pi^+\pi^-\pi^0\pi^0$						[19]			
$K_SK^\pm\pi^\mp$				[55]	[51]				
$K^+K^-\pi^+\pi^-$				[56]	[51]				
$K^*K^\pm\pi^\mp$					[51]				
$\omega\pi^0$			[29]						[28]
$\eta\pi^+\pi^-$					[57]				
$\omega\pi\pi$				[58]	[59]				
$p\overline{p}$				[60]	[61]				
$R(s)$ 2.0 – 3.1 GeV						[18,19]	[21]		[20]

in this energy region can be obtained with these errors then the $\gamma\gamma2$ contribution to the total error can be significantly reduced.

Taking the new hypothetical data together [including τ decays and $R(s)$ measurements], the total error from the energy range above 1.4 GeV [including resonances and QCD] then becomes 10^{10} (stat)(syst) = (0.6)(1.6).

D. Conclusion

Combining the hypothetical results from above and below 1.4 GeV the total error becomes 10^{10} (stat) (syst)=(2.2)(3.3) which is equivalent to a total error of 3.98×10^{-10}. Therefore, new measurements of $e^+e^-\rightarrow$hadrons [including the cross section ratio $R(s)$] and τ decay spectral functions with assumed errors of (stat)(syst)=(3%)(0.5%) and (3%)(1.0%), respectively, are sufficient, to reduce the total error on the lowest order contribution to $a_\mu^{\rm had}$ in *all energy regions* to the level of the expected error of the new measurement of a_μ by the BNL E821 Collaboration.

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