Helicity decomposition for inclusive J/ψ production

Eric Braaten
Physics Department, Ohio State University, Columbus, Ohio 43210

Yu-Qi Chen
Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208

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We present a general method for calculating inclusive cross sections for the production of heavy quarkonium states with definite polarization within the NRQCD factorization approach. Cross sections for polarized production can involve additional matrix elements that do not contribute to cross sections for unpolarized production. They can also include interference terms between parton processes that produce $Q\bar{Q}$ pairs with different total angular momentum. The interference terms cancel upon summing over polarizations. Our method can be generalized to N dimensions and is, therefore, compatible with the use of dimensional regularization to calculate radiative corrections. We illustrate the method by applying it to the production of J/ψ via the parton processes $q\bar{q}\rightarrow c\bar{c}$ and $gg\rightarrow c\bar{c}$. [S0556-2821(96)05717-7]

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I. INTRODUCTION

Calculations of inclusive production rates of heavy quarkonium have recently been placed on a firm theoretical foundation by the development of a factorization approach based on nonrelativistic quantum chromodynamics (NRQCD) [1]. In this formalism, the cross section is expressed as a sum of products of short-distance coefficients and NRQCD matrix elements. The short-distance coefficients can be calculated as perturbation series in the coupling constant α_s at the scale of the heavy quark mass. The matrix elements scale in a definite way with the typical relative velocity v of the heavy quark in the quarkonium state. Thus the production cross sections can be calculated systematically to any desired order in α_s and v^2 in terms of well-defined NRQCD matrix elements. There have been many recent applications of this formalism to quarkonium production in various high-energy processes [2].

Most calculations of quarkonium production have been carried out using a covariant projection formalism developed for calculations in the color-singlet model [3]. In the amplitude for producing a $c\bar{c}$ pair with total momentum P and relative momentum q, the spinor factor $u\bar{v}$ is replaced by an appropriate Dirac matrix that projects out a state with the desired angular momentum quantum numbers. The resulting prescription is relatively simple for S-wave states. The Dirac matrix is proportional to $\gamma_5(\not{P}-2m_c)$ for a ${}^1S_0$ state and $\not{\epsilon}(\not{P}-2m_c)$ for a ${}^3S_1$ state, and the relative momentum q is set to zero. For P waves, the Dirac matrix is more complicated, and the amplitude must be differentiated with respect to the relative momentum. The resulting expression for the amplitude can be very complicated.

The covariant projection method also has a number of other drawbacks. For each angular momentum state, a separate calculation of the cross section is required, beginning at the level of the amplitude for $c\bar{c}$ production. Another drawback is that it is difficult to generalize the projection method to allow the calculation of relativistic corrections to the cross section. There is also a potential difficulty in calculating radiative corrections using this method. The Dirac matrices that are used to project onto the appropriate angular momentum states are specific to three space dimensions. However, the most convenient method for regularizing the ultraviolet and infrared divergences that arise in higher-order calculations is dimensional regularization. It is not easy to generalize the projection matrices to N dimensions, since the representations of the rotational symmetry group are different for each integer value of N.

In this paper, we develop a method for calculating quarkonium production rates that fully exploits the NRQCD factorization framework. The short-distance coefficients are calculated in a form that holds for every quarkonium state. The corresponding NRQCD matrix elements are then simplified using rotational symmetry and the approximate heavy-quark spin symmetry of NRQCD. It is only at this stage that the angular momentum quantum numbers of the quarkonium state come into play. Relativistic corrections are easily calculated in this framework. Our method for calculating the short-distance coefficients is readily generalized to N spacial dimensions, so that dimensional regularization can be used to cut off infrared and ultraviolet divergences. After removing the divergences, one can specialize to $N=3$ and use rotational symmetry and spin symmetry to simplify the NRQCD matrix elements. This approach allows the consistent use of dimensional regularization to calculate inclusive heavy quarkonium production and decays.

Our method has important implications for the production of quarkonium states with definite polarization. We demonstrate that cross sections for polarized production involve new matrix elements that do not appear in cross sections for unpolarized production. Thus, measurements of cross sections for unpolarized production are not necessarily sufficient to predict production rates for polarized quarkonium states. We also show that cross sections for polarized production can involve interference between parton processes that produce $c\bar{c}$ pairs with different total angular momentum J. The interference terms vanish upon summing over polarizations. Thus the cross sections for producing $c\bar{c}$ states with definite total angular momentum are not sufficient to determine all the short-distance coefficients in cross sections for polarized production.

The NRQCD factorization approach is summarized in Sec. II. In Sec. III, we present a general matching procedure for calculating the short-distance coefficients in the factorization formula. We illustrate the method by applying it to the parton processes $q\bar{q}\to c\bar{c}$ and $gg\to c\bar{c}$. The resulting expressions for the cross sections hold for every quarkonium state. In Sec. IV, we show how the NRQCD matrix elements can be simplified by using rotational symmetry and the approximate heavy-quark spin symmetry of NRQCD. For simplicity, we focus on the matrix elements for J/ψ production. In Sec. V, we calculate the cross section and the spin alignment of the ψ from the parton processes $q\bar{q}\to c\bar{c}$ and $gg\to c\bar{c}$. We point out that several previous calculations of quarkonium production need to be reconsidered in light of our results.

II. NRQCD FACTORIZATION FORMALISM

We consider the inclusive production of a quarkonium state H with momentum P and helicity λ via a parton process of the form $12\to H(P,\lambda)+X$. The differential cross section, summed over additional final states X, can be written

$$\sum_X d\sigma(12\to H(P,\lambda)+X) = \frac{1}{4E_1E_2v_{12}}\frac{d^3P}{(2\pi)^3 2E_P} \times \sum_X (2\pi)^4\delta^4(k_1+k_2-P-k_X)|\mathcal{T}_{12\to H(P,\lambda)+X}|^2, \quad (2.1)$$

where $E_P=\sqrt{M_H^2+\mathbf{P}^2}$ and the sum over X includes integration over the Lorentz-invariant phase space for the additional particles.

The NRQCD factorization formalism can be used to factor the cross section (2.1) into short-distance coefficients and long-distance matrix elements [1]:

$$\sum_X d\sigma(12\to H(P,\lambda)+X) = \frac{1}{4E_1E_2v_{12}}\frac{d^3P}{(2\pi)^3 2E_P}\sum_{mn} C_{mn}(P,k_1,k_2) \times\langle\mathcal{O}_{mn}^{H(\lambda)}\rangle. \quad (2.2)$$

The coefficients C_{mn} are functions of the kinematic variables P, k_1, and k_2. They take into account the effects of short distances of order $1/m_c$ or smaller, and therefore can be calculated as perturbation series in the QCD coupling constant $\alpha_s(m_c)$. The matrix elements $\langle\mathcal{O}_{mn}^{H(\lambda)}\rangle$ are expectation values in the NRQCD vacuum of local four-fermion operators that have the structure

$$\mathcal{O}_{mn}^{H(\lambda)}=\chi^\dagger\mathcal{K}'^\dagger_m\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\mathcal{K}_n\chi, \quad (2.3)$$

where ψ and χ are the field operators for the heavy quark and antiquark in NRQCD, and $\mathcal{K}_n$ and $\mathcal{K}'^\dagger_m$ are products of a color matrix (1 or T^a), a spin matrix (1 or σ^i), and a polynomial in the gauge covariant derivative $\mathbf{D}$. The projection operator $\mathcal{P}_{H(\lambda)}$ can be written

$$\mathcal{P}_{H(\lambda)}=\sum_S |H(\mathbf{P}=0,\lambda)+S\rangle\langle H(\mathbf{P}=0,\lambda)+S|. \quad (2.4)$$

The sum is over soft hadron states S whose total energy is less than the ultraviolet cutoff Λ of NRQCD. Thus this operator projects onto the subspace of states that in the asymptotic future include the quarkonium state $H(\lambda)$ at rest plus soft hadrons. The normalization of the meson states in Eq. (2.4) must coincide with those in the T-matrix element in Eq. (2.1). The standard relativistic normalization is

$$\langle H(\mathbf{P}',\lambda')|H(\mathbf{P},\lambda)\rangle=2E_P(2\pi)^3\delta^3(\mathbf{P}-\mathbf{P}')\delta_{\lambda\lambda'}. \quad (2.5)$$

With this normalization of states, the projection operator $\mathcal{P}_{H(\lambda)}$ has energy dimension -2.

If the colliding particles are leptons, the cross section for quarkonium production is given directly by the factorization formula (2.2). If the colliding particles are hadrons, the cross section (2.2) must be folded with parton distributions for partons 1 and 2 in the colliding hadrons. In this case, the derivation of the factorization formula requires that the transverse momentum p_T of the quarkonium be large compared to $\Lambda_{\rm QCD}$, the scale of nonperturbative effects in QCD. This restriction to large p_T follows from the diagrammatic analysis that underlies the factorization formula [1]. This analysis shows that the dominant contributions to the cross section can be factored into (a) hard-scattering amplitudes for parton processes of the form $12\to c\bar{c}+34\ldots$, (b) jet-like subdiagrams for the incoming partons 1 and 2 and the outgoing partons 3, 4, etc., (c) a subdiagram involving a $c\bar{c}$ pair with relative momentum that is small compared to the quark mass m_c, and (d) a soft part. The soft part involves soft gluons that couple to the jetlike subdiagrams and to the $c\bar{c}$ subdiagram, but not to the hard-scattering subdiagram. The effects of soft partons that are exchanged between the various subdiagrams cancel upon summing over all possible connections of the soft partons. This cancellation is effective provided that the $c\bar{c}$ pair has large transverse momentum relative to the incoming hadrons. All the effects of soft partons can then be factored into parton distributions associated with the incoming hadrons, fragmentation functions associated with the outgoing partons produced by the hard scattering, and a factor associated with the $c\bar{c}$ pair that depends on their relative momentum q. In this step in the derivation of the factorization formula, the short-distance scales p_T and m_c are separated from the long-distance scale $\Lambda_{\rm QCD}$. The effects of the short-distance scales appear only in the hard-scattering amplitudes, and all effects of the scale $\Lambda_{\rm QCD}$ are factored into parton distributions, fragmentation functions, and $c\bar{c}$ factors.

The remaining step in the derivation of the factorization formula involves separating the scale m_c in the hard-scattering amplitude from the scale m_cv of the relative momentum in the charmonium state. This separation can be accomplished by Taylor expanding the hard-scattering amplitude in powers of q. A naive Taylor expansion generates ultraviolet divergences in the $c\bar{c}$ factor. The effective field theory framework provided by NRQCD makes this Taylor expansion meaningful. The long-distance factors generated by the Taylor expansion can be identified with matrix elements of local operators in NRQCD. The renormalization framework of NRQCD allows the ultraviolet divergences in these matrix elements to be systematically removed. The resulting expression for the cross section has the form of the NRQCD factorization formula (2.2). The effects of the scale m_c appear only in the short-distance coefficients C_{nm}. All effects involving momentum scales of order $m_c v$ or smaller have been factored into the NRQCD matrix elements $\langle \mathcal{O}_{mn}^{H(\lambda)} \rangle$.

The breakdown of the factorization formula at small p_T when the colliding particles are hadrons can be attributed to ''higher-twist'' processes. In the factorization formula (2.2), the c and $\bar{c}$ that form charmonium are assumed to be produced by a hard-scattering collision involving a single parton from each of the colliding hadrons. Higher-twist processes can involve more than one parton from a single hadron. An example of a higher-twist process for charmonium production is one in which the c and $\bar{c}$ that bind to form charmonium are produced by a quantum fluctuation of the hadron into a state containing a $c\bar{c}$ pair [4]. The transition of the $c\bar{c}$ to a charmonium state can then be induced by the collision with the other hadron. The short-distance part of this process is the fluctuation of the hadron into a state containing the $c\bar{c}$ pair. The cross section for the scattering of the $c\bar{c}$ pair from the other hadron involves long-distance effects that cannot be expressed in terms of matrix elements of local NRQCD operators.

III. SHORT-DISTANCE COEFFICIENTS

The short-distance coefficients C_{mn} in Eq. (2.2) can be determined by matching perturbative calculations in full QCD with the corresponding perturbative calculations in NRQCD. In most previous calculations, the coefficients were determined by matching cross sections for producing $c\bar{c}$ pairs with definite total angular momentum. However, $c\bar{c}$ states with different angular momentum quantum numbers can have a nonzero overlap with the same final state $|H+S\rangle$. The resulting interference terms cannot always be obtained by matching cross sections. Below we present a matching prescription that is sufficiently general to provide these interference terms.

A. General matching prescription

Let $c\bar{c}(\mathbf{q},\xi,\eta)$ represent a state that consists of a c and a $\bar{c}$ that have total momentum P, three-momenta $\pm\mathbf{q}$ in the $c\bar{c}$ rest frame, and spin and color states specified by the spinors ξ and η. These spinors are two-component Pauli spinors with a color index. Using the abbreviated notation $c\bar{c}\equiv c\bar{c}(\mathbf{q},\xi,\eta)$ and $c\bar{c}'\equiv c\bar{c}(\mathbf{q}',\xi',\eta')$, the matching condition is

$$\sum_X (2\pi)^4\delta^4(k_1+k_2-P-k_X)\mathcal{T}^*_{12\to c\bar{c}'+X}\mathcal{T}_{12\to c\bar{c}+X}$$

$$=\sum_{mn} C_{mn}(P,k_1,k_2)\langle \chi^\dagger \mathcal{K}'^\dagger_m \psi \mathcal{P}_{c\bar{c}',c\bar{c}}\psi^\dagger \mathcal{K}_n \chi\rangle. \quad (3.1)$$

The operator $\mathcal{P}_{c\bar{c}',c\bar{c}}$ in the matrix element in Eq. (3.1) is defined by

$$\mathcal{P}_{c\bar{c}',c\bar{c}}=\sum_S |c(\mathbf{q}',\xi')\bar{c}(-\mathbf{q}',\eta')+S\rangle$$
$$\times\langle c(\mathbf{q},\xi)\bar{c}(-\mathbf{q},\eta)+S|. \quad (3.2)$$

The sum is over soft parton states whose total energy is less than the ultraviolet cutoff Λ of NRQCD. The normalization of the $c\bar{c}$ states in Eq. (3.2) must coincide with those in the T-matrix elements in Eq. (2.1). The standard relativistic normalization is

$$\langle c(\mathbf{q}_1',\xi')\bar{c}(\mathbf{q}_2',\eta')|c(\mathbf{q}_1,\xi)\bar{c}(\mathbf{q}_2,\eta)\rangle$$
$$=4E_{q_1}E_{q_2}(2\pi)^6\delta^3(\mathbf{q}_1-\mathbf{q}_1')\delta^3(\mathbf{q}_2-\mathbf{q}_2')\xi^\dagger\xi'\,\eta'^\dagger\eta, \quad (3.3)$$

where the spinors are normalized so that $\xi^\dagger\xi=\eta^\dagger\eta=1$, and similarly for ξ' and η'. In expressions like $\xi^\dagger\xi'$, both the spin and color indices are contracted. Note that with the normalization (3.3), the operator $P_{cc',cc}$ in Eq. (3.2) has dimension -4. The difference in the dimensions of the operators P_H and $P_{cc',cc}$ matches the difference in the dimensions of the T-matrix elements in Eqs. (2.1) and (3.1).

To carry out the matching procedure, the left side of Eq. (3.1) is calculated using perturbation theory in full QCD, and then expanded as a Taylor series in $\mathbf{q}$ and $\mathbf{q}'$. The matrix elements on the right side of Eq. (3.1) are calculated using perturbation theory in NRQCD, and then expanded as Taylor series in $\mathbf{q}$ and $\mathbf{q}'$. The short-distance coefficients C_{mn} are obtained by matching the terms in these Taylor expansions order by order in α_s.

B. Matching for $q\bar{q}\to c\bar{c}$

We first illustrate the matching procedure by applying it to the process $q\bar{q}\to c\bar{c}$. The T-matrix element in Feynman gauge is

$$\mathcal{T}_{12\to c\bar{c}}=g^2\frac{1}{P^2}\overline{v}(k_2)\gamma_\mu T^a u(k_1)\overline{u}(p)\gamma^\mu T^a v(\bar{p}). \quad (3.4)$$

Dirac and color indices are implicit in the spinors. The four-momenta of the outgoing c and $\bar{c}$ can be expressed as $p=\frac{1}{2}P+L\mathbf{q}$ and $\bar{p}=\frac{1}{2}P-L\mathbf{q}$, where P is the total four-momentum of the $c\bar{c}$ pair, $\mathbf{q}$ is the relative three-momentum of the c in the center-of-momentum (c.m.) frame of the pair, and L^μ_i is a Lorentz boost matrix. The boost matrix L^μ_i transforms a purely spacelike four-vector, such as $(0,\mathbf{q})$, from the c.m. frame where the components of P are $(2E_q,\mathbf{0})$ to the frame in which its components are P^μ. The matrix is given explicitly in Appendix A. The expressions for the Dirac spinors $u(p)$ and $v(\bar{p})$ in terms of Pauli spinors ξ and η are also given in Appendix A. Identities in Appendix A can be used to expand $\overline{u}(p)\gamma^\mu T^a v(\bar{p})$ as a Taylor series in $\mathbf{q}$. To linear order in $\mathbf{q}$, the T-matrix element (3.4) is

$$\mathcal{T}_{12\to c\bar{c}}=\frac{g^2}{2m_c}\overline{v}(k_2)\gamma_\mu T^a u(k_1)L^\mu_i\xi^\dagger\sigma^i T^a\eta. \quad (3.5)$$

The components of the boost matrix to linear order in $\mathbf{q}$ are

$$L_i^0 = \frac{1}{2m_c} P^i, \tag{3.6a}$$

$$L_i^j = \delta^{ji} - \hat{P}^j \hat{P}^i + \frac{E_P}{2m_c} \hat{P}^j \hat{P}^i, \tag{3.6b}$$

where $E_P = \sqrt{4m_c^2 + \mathbf{P}^2}$.

In order to carry out the matching procedure, we first calculate the left side of Eq. (3.1). In this case, there is no sum over X. Multiplying Eq. (3.5) by the complex conjugate of $\mathcal{T}_{12\to c\bar{c}'}$ and averaging over initial spins and colors, we obtain

$$\begin{aligned}(2\pi)^4 \delta^4(k_1+k_2-P) \overline{\sum} \, \mathcal{T}^*_{12\to c\bar{c}'} \mathcal{T}_{12\to c\bar{c}} \\ = (2\pi)^4 \delta^4(k_1+k_2-P) \frac{4\pi^2 \alpha_s^2}{9} \\ \times [\delta^{ji} - \hat{n}^j \hat{n}^i] \eta'^\dagger \sigma^j T^a \xi' \xi^\dagger \sigma^i T^a \eta. \end{aligned} \tag{3.7}$$

We have simplified the expression by using the identities (A.5) and (A.6a) in Appendix A. We have also used the fact that $(k_1-k_2)_\mu L_i^\mu = -\sqrt{P^2} \hat{n}^i$, where $\hat{\mathbf{n}}$ is a unit vector, which follows from Eq. (A.6b). In a frame in which $\mathbf{P}$, $\mathbf{k_1}$, and $\mathbf{k_2}$ are collinear, they are also collinear with $\hat{\mathbf{n}}$.

We next consider the right side of the matching equation (3.1). We must construct an operator of the form (2.3) whose vacuum matrix element, when calculated to leading order in α_s, reproduces the spinor factor in Eq. (3.7) upon expanding to linear order in $\mathbf{q}$ and $\mathbf{q}'$. One can see by inspection that the appropriate operator is $\chi^\dagger \sigma^j T^a \psi \mathcal{P}_{c\bar{c}',c\bar{c}} \psi^\dagger \sigma^i T^a \chi$. At leading order in α_s, only the $c\bar{c}$ states in the projection operator $P_{cc',cc}$ contribute:

$$\begin{aligned}\langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{c\bar{c}',c\bar{c}} \psi^\dagger \sigma^i T^a \chi \rangle = \langle 0 | \chi^\dagger \sigma^j T^a \psi | c\bar{c}(\mathbf{q}', \xi', \eta') \rangle \\ \times \langle c\bar{c}(\mathbf{q}, \xi, \eta) | \psi^\dagger \sigma^i T^a \chi | 0 \rangle. \end{aligned} \tag{3.8}$$

The vacuum-to-$c\bar{c}$ matrix element reduces at leading order in α_s to

$$\langle c\bar{c}(\mathbf{q}, \xi, \eta) | \psi^\dagger \sigma^i T^a \chi | 0 \rangle = 2E_q \xi^\dagger \sigma^i T^a \eta, \tag{3.9}$$

where the factor of $2E_q$ arises from the relativistic normalization of states. Expanding to linear order in $\mathbf{q}$ and $\mathbf{q}'$, the matrix element (3.8) reduces to

$$\langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{c\bar{c}',c\bar{c}} \psi^\dagger \sigma^i T^a \chi \rangle = 4m_c^2 \eta'^\dagger \sigma^j T^a \xi' \xi^\dagger \sigma^i T^a \eta. \tag{3.10}$$

Comparing Eqs. (3.7) and (3.10), we can read off the short-distance coefficient for the matrix element:

$$C^{ji} = (2\pi)^4 \delta^4(k_1+k_2-P) \frac{\pi^2 \alpha_s^2}{9m_c^2} [\delta^{ji} - \hat{n}^j \hat{n}^i]. \tag{3.11}$$

Having determined the short-distance coefficient, we can insert it into Eq. (2.2) to obtain an expression for the differential cross section for producing a meson H with helicity λ via the parton process $q\bar{q} \to c\bar{c}$:

$$\begin{aligned} d\sigma(q\bar{q} \to H(P,\lambda)) = \frac{1}{4E_1 E_2 v_{12}} \frac{d^3P}{(2\pi)^3 2E_P} (2\pi)^4 \delta^4 \\ \times (k_1+k_2-P) \frac{\pi^2 \alpha_s^2}{9m_c^2} [\delta^{ji} - \hat{n}^j \hat{n}^i] \\ \times \langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{H(\lambda)} \psi^\dagger \sigma^i T^a \chi \rangle. \end{aligned} \tag{3.12}$$

Using the delta function to integrate over the phase space of P, we find that the cross section is

$$\begin{aligned}\sigma(q\bar{q} \to H(\lambda)) = \delta(s - 4m_c^2) \frac{\pi^3 \alpha_s^2}{36 m_c^4} [\delta^{ji} - \hat{z}^j \hat{z}^i] \\ \times \langle \chi^\dagger \sigma^j T^a \psi \mathcal{P}_{H(\lambda)} \psi^\dagger \sigma^i T^a \chi \rangle, \end{aligned} \tag{3.13}$$

where $s = (k_1+k_2)^2$ and we have assumed that the three-momentum of the colliding partons is along the z axis. Note that the expression (3.13) is dimensionally correct. Each of the quark fields has dimension $\frac{3}{2}$ and the projection operator has dimension -2, so the dimension of the matrix element is 4. The expression (3.13) gives a contribution to the cross section for every quarkonium state H.

C. Matching for $gg \to c\bar{c}$

We next illustrate the matching procedure by applying it to the process $gg \to c\bar{c}$. The T-matrix element can be decomposed into three independent color structures:

$$\begin{aligned}\mathcal{T}_{gg\to c\bar{c}} = -g^2 \epsilon_\mu^a(k_1) \epsilon_\nu^b(k_2) \Bigg(\frac{1}{6} \delta^{ab} S^{\mu\nu} + \frac{1}{2} d^{abc} D^{\mu\nu c} \\ + \frac{i}{2} f^{abc} F^{\mu\nu c} \Bigg). \end{aligned} \tag{3.14}$$

The Dirac factor for the δ^{ab} term is

$$S^{\mu\nu} = \overline{u(p)} \left[\frac{\gamma^\mu (\not{p} - \not{k}_1 + m_c) \gamma^\nu}{2p\cdot k_1} + \frac{\gamma^\nu (\not{p} - \not{k}_2 + m_c) \gamma^\mu}{2p\cdot k_2} \right] v(\bar{p}). \tag{3.15}$$

The term $D^{\mu\nu c}$ in Eq. (3.14) differs from Eq. (3.15) only by inserting the color matrix T^c between the spinors. The Dirac factor for the f^{abc} term in Feynman gauge is

$$\begin{aligned} F^{\mu\nu c} = \overline{u(p)} \Bigg[\frac{\gamma^\mu (\not{p} - \not{k}_1 + m_c) \gamma^\nu}{2p\cdot k_1} - \frac{\gamma^\nu (\not{p} - \not{k}_2 + m_c) \gamma^\mu}{2p\cdot k_2} \\ - 2 \frac{(P+k_2)^\mu \gamma^\nu - (P+k_1)^\nu \gamma^\mu + g^{\mu\nu}(\not{k}_1 - \not{k}_2)}{P^2} \Bigg] \\ \times T^c v(\bar{p}). \end{aligned} \tag{3.16}$$

Using the identities in Appendix A, we expand Eqs. (3.15) and (3.16) to linear order in $\mathbf{q}$:

$$S^{\mu\nu}=\frac{i}{2m_c^2}\epsilon^{\mu\nu\lambda\rho}(k_1-k_2)_\lambda P_\rho\xi^\dagger\eta+\left[\frac{1}{m_c^3}(k_1\cdot L)_n((k_1-k_2)^\mu L_j^\nu+(k_1-k_2)^\nu L_j^\mu)-\frac{1}{m_c^3}(k_1\cdot L)_j(P^\mu L_n^\nu-P^\nu L_n^\mu)\right.$$
$$\left.-\frac{2}{m_c^3}g^{\mu\nu}(k_1\cdot L)_n(k_1\cdot L)_j+\frac{2}{m_c}(L_n^\mu L_j^\nu+L_n^\nu L_j^\mu)\right]q^n\xi^\dagger\sigma^j\eta, \tag{3.17}$$

$$F^{\mu\nu c}=-\frac{1}{m}(k_1^\mu L_j^\nu-k_2^\nu L_j^\mu)\xi^\dagger\sigma^jT^a\eta-\frac{i}{2m_c^4}\epsilon^{\mu\nu\lambda\rho}(k_1-k_2)_\lambda P_\rho(k_1\cdot L)_nq^n\xi^\dagger T^c\eta. \tag{3.18}$$

The first term in $F^{\mu\nu c}$ vanishes when contracted with the polarization vectors $\epsilon_\mu^a(k_1)$ and $\epsilon_\nu^b(k_2)$ in Eq. (3.14). Multiplying Eq. (3.14) by the complex conjugate of $\mathcal{T}_{gg\to c\bar{c}'}$ and averaging over the spins and colors of the initial gluons, we obtain

$$\overline{\sum}\mathcal{T}^*_{gg\to c\bar{c}'}\mathcal{T}_{gg\to c\bar{c}}=\frac{\pi^2\alpha_s^2}{9}\left\{\eta'^\dagger\xi'\xi^\dagger\eta+\frac{15}{8}\eta'^\dagger T^a\xi'\xi^\dagger T^a\eta+\frac{1}{m_c^2}[(\delta^{mn}-\hat{n}^m\hat{n}^n)(\delta^{ij}-\hat{n}^i\hat{n}^j)+(\delta^{mj}-\hat{n}^m\hat{n}^j)(\delta^{ni}-\hat{n}^n\hat{n}^i)\right.$$
$$+\delta^{mi}\delta^{nj}-(\delta^{mi}-\hat{n}^m\hat{n}^i)(\delta^{nj}-\hat{n}^n\hat{n}^j)]q'^mq^n\left(\eta'^\dagger\sigma^i\xi'\xi^\dagger\sigma^j\eta+\frac{15}{8}\eta'^\dagger\sigma^iT^a\xi'\xi^\dagger\sigma^jT^a\eta\right)$$
$$\left.+\frac{27}{8m_c^2}\hat{n}^m\hat{n}^nq'^mq^n\eta'^\dagger T^a\xi'\xi^\dagger T^a\eta\right\}. \tag{3.19}$$

At leading order in α_s and to linear order in $\mathbf{q}$ and $\mathbf{q}'$, the spinor factors on the right side of Eq. (3.19) can be identified with the following matrix elements:

$$\langle\chi^\dagger\psi\mathcal{P}_{c\bar{c}',c\bar{c}}\psi^\dagger\chi\rangle=4m_c^2\eta'^\dagger\xi'\xi^\dagger\eta, \tag{3.20a}$$

$$\langle\chi^\dagger T^a\psi\mathcal{P}_{c\bar{c}',c\bar{c}}\psi^\dagger T^a\chi\rangle=4m_c^2\eta'^\dagger T^a\xi'\xi^\dagger T^a\eta, \tag{3.20b}$$

$$\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^m\right)T^a\psi\mathcal{P}_{c\bar{c}',c\bar{c}}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^n\right)T^a\chi\right\rangle=4m_c^2q'^mq^n\eta'^\dagger T^a\xi'\xi^\dagger T^a\eta, \tag{3.20c}$$

$$\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^m\right)\sigma^iT^a\psi\mathcal{P}_{c\bar{c}',c\bar{c}}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^n\right)\sigma^jT^a\chi\right\rangle=4m_c^2q'^mq^n\eta'^\dagger\sigma^iT^a\xi'\xi^\dagger\sigma^jT^a\eta, \tag{3.20d}$$

where $\psi^\dagger\overleftrightarrow{\mathbf{D}}\chi=\psi^\dagger\mathbf{D}\chi-(\mathbf{D}\psi)^\dagger\chi$.

Matching terms on the left side and right side of Eq. (3.1), we determine the short-distance coefficient C_{mn} for each of the matrix elements. Inserting these coefficients into Eq. (2.2), we obtain the differential cross section for production of $H(\lambda)$ from the parton process $gg\to c\bar{c}$. The final result, after integrating over phase space, is

$$\sigma(gg\to H(\lambda))=\delta(s-4m_c^2)\frac{\pi^3\alpha_s^2}{144m_c^4}\left\{\langle\chi^\dagger\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\chi\rangle+\frac{15}{8}\langle\chi^\dagger T^a\psi\mathcal{P}_{H(\lambda)}\psi^\dagger T^a\chi\rangle+\frac{1}{m_c^2}[(\delta^{mn}-\hat{z}^m\hat{z}^n)(\delta^{ij}-\hat{z}^i\hat{z}^j)\right.$$
$$+(\delta^{mj}-\hat{z}^m\hat{z}^j)(\delta^{ni}-\hat{z}^n\hat{z}^i)+\delta^{mi}\delta^{nj}-(\delta^{mi}-\hat{z}^m\hat{z}^i)(\delta^{nj}-\hat{z}^n\hat{z}^j)]$$
$$\times\left(\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^m\right)\sigma^i\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^n\right)\sigma^j\chi\right\rangle+\frac{15}{8}\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^m\right)\sigma^iT^a\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^n\right)\sigma^jT^a\chi\right\rangle\right)$$
$$\left.+\frac{27}{8m_c^2}\hat{z}^m\hat{z}^n\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^m\right)T^a\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^n\right)T^a\chi\right\rangle\right\}. \tag{3.21}$$

The expression (3.21) for the cross section applies equally well to any quarkonium state H. The relative importance of the various terms for a given state H depends on the magnitude of the matrix elements.

IV. REDUCING THE MATRIX ELEMENTS

The cross sections (3.13) and (3.21) hold for any quarkonium state H, with the dependence on H appearing only in the matrix elements. The matrix elements can be simplified by using rotational symmetry and the approximate heavy-quark spin symmetry of NRQCD. Their magnitudes can be estimated using velocity-scaling rules for NRQCD matrix elements. In this section, we apply these methods to simplify the matrix elements that appear in inclusive cross sections for the production of J/ψ. The extension to other quarkonium states will be presented elsewhere [5].

A. Rotational symmetry

Under rotations, each of the matrix elements in Eqs. (3.13) and (3.21) transforms as a component of a tensor. The indices of the tensor include vector indices, such as i, j, m, and n, and the helicity λ in the projection operator $\mathcal{P}_{H(\lambda)}$. The helicity λ appears both in the ket and in the bra in the expression for the projection operator given in Eq. (2.4). If the meson has total angular momentum J, then $\mathcal{P}_{H(\lambda)}$ transforms like the $(\lambda,-\lambda)$ component of the direct product representation $\underline{J}\otimes\underline{J}$. Thus the entire operator transforms as an element of the representation $\underline{J}\otimes\underline{J}\otimes\underline{1}\otimes\underline{1}\otimes\cdots$, where there is a $\underline{1}$ for each vector index of the operator. The vacuum matrix element of the operator can be expressed in terms of N independent matrix elements, where N is the number of times the trivial representation $\underline{0}$ appears in the decomposition of the direct product $\underline{J}\otimes\underline{J}\otimes\underline{1}\otimes\underline{1}\otimes\cdots$ into irreducible representations.

In the case of the J/ψ, the independent matrix elements can be determined by elementary tensor analysis. The ψ has total angular momentum $J=1$, and the helicity label λ is just a vector index in a spherical basis. The spherical basis and the Cartesian basis are related by a unitary transformation with matrix

$$U_{\lambda i}=\begin{pmatrix} -1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -i/\sqrt{2} & 0 \end{pmatrix}_{\lambda i}. \tag{4.1}$$

The helicity labels λ in the ket and in the bra in Eq. (2.4) are related to the corresponding Cartesian indices i and j by the transformation matrices $U_{\lambda i}$ and $U^\dagger_{j\lambda}$, respectively. Since there are no momentum vectors that the matrix element can depend on, it must be expressible in terms of the invariant tensors δ^{mn} and ϵ^{lmn}. The number of independent matrix elements is the number of invariant tensors that can be formed out of the available indices.

We first consider operators with no vector indices, such as $\langle\chi^\dagger\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\chi\rangle$. The only indices that are available are λ and λ from the projection operator $\mathcal{P}_{\psi(\lambda)}$. The only invariant tensor is $\delta_{\lambda\lambda}=1$. Therefore, we have

$$\langle\chi^\dagger\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\chi\rangle=\frac{1}{3}\langle\chi^\dagger\psi\mathcal{P}_{\psi}\psi^\dagger\chi\rangle, \tag{4.2}$$

where $\mathcal{P}_\psi=\Sigma_\lambda\mathcal{P}_{\psi(\lambda)}$. Standard production matrix elements $\mathcal{O}_1^\psi({}^{2S+1}L_J)$ and $\mathcal{O}_8^\psi({}^{2S+1}L_J)$ were defined in Ref. [1] using a projection operator analogous to Eq. (2.4), but constructed out of states with the standard nonrelativistic normalization. As discussed in Appendix B, the two projection operators differ at leading order simply by a normalization factor of $4m_c$. Thus, we can write

$$\langle\chi^\dagger\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\chi\rangle=\frac{4}{3}m_c\langle\mathcal{O}_1^\psi({}^1S_0)\rangle, \tag{4.3a}$$

$$\langle\chi^\dagger T^a\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger T^a\chi\rangle=\frac{4}{3}m_c\langle\mathcal{O}_8^\psi({}^1S_0)\rangle, \tag{4.3b}$$

where the matrix elements on the right sides are defined in Appendix B.

We next consider matrix elements with two vector indices, such as $\langle\chi^\dagger\sigma^i\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\sigma^j\chi\rangle$. Rotational symmetry implies that such a matrix element can be expressed in terms of three independent tensors: δ^{ij}, $U_{\lambda i}U^\dagger_{j\lambda}$, and $U_{\lambda j}U^\dagger_{i\lambda}$. Thus there are three independent matrix elements. If we sum over the helicities λ, the only possible tensor is δ^{ij}, so there is only one possible matrix element. Thus, up to corrections of relative order v^2, we have

$$\sum_\lambda\langle\chi^\dagger\sigma^i\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\sigma^j\chi\rangle=\frac{4}{3}\delta^{ij}m_c\langle\mathcal{O}_1^\psi({}^3S_1)\rangle, \tag{4.4a}$$

$$\sum_\lambda\langle\chi^\dagger\sigma^iT^a\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\sigma^jT^a\chi\rangle=\frac{4}{3}\delta^{ij}m_c\langle\mathcal{O}_8^\psi({}^3S_1)\rangle, \tag{4.4b}$$

$$\sum_\lambda\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^j\right)\chi\right\rangle=\frac{4}{3}\delta^{ij}m_c\langle\mathcal{O}_1^\psi({}^1P_1)\rangle, \tag{4.4c}$$

$$\sum_\lambda\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)T^a\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^j\right)T^a\chi\right\rangle=\frac{4}{3}\delta^{ij}m_c\langle\mathcal{O}_8^\psi({}^1P_1)\rangle, \tag{4.4d}$$

where the matrix elements on the right sides are defined in Appendix B. In the cross section (3.21), we also require the matrix element $\langle\chi^\dagger(-(i/2)\overleftrightarrow{D}^j)T^a\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger(-(i/2)\overleftrightarrow{D}^j)T^a\chi\rangle$ contracted with $\hat{z}^i\hat{z}^j$. This can be expressed in terms of $\langle\mathcal{O}_8^\psi({}^1P_1)\rangle$ and one additional matrix element

$$\hat{z}^i\hat{z}^j\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)T^a\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^j\right)T^a\chi\right\rangle=\frac{2(1-\delta_{\lambda 0})}{3}m_c\langle\mathcal{O}_8^\psi({}^1P_1)\rangle+\frac{3\delta_{\lambda 0}-1}{2}\times\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^3\right)T^a\psi\mathcal{P}_{\psi(0)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^3\right)T^a\chi\right\rangle. \tag{4.5}$$

Upon summing over helicities, we recover Eq. (4.4d).

Matrix elements with four vector indices, such as $\langle\chi^\dagger\sigma^i(-(i/2)\overleftrightarrow{D}^m)\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\sigma^j(-(i/2)\overleftrightarrow{D}^n)\chi\rangle$, can be reduced by rotational symmetry to 15 independent matrix elements. Upon summing over helicities, the number of independent matrix elements is reduced to three.

B. Vacuum-saturation approximation

The vacuum matrix element $\langle\chi^\dagger\sigma^i\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\sigma^j\chi\rangle$ can be simplified by using the *vacuum-saturation approximation*. In this approximation, only the NRQCD vacuum is retained in the sum over soft states in the projection operator (2.4). The matrix element then reduces to

$$\langle \chi^\dagger \sigma^i \psi \mathcal{P}_{\psi(\lambda)} \psi^\dagger \sigma^j \chi \rangle \approx \langle 0 | \chi^\dagger \sigma^i \psi | \psi(\mathbf{P}=0,\lambda) \rangle \times \langle \psi(\mathbf{P}=0,\lambda) | \psi^\dagger \sigma^j \chi | 0 \rangle. \quad (4.6)$$

As pointed out in Ref. [1], the vacuum-saturation approximation is a controlled approximation in the case of heavy quarkonium, with a relative error of order v^4. The reason for this is that the matrix element involves a transition from a color-singlet $c\bar{c}$ state to states of the form $|\psi(\lambda)+S\rangle$, where S is a soft state. Since the ψ is a color singlet, the state S must also be a color singlet. Suppose it is not the NRQCD vacuum. Since a single chromoelectric dipole transition will produce a color-octet state, at least two chromoelectric dipole transitions are required to produce the state S. The amplitude is suppressed by a power of v for each such transition. There is another power of v^2 from the complex conjugate amplitude, and this gives an overall suppression factor of v^4.

After using the vacuum-saturation approximation, the matrix element (4.6) can be further simplified by using rotational symmetry. The vacuum-to-ψ matrix element must have the form

$$\langle \psi(\mathbf{P}=0,\lambda) | \psi^\dagger \sigma^j \chi | 0 \rangle = U^\dagger_{j\lambda} \sqrt{2M_\psi} \sqrt{\frac{3}{2\pi}} \overline{R_\psi}. \quad (4.7)$$

The factor of $\sqrt{2M_\psi}$ comes from the relativistic normalization of the charmonium state. The factor $\sqrt{3/2\pi}$ has been inserted so that $\overline{R_\psi}$ can be interpreted as the nonrelativistic radial wavefunction evaluated at the origin. Inserting Eq. (4.7) into Eq. (4.6), it reduces to

$$\langle \chi^\dagger \sigma^i \psi \mathcal{P}_{\psi(\lambda)} \psi^\dagger \sigma^j \chi \rangle = U_{\lambda i} U^\dagger_{j\lambda} \frac{3}{\pi} M_\psi |\overline{R_\psi}|^2 + O(v^4 m_c |\overline{R_\psi}|^2). \quad (4.8)$$

C. Heavy-quark spin symmetry

The most powerful tool at our disposal for simplifying the production matrix elements of heavy quarkonium is the approximate heavy-quark spin symmetry of NRQCD. The spin symmetry transformations of the heavy-quark and antiquark fields are

$$\psi(\mathbf{x},t) \to V \psi(\mathbf{x},t), \quad (4.9a)$$

$$\chi(\mathbf{x},t) \to W \chi(\mathbf{x},t), \quad (4.9b)$$

where V and W are SU(2) matrices. Under a rotation, the transformation of the fields is

$$\psi(\mathbf{x},t) \to V \psi(O \cdot \mathbf{x},t), \quad (4.10a)$$

$$\chi(\mathbf{x},t) \to V \chi(O \cdot \mathbf{x},t), \quad (4.10b)$$

where O is the O(3) matrix whose elements are

$$O^{ij} = \frac{1}{2} \mathrm{tr}(V^\dagger \sigma^i V \sigma^j). \quad (4.11)$$

While rotational symmetry is an exact symmetry of NRQCD, spin symmetry is only an approximate symmetry. NRQCD is equivalent to full QCD in the sense that the coefficients of the terms in the NRQCD Lagrangian can be tuned so that the effective theory reproduces the quarkonium spectrum and low-energy quarkonium matrix elements to any desired order in v. Minimal NRQCD reproduces the spectrum and matrix elements to a relative accuracy of order v^2. This theory has exact heavy-quark spin symmetry. With the v^2-improved NRQCD Lagrangian, the accuracy is improved to v^4. However the v^2-improved Lagrangian includes a term $\psi^\dagger \mathbf{B} \cdot \boldsymbol{\sigma} \psi - \chi^\dagger \mathbf{B} \cdot \boldsymbol{\sigma} \chi$ that breaks the spin symmetry. Thus spin symmetry gives relations between matrix elements that are accurate up to corrections of relative order v^2.

The consequences of spin symmetry are particularly simple for spin-triplet matrix elements of the ψ, such as $\langle \chi^\dagger \sigma^i \psi \mathcal{P}_{\psi(\lambda)} \psi^\dagger \sigma^j \chi \rangle$. Since the ψ is an S-wave state, its helicity λ is a spin index. Spin symmetry implies that the helicity label λ in the ket of the projection operator $\mathcal{P}_{\psi(\lambda)}$ must be matched with the vector index i, while the λ in the bra must be matched with the index j. Thus the matrix element must be proportional to $U_{\lambda i} U^\dagger_{j\lambda}$ The proportionality constant is obtained by summing over λ and contracting with δ^{ij}. We find that, up to corrections of relative order v^2,

$$\langle \chi^\dagger \sigma^i \psi \mathcal{P}_{\psi(\lambda)} \psi^\dagger \sigma^j \chi \rangle = \frac{4}{3} U_{\lambda i} U^\dagger_{j\lambda} m_c \langle \mathcal{O}_1^\psi(^3S_1) \rangle, \quad (4.12a)$$

$$\langle \chi^\dagger \sigma^i T^a \psi \mathcal{P}_{\psi(\lambda)} \psi^\dagger \sigma^j T^a \chi \rangle = \frac{4}{3} U_{\lambda i} U^\dagger_{j\lambda} m_c \langle \mathcal{O}_8^\psi(^3S_1) \rangle. \quad (4.12b)$$

The relative error of v^2 in Eq. (4.12a) is larger than the relative error of v^4 obtained by using the vacuum saturation approximation (4.8). One can easily show that the error in Eq. (4.12a) can be improved to v^4 by replacing $4m_c$ by $2M_\psi$.

The matrix element

$$\left\langle \chi^\dagger \sigma^i \left(-\frac{i}{2} \overleftrightarrow{D}^m \right) \psi \mathcal{P}_{\psi(\lambda)} \psi^\dagger \sigma^j \left(-\frac{i}{2} \overleftrightarrow{D}^n \right) \chi \right\rangle$$

can also be reduced to a single matrix element by using spin symmetry. This symmetry implies that the tensor structure in the indices λ, i, and j must be the same as in Eq. (4.12b). Rotational symmetry then implies that the matrix element must also be proportional to δ^{mn}. Thus, up to corrections of relative order v^2, we have

$$\left\langle \chi^\dagger \sigma^i \left(-\frac{i}{2} \overleftrightarrow{D}^m \right) \psi \mathcal{P}_{\psi(\lambda)} \psi^\dagger \sigma^j \left(-\frac{i}{2} \overleftrightarrow{D}^n \right) \chi \right\rangle = 4 U_{\lambda i} U^\dagger_{j\lambda} \delta^{mn} m_c \langle \mathcal{O}_1^\psi(^3P_0) \rangle, \quad (4.13a)$$

$$\left\langle \chi^\dagger \sigma^i \left(-\frac{i}{2} \overleftrightarrow{D}^m \right) T^a \psi \mathcal{P}_{\psi(\lambda)} \psi^\dagger \sigma^j \left(-\frac{i}{2} \overleftrightarrow{D}^n \right) T^a \chi \right\rangle = 4 U_{\lambda i} U^\dagger_{j\lambda} \delta^{mn} m_c \langle \mathcal{O}_8^\psi(^3P_0) \rangle. \quad (4.13b)$$

Spin symmetry can also be applied to spin-singlet matrix elements of the ψ, such as

$$\left\langle \chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\psi\mathcal{P}_{\psi(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^j\right)\chi\right\rangle.$$

A spin-singlet operator like $\psi^\dagger(-(i/2)\overleftrightarrow{D}^j)\chi$ creates a $c\bar{c}$ pair in a spin-singlet state. At leading order in v^2, the transition to a final state of the form $|\psi+S\rangle$ must involve the term $\psi^\dagger\mathbf{B}\cdot\boldsymbol{\sigma}\psi-\chi^\dagger\mathbf{B}\cdot\boldsymbol{\sigma}\chi$ from the v^2-improved NRQCD Lagrangian. The resulting constraints on the matrix element can be deduced using angular momentum theory. These constraints are simply those that are already provided by rotational invariance. Spin symmetry does give new relations between the production matrix elements of the ψ and those of the η_c, but they are not of great practical significance.

D. Velocity-scaling rules

The relative importance of the NRQCD matrix elements that appear in cross sections for ψ production are determined by how they scale with the relative velocity v of the charm quark in the meson [1]. When applied to the matrix elements $\langle\mathcal{O}_n({}^{2S+1}L_J)\rangle$, the scaling rules are fairly simple. The operator $\mathcal{O}_n({}^{2S+1}L_J)$ creates and annihilates a pointlike $c\bar{c}$ pair in the color state $\underline{n}$ and in the angular momentum state ${}^{2S+1}L_J$. The scaling of the matrix element is determined by the orbital angular momentum quantum number L and by the number of chromoelectric and chromomagnetic dipole transitions that are required for the $c\bar{c}$ pair to reach the state $c\bar{c}(\underline{n},{}^{2S+1}L_J)$ from the dominant Fock state of the meson. If the minimum number of these transitions is E and M, respectively, the matrix element $\langle\mathcal{O}_n({}^{2S+1}L_J)\rangle$ scales as $v^{3+2L+2E+4M}$.

The dominant Fock state of the ψ consists of a $c\bar{c}$ pair in a color-singlet ${}^3S_1$ state. Thus the matrix element $\langle\mathcal{O}_1({}^3S_1)\rangle$ scales as v^3. The $c\bar{c}$ pair can reach the color-octet states $c\bar{c}(\underline{8},{}^3P_J)$, $c\bar{c}(\underline{8},{}^3S_1)$, and $c\bar{c}(\underline{8},{}^1S_0)$) through a single chromoelectric, a double chromoelectric, and a single chromomagnetic dipole transition, respectively. Thus the matrix elements $\langle\mathcal{O}_8({}^3P_J)\rangle$, $\langle\mathcal{O}_8({}^3S_1)\rangle$, and $\langle\mathcal{O}_8({}^1S_0)\rangle$ all scale as v^7. All the other matrix elements $\langle\mathcal{O}_n({}^{2S+1}L_J)\rangle$ scale as v^{11} or smaller. Those that scale like v^{11} include $\langle\mathcal{O}_1({}^1P_1)\rangle$, $\langle\mathcal{O}_8({}^1P_1)\rangle$, $\langle\mathcal{O}_1({}^1S_0)\rangle$, and $\langle\mathcal{O}_1({}^3P_J)\rangle$. The matrix element

$$\left\langle \chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^3\right)T^a\psi\mathcal{P}_{\psi(0)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^3\right)T^a\chi\right\rangle$$

also scales as v^{11} like $\langle\mathcal{O}_8({}^1P_1)\rangle$.

V. CROSS SECTIONS FOR POLARIZED ψ

In this section, we combine the results of Sec. III and IV to obtain the cross sections for production of ψ with definite helicity via the order-α_s^2 parton processes $q\bar{q}\to c\bar{c}$ and $gg\to c\bar{c}$. These cross sections can be folded with parton distributions to obtain cross sections for ψ production in hadron-hadron scattering. The resulting cross sections should not be taken too seriously, since these parton processes produce a ψ with zero transverse momentum. The factorization formula (2.2) is therefore not strictly applicable, since its derivation requires the transverse momentum of the ψ to be large compared to $\Lambda_{\rm QCD}$. We choose to ignore this difficulty, since our primary purpose is to illustrate the calculation of cross sections for polarized production.

The cross section for ψ production via the parton process $q\bar{q}\to c\bar{c}$ is obtained by inserting the matrix element (4.12b) into Eq. (3.13):

$$\sigma(q\bar{q}\to\psi(\lambda))=\delta(s-4m_c^2)\frac{\pi^3\alpha_s^2}{27m_c^3}(1-\delta_{\lambda 0})\langle O_8^\psi({}^3S_1)\rangle. \tag{5.1}$$

We have used the fact that the matrix U in Eq. (4.1) satisfies $\Sigma_i U_{\lambda i}U^\dagger_{i\lambda}=1$ and $\Sigma_j U^\dagger_{j\lambda}\hat{z}^j=\delta_{\lambda 0}$. The cross section (5.1) implies that ψ's produced by this process are transversely polarized, which could be anticipated from the fact that the gluon interaction with the light quark conserves helicity. Summing over helicities, we obtain

$$\sigma(q\bar{q}\to\psi)=\delta(s-4m_c^2)\frac{2\pi^3\alpha_s^2}{27m_c^3}\langle O_8^\psi({}^3S_1)\rangle. \tag{5.2}$$

The cross section for producing a ψ with helicity λ via the parton process $gg\to c\bar{c}$ is obtained by inserting the matrix elements (4.3), (4.5), and (4.13) into Eq. (3.21):

$$\sigma(gg\to\psi(\lambda))=\delta(s-4m_c^2)\frac{\pi^3\alpha_s^2}{108m_c^3}\Bigg\{\langle O_1^\psi({}^1S_0)\rangle+\frac{15}{8}\langle O_8^\psi({}^1S_0)\rangle+\frac{3(3-2\delta_{\lambda0})}{m_c^2}\langle O_1^\psi({}^3P_0)\rangle+\frac{45(3-2\delta_{\lambda0})}{8m_c^2}\langle O_8^\psi({}^3P_0)\rangle$$
$$+\frac{27(1-\delta_{\lambda0})}{16m_c^2}\langle O_8^\psi({}^1P_1)\rangle+\frac{81(1-3\delta_{\lambda0})}{64m_c^3}\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^3\right)T^a\psi\mathcal{P}_{\psi(0)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^3\right)T^a\chi\right\rangle\Bigg\}. \tag{5.3}$$

Summing over helicities, this reduces to

$$\sigma(gg\to\psi)=\delta(s-4m_c^2)\frac{\pi^3\alpha_s^2}{36m_c^3}\Bigg\{\langle O_1^\psi({}^1S_0)\rangle+\frac{15}{8}\langle O_8^\psi({}^1S_0)\rangle+\frac{7}{m_c^2}\langle O_1^\psi({}^3P_0)\rangle+\frac{105}{8m_c^2}\langle O_8^\psi({}^3P_0)\rangle+\frac{9}{8m_c^2}\langle O_8^\psi({}^1P_1)\rangle\Bigg\}. \tag{5.4}$$

Note that the contribution from the matrix element $\langle\chi^\dagger(-(i/2)\overleftrightarrow{D}^3)T^a\psi\mathcal{P}_{\psi(0)}\psi^\dagger(-(i/2)\overleftrightarrow{D}^3)T^a\chi\rangle$ drops out of the cross section for unpolarized production. If we retain only those matrix elements that are of leading order in v^2, the cross section (5.3) reduces to

$$\sigma(gg\to\psi(\lambda))\approx\delta(s-4m_c^2)\frac{5\pi^3\alpha_s^2}{288m_c^3}\Bigg\{\langle O_8^\psi(^1S_0)\rangle + \frac{3(3-2\delta_{\lambda0})}{m_c^2}\langle O_8^\psi(^3P_0)\rangle\Bigg\}. \tag{5.5}$$

The $\langle O_8^\psi(^1S_0)\rangle$ term gives ψ's that are unpolarized, while the $\langle O_8^\psi(^3P_0)\rangle$ term gives ψ's with helicity -1, 0, and $+1$ in the proportions $3:1:3$.

The production of J/ψ from the color-octet parton processes $q\bar{q}\to c\bar{c}$ and $gg\to c\bar{c}$ has been studied previously by several groups [6–8]. In particular, the spin alignment of the ψ from these processes has been calculated by Tang and Väntinnen [6] and by Cho and Leibovich [7]. We proceed to compare our results with this previous work.

Tang and Väntinnen [6] have used the covariant projection method to calculate the cross sections for polarized ψ and ψ' in fixed-target hadron-hadron collisions from the order-α_s^2 parton processes. Our result (5.1) for the cross section from $q\bar{q}\to c\bar{c}$ is in agreement with theirs. Our result (5.5) for the cross section from $gg\to c\bar{c}$ agrees with theirs only after summing over helicities. We disagree on the helicity dependence of the $\langle O_8^\psi(^1S_0)\rangle$ term. Their result for this term is proportional to $1-\delta_{\lambda0}$, which implies that the ψ is produced with transverse polarization. This is clearly incorrect, because a pointlike $c\bar{c}$ pair in a $^1S_0$ state is rotationally invariant in its rest frame. The ψ's that are produced by the binding of the $c\bar{c}$ pair must therefore be unpolarized, as in our result (5.5).

Cho and Leibovich [7] have used the covariant projection method to calculate the cross sections for ψ and ψ' in $p\bar{p}$ collisions, including not only the order-α_s^2 parton processes but also the order-α_s^3 processes of the form $ij\to c\bar{c}+k$. Our results (5.1) and (5.5) for the cross sections from order-α_s^2 parton processes agree with theirs. Cho and Leibovich determined the short-distance coefficients by matching cross sections for producing color-octet $c\bar{c}$ pairs with vanishing relative momentum and in specific angular momentum states. This method gives the correct cross sections for unpolarized production, but it fails in general for polarized production. In particular, as we show below, it fails to give the correct short-distance coefficients for spin-triplet P-wave matrix elements. In the matrix element (4.13b), the indices i and m can be decomposed into contributions from total angular momentum $J=0,1,2$ by using the identity

$$\delta_{ii'}\delta_{mm'}=U_{i\alpha}^\dagger U_{m\beta}^\dagger\left(\sum_{Jh}\langle 1\alpha;1\beta|Jh\rangle\langle Jh|1\alpha';1\beta'\rangle\right)\times U_{\alpha'i'}U_{\beta'm'}, \tag{5.6}$$

where a sum over α, β, α', and β' is implied. A similar identity can be used to decompose the indices j and n into contributions from total angular momentum $J'=0,1,2$. The matrix element (4.13b) is thereby expressed as a linear combination of matrix elements of the form $\langle\mathcal{J}^\dagger(J,h)\mathcal{P}_{\psi(\lambda)}\mathcal{J}(J',h')\rangle$, where the operator $\mathcal{J}(J,h)$ creates a pointlike $c\bar{c}$ pair with total angular momentum quantum numbers J and h. The projection operator $\mathcal{P}_{\psi(\lambda)}$ projects onto states that contain a ψ with helicity λ plus soft hadrons. If the soft hadrons have total angular momentum J_S, then the angular momentum structure of the matrix element is

$$\sum_{\lambda_S}\langle Jh|1\lambda;J_S\lambda_S\rangle\langle 1\lambda;J_S\lambda_S|J'h'\rangle. \tag{5.7}$$

Rotational symmetry requires that $h=h'$ and that both J and J' lie in the irreducible decomposition of $1\otimes J_S$. It does not require that $J=J'$, so there can be interference terms involving matrix elements with $J\neq J'$. If we sum Eq. (5.7) over the helicities λ, the orthogonality relations for Clebsch-Gordan coefficients imply that $J=J'$. Thus the interference terms cancel upon summing over helicities. The method used by Cho and Leibovich therefore gives the correct cross sections for unpolarized production. In the specific case of $gg\to c\bar{c}$, Cho and Leibovich found that the only amplitudes for the production of spin-triplet P-wave $c\bar{c}$ pairs that are nonzero are $gg\to c\bar{c}(\underline{8},{}^3P_0)$ and $gg\to c\bar{c}(\underline{8},{}^3P_2,|h|=2)$. Since there is at most one nonzero amplitude for any given value of h, there cannot be any interference terms. Thus their method gives the correct cross sections for polarized production from the order-α_s^2 parton processes. However, for the order-α_s^3 parton processes $ij\to c\bar{c}+k$, there are nonvanishing amplitudes with different values of J and the same helicity h, so the interference terms are nonzero. These interference terms must be taken into account in calculating the term proportional to $\langle O_8^\psi(^3P_0)\rangle$ in the inclusive cross section for polarized ψ. The only calculation thus far in which these interference terms have been correctly included is a calculation of the gluon fragmentation function for the production of a longitudinally polarized ψ by Beneke and Rothstein [9].

Fleming and Maksymyk [8] recently presented a method for calculating cross sections for the production of unpolarized heavy quarkonium. To determine the short-distance coefficients of the NRQCD matrix elements, they matched cross sections for producing $c\bar{c}$ pairs with specific color and angular momentum quantum numbers. The cross sections were expressed as integrals over the relative three-momentum $\mathbf{q}$ of the $c\bar{c}$ pair, and they used the nonrelativistic expansions of the Dirac spinors given in Appendix A to expand the integrands as Taylor series in $\mathbf{q}$. In general, their method would fail if applied to cross sections for polarized production, because it does not give the interference terms between parton processes that produce $c\bar{c}$ pairs with different total angular momentum J. Fleming and Maksymyk applied their method to the production of unpolarized ψ from the parton processes $q\bar{q}\to c\bar{c}$ and $gg\to c\bar{c}$. Our cross sections agree with theirs after summing over helicities.

VI. SUMMARY

The NRQCD factorization formalism [1] is a powerful tool for analyzing the production of heavy quarkonium. Cross sections are factored into short-distance coefficients that can be calculated using perturbative QCD and long-distance matrix elements that scale in a definite way with v. In this paper, we presented a general matching prescription for calculating the short-distance coefficients in the inclusive cross sections for quarkonium states. Using this matching prescription, the cross sections are obtained in a form that applies equally well to any quarkonium state. The specific state enters only in the reduction of the NRQCD matrix elements using rotational symmetry and heavy-quark spin symmetry. For simplicity, we discussed the reduction of the NRQCD matrix elements only for the case of the ψ. The generalization to other states will be presented elsewhere [5].

Our approach has interesting implications for cross sections for producing quarkonium states with definite polarization. We showed that cross sections for polarized production can involve new matrix elements that do not contribute to cross sections for unpolarized production. We also showed that there are interference terms involving the production of $c\bar{c}$ states with different total angular momentum J. These interference terms must be taken into account in the cross sections for polarized production from order-α_s^3 parton processes.

Our method has a significant advantage over the covariant projection method in that it can be straightforwardly generalized to N space dimensions. This is important, because it allows the use of dimensional regularization in calculations of radiative corrections. There are potential inconsistencies in combining the covariant projection method with dimensional regularization for calculations involving orbital angular momentum $L=1$ or higher, because the projections onto states with definite total angular momentum J are specific to three dimensions. Our method for calculating the short-distance coefficients of NRQCD matrix elements involves matching Taylor expansions in the relative momentum, which can be readily calculated in N dimensions. After using renormalization to remove the poles in $1/(N-3)$ from the short-distance coefficients, we can specialize to $N=3$ dimensions, and use rotational symmetry and heavy-quark spin symmetry to simplify the matrix elements. Our approach thus allows the convenience of dimensional regularization to be combined with the full power of the NRQCD factorization approach.

While this paper was being completed, Beneke and Rothstein [10] presented a paper that also points out that there are interference terms involving $c\bar{c}$ pairs with different total angular momentum. They presented a thorough phenomenological analysis of the production in fixed-target hadron-hadron collisions of ψ, ψ', and χ_{cJ} via order-α_s^2 parton processes.

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APPENDIX A: NONRELATIVISTIC EXPANSION OF SPINORS

In this Appendix, we give the nonrelativistic expansions for the spinors of a heavy quark c and antiquark $\bar{c}$ with arbitrary total four-momentum P. We assume that the relative three-momentum $\mathbf{q}$ of the c in the center-of-momentum (c.m.) frame of the $c\bar{c}$ pair is small compared to the quark mass m_c. The four-momenta p and $\bar{p}$ of the c and $\bar{c}$ can be written

$$p=\tfrac{1}{2}P+L\mathbf{q}, \tag{A1a}$$

$$\bar{p}=\tfrac{1}{2}P-L\mathbf{q}, \tag{A1b}$$

where P is the total four-momentum and L is a Lorentz boost matrix. From the mass-shell conditions, $p^2=\bar{p}^2=m_c^2$, we have $P\cdot L\mathbf{q}=0$ and $P^2=4E_q^2$, where $E_q=\sqrt{m_c^2+\mathbf{q}^2}$. The components of the four-momenta P and $L\mathbf{q}$ in the c.m. frame of the pair are

$$P^\mu|_{\rm CM}=(2E_q,\mathbf{0}), \tag{A2a}$$

$$(L\mathbf{q})^\mu|_{\rm CM}=(0,\mathbf{q}). \tag{A2b}$$

When boosted to an arbitrary frame in which the pair has total three-momentum $\mathbf{P}$, these four-momenta are

$$P^\mu=(\sqrt{4E_q^2+\mathbf{P}^2},\mathbf{P}), \tag{A3a}$$

$$(L\mathbf{q})^\mu=L^\mu_j q^j. \tag{A3b}$$

The boost matrix L^μ_j, which has one Lorentz index and one Cartesian index, has components

$$L^0_j=\frac{1}{2E_q}P^j, \tag{A4a}$$

$$L^i_j=\delta^{ij}-\frac{P^iP^j}{\mathbf{P}^2}+\frac{P^0}{2E_q}\frac{P^iP^j}{\mathbf{P}^2}. \tag{A4b}$$

The boost tensor L^μ_i has many useful properties. Its contraction with the Lorentz vector P vanishes:

$$P_\mu L^\mu_j=0. \tag{A5}$$

The contractions of two boost matrices in their Lorentz indices or in their Cartesian indices have simple forms:

$$g_{\mu\nu}L^\mu_i L^\nu_j=-\delta^{ij}, \tag{A6a}$$

$$L^\mu_i L^\nu_i=-g^{\mu\nu}+\frac{P^\mu P^\nu}{P^2}. \tag{A6b}$$

There are also simple identities involving contractions of boost matrices with the Levi-Civita tensors $\epsilon_{\mu\nu\lambda\rho}$ and ϵ^{ijk}:

$$\epsilon_{\mu\nu\lambda\rho}L^\mu_i L^\nu_j L^\lambda_k=\epsilon^{ijk}\hat{P}_\rho, \tag{A7a}$$

$$\epsilon_{\mu\nu\lambda\rho}L^\mu_i L^\nu_j \hat{P}^\lambda=\epsilon^{ijk}L_{\rho k}, \tag{A7b}$$

$$\epsilon_{\mu\nu\lambda\rho}L^\mu_i L^\nu_j=\epsilon^{ijk}(\hat{P}_\lambda L_{\rho k}-\hat{P}_\rho L_{\lambda k}), \tag{A7c}$$

$$\epsilon_{\mu\nu\lambda\rho}L^\mu_i \hat{P}^\nu=\epsilon^{ijk}L_{\lambda j}L_{\rho k}, \tag{A7d}$$

$$\epsilon_{\mu\nu\lambda\rho}L^\mu_i=\epsilon^{ijk}(\hat{P}_\nu L_{\lambda j}L_{\rho k}+\hat{P}_\lambda L_{\rho j}L_{\nu k}+\hat{P}_\rho L_{\nu j}L_{\lambda k}), \tag{A7e}$$

$$\epsilon_{\mu\nu\lambda\rho}\hat{P}^{\mu}=\epsilon^{ijk}L_{\nu i}L_{\lambda j}L_{\rho k}, \quad \text{(A7f)}$$

where $\hat{P}^{\mu}=P^{\mu}/\sqrt{P^2}$. Our sign convention is $\epsilon^{0ijk}=-\epsilon_{0ijk}=\epsilon^{ijk}$.

The representation for gamma matrices that is most convenient for carrying out the nonrelativistic expansion of a spinor is the Dirac representation:

$$\gamma^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \gamma^i=\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}. \quad \text{(A8)}$$

In the c.m. frame of the $c\bar{c}$ pair, the spinors for the c and the $\bar{c}$ are

$$u(p)|_{\text{CM}}=\frac{1}{\sqrt{E_q+m_c}}\begin{pmatrix}(E_q+m_c)\xi\\ \mathbf{q}\cdot\boldsymbol{\sigma}\xi\end{pmatrix}, \quad \text{(A9a)}$$

$$v(\bar{p})|_{\text{CM}}=\frac{1}{\sqrt{E_q+m_c}}\begin{pmatrix}-\mathbf{q}\cdot\boldsymbol{\sigma}\eta\\ (E_q+m_c)\eta\end{pmatrix}. \quad \text{(A9b)}$$

Color and spin quantum numbers on the Dirac spinors and on the two-component Pauli spinors ξ and η are suppressed. When boosted to a frame in which the pair has total three-momentum $\mathbf{P}$, the spinors for the c and $\bar{c}$ are

$$u(p)=\frac{1}{\sqrt{4E_q(E_P+2E_q)(E_q+m_c)}}(2E_q+\not{P}\gamma_0)\times\begin{pmatrix}(E_q+m_c)\xi\\ \mathbf{q}\cdot\boldsymbol{\sigma}\xi\end{pmatrix}, \quad \text{(A10a)}$$

$$v(\bar{p})=\frac{1}{\sqrt{4E_q(E_P+2E_q)(E_q+m_c)}}(2E_q+\not{P}\gamma_0)\times\begin{pmatrix}-\mathbf{q}\cdot\boldsymbol{\sigma}\eta\\ (E_q+m_c)\eta\end{pmatrix}. \quad \text{(A10b)}$$

These spinors are normalized so that $\bar{u}u=-\bar{v}v=2m_c$ if the Pauli spinors are normalized so that $\xi^{\dagger}\xi=\eta^{\dagger}\eta=1$.

There are 16 independent quantities that can be formed by sandwiching Dirac matrices between $\overline{u(p)}$ and $v(\bar{p})$. They are

$$\overline{u(p)}v(\bar{p})=-2\xi^{\dagger}(\mathbf{q}\cdot\boldsymbol{\sigma})\eta, \quad \text{(A11a)}$$

$$\overline{u(p)}\gamma^{\mu}v(\bar{p})=L^{\mu}_{j}\left(2E_q\xi^{\dagger}\sigma^{j}\eta-\frac{2}{E_q+m_c}q^{j}\xi^{\dagger}(\mathbf{q}\cdot\boldsymbol{\sigma})\eta\right), \quad \text{(A11b)}$$

$$\overline{u(p)}\Sigma^{\mu\nu}v(\bar{p})=(P^{\mu}L^{\nu}_{j}-P^{\nu}L^{\mu}_{j})\left(\frac{im_c}{E_q}\xi^{\dagger}\sigma^{j}\eta+\frac{i}{E_q(E_q+m_c)}q^{j}\xi^{\dagger}(\mathbf{q}\cdot\boldsymbol{\sigma})\eta\right)-2L^{\mu}_{j}L^{\nu}_{k}\epsilon^{jkl}q^{l}\xi^{\dagger}\eta, \quad \text{(A11c)}$$

$$\overline{u(p)}\gamma^{\mu}\gamma_5 v(\bar{p})=\frac{m_c}{E_q}P^{\mu}\xi^{\dagger}\eta-2iL^{\mu}_{j}\xi^{\dagger}(\mathbf{q}\times\boldsymbol{\sigma})^{j}\eta, \quad \text{(A11d)}$$

$$\overline{u(p)}\gamma_5 v(\bar{p})=2E_q\xi^{\dagger}\eta. \quad \text{(A11e)}$$

From these expressions, it is trivial to carry out the nonrelativistic expansions in powers of $\mathbf{q}$. For example, through linear order in $\mathbf{q}$, we have

$$\overline{u(p)}v(\bar{p})=-2\xi^{\dagger}(\mathbf{q}\cdot\boldsymbol{\sigma})\eta, \quad \text{(A12a)}$$

$$\overline{u(p)}\gamma^{\mu}v(\bar{p})\approx 2m_c L^{\mu}_{j}\xi^{\dagger}\sigma^{j}\eta, \quad \text{(A12b)}$$

$$\overline{u(p)}\Sigma^{\mu\nu}v(\bar{p})\approx i(P^{\mu}L^{\nu}_{j}-P^{\nu}L^{\mu}_{j})\xi^{\dagger}\sigma^{j}\eta-2L^{\mu}_{j}L^{\nu}_{k}\epsilon^{jkl}q^{l}\xi^{\dagger}\eta, \quad \text{(A12c)}$$

$$\overline{u(p)}\gamma^{\mu}\gamma_5 v(\bar{p})\approx P^{\mu}\xi^{\dagger}\eta-2iL^{\mu}_{j}\xi^{\dagger}(\mathbf{q}\times\boldsymbol{\sigma})^{j}\eta, \quad \text{(A12d)}$$

$$\overline{u(p)}\gamma_5 v(\bar{p})\approx 2m_c\xi^{\dagger}\eta. \quad \text{(A12e)}$$

APPENDIX B: PRODUCTION MATRIX ELEMENTS

In this Appendix, we define some of the standard NRQCD production matrix elements that were introduced in Ref. [1]. In order to establish the notation for the fields, we give the Lagrangian for minimal NRQCD:

$$\mathcal{L}=\psi^{\dagger}\left(iD_t+\frac{\mathbf{D}^2}{2M}\right)\psi+\chi^{\dagger}\left(iD_t-\frac{\mathbf{D}^2}{2M}\right)\chi+\mathcal{L}_{\text{QCD}}, \quad \text{(B1)}$$

where ψ is the Pauli spinor field that annihilates a heavy quark, χ is the Pauli spinor field that creates a heavy antiquark, and $\mathcal{L}_{\text{QCD}}$ is the usual QCD Lagrangian for the gluons and the light quarks and antiquarks.

The production operators in Ref. [1] were defined using a projection operator $a^{\dagger}_{H}a_{H}\equiv P^{\text{NR}}_{M(\lambda)}$ defined by

$$\mathcal{P}^{\text{NR}}_{H(\lambda)}=\sum_{S}|H(\mathbf{P}=0,\lambda)+S\rangle\langle H(\mathbf{P}=0,\lambda)+S|, \quad \text{(B2)}$$

where the states in the sum have the standard nonrelativistic normalization. For example, the normalization of the quarkonium state is

$$\langle H(\mathbf{P}',\lambda')|H(\mathbf{P},\lambda)\rangle=(2\pi)^3\delta^3(\mathbf{P}-\mathbf{P}')\delta_{\lambda\lambda'}. \quad \text{(B3)}$$

Thus the projection operator (B2) has energy dimension -3.

The production operators of dimension six are

$$\mathcal{O}^{H}_{1}(^1S_0)=\chi^{\dagger}\psi\mathcal{P}^{\text{NR}}_{H(\lambda)}\psi^{\dagger}\chi, \quad \text{(B4a)}$$

$$\mathcal{O}^{H}_{1}(^3S_1)=\chi^{\dagger}\sigma^{i}\psi\mathcal{P}^{\text{NR}}_{H(\lambda)}\psi^{\dagger}\sigma^{i}\chi, \quad \text{(B4b)}$$

$$\mathcal{O}^{H}_{8}(^1S_0)=\chi^{\dagger}T^{a}\psi\mathcal{P}^{\text{NR}}_{H(\lambda)}\psi^{\dagger}T^{a}\chi, \quad \text{(B4c)}$$

$$\mathcal{O}^{H}_{8}(^3S_1)=\chi^{\dagger}\sigma^{i}T^{a}\psi\mathcal{P}^{\text{NR}}_{H(\lambda)}\psi^{\dagger}\sigma^{i}T^{a}\chi. \quad \text{(B4d)}$$

Some of the production operators of dimension eight are

$$\mathcal{O}_1^H(^1P_1)=\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\psi\mathcal{P}_{H(\lambda)}^{\rm NR}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\chi, \quad \text{(B5a)}$$

$$\mathcal{O}_1^H(^3P_0)=\frac{1}{3}\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)\psi\mathcal{P}_{H(\lambda)}^{\rm NR}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)\chi, \quad \text{(B5b)}$$

$$\mathcal{O}_8^H(^1P_1)=\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)T^a\psi\mathcal{P}_{H(\lambda)}^{\rm NR}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)T^a\chi, \quad \text{(B5c)}$$

$$\mathcal{O}_8^H(^3P_0)=\frac{1}{3}\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)T^a\psi\mathcal{P}_{H(\lambda)}^{\rm NR}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)T^a\chi, \quad \text{(B5d)}$$

where $\psi^\dagger\overleftrightarrow{\mathbf{D}}\chi=\psi^\dagger\mathbf{D}\chi-(\mathbf{D}\psi)^\dagger\chi$.

Note that the projection operator defined by Eq. (B2) differs from the corresponding projection operator (2.4), which is defined using states with the standard relativistic normalization. At leading order in v^2, these operators differ simply by an overall factor: $\mathcal{P}_{H(\lambda)}\approx 4m_c\mathcal{P}_{H(\lambda)}^{\rm NR}$. Beyond leading order, the relation between the two projectors is complicated because the normalization differs for each term in the sum over soft states S. For physical quantities, such as the cross section in Eq. (3.1), the difference between the projection operators is compensated by the short-distance coefficients:

$$\sum_{mn} C_{mn}\langle\chi^\dagger\mathcal{K}'^\dagger_m\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\mathcal{K}_n\chi\rangle = \sum_{mn} C_{mn}^{\rm NR}\langle\chi^\dagger\mathcal{K}'^\dagger_m\psi\mathcal{P}_{H(\lambda)}^{\rm NR}\psi^\dagger\mathcal{K}_n\chi\rangle. \quad \text{(B6)}$$

We now list the relations between scalar matrix elements defined with the projection operator (2.4) and the matrix elements of the operators defined above. At leading order in α_s and to leading order in v^2, we have

$$\langle\chi^\dagger\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\chi\rangle=4m_c\mathcal{O}_1^H(^1S_0), \quad \text{(B7a)}$$

$$\langle\chi^\dagger\sigma^i\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\sigma^i\chi\rangle=4m_c\mathcal{O}_1^H(^3S_1), \quad \text{(B7b)}$$

$$\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\chi\right\rangle=4m_c\mathcal{O}_1^H(^1P_1), \quad \text{(B7c)}$$

$$\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)\psi\mathcal{P}_{H(\lambda)}^{\rm NR}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)\chi\right\rangle=12m_c\mathcal{O}_1^H(^3P_0). \quad \text{(B7d)}$$

The corresponding color-octet matrix elements are

$$\langle\chi^\dagger T^a\psi\mathcal{P}_{H(\lambda)}\psi^\dagger T^a\chi\rangle=4m_c\mathcal{O}_8^H(^1S_0), \quad \text{(B8a)}$$

$$\langle\chi^\dagger\sigma^i T^a\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\sigma^i T^a\chi\rangle=4m_c\mathcal{O}_8^H(^3S_1), \quad \text{(B8b)}$$

$$\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)T^a\psi\mathcal{P}_{H(\lambda)}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)T^a\chi\right\rangle=4m_c\mathcal{O}_8^H(^1P_1), \quad \text{(B8c)}$$

$$\left\langle\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)T^a\psi\mathcal{P}_{H(\lambda)}^{\rm NR}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)T^a\chi\right\rangle =12m_c\mathcal{O}_8^H(^3P_0). \quad \text{(B8d)}$$

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