Static Einstein-Maxwell solutions in 2+1 dimensions

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We obtain the Einstein-Maxwell equations for (2+1)-dimensional static space-time, which are invariant under the transformation $q_0=iq_2$, $q_2=iq_0$, $\alpha \rightleftharpoons \gamma$. It is shown that the magnetic solution obtained with the help of the procedure used by Cataldo and Salgado can be obtained from the static BTZ solution using an appropriate transformation. Superpositions of a perfect fluid and an electric or a magnetic field are separately studied and their corresponding solutions found. [S0556-2821(96)04814-X]

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It is well known that in (2+1)-dimensional space-time the metric around a point mass is given by

$$ds^2=dt^2-\frac{dr^2}{(1-kM/2\pi)^2}-r^2d\phi^2. \tag{1}$$

This space-time is analogous to the Schwarzschild metric around a point mass in 3+1 dimensions. The metric (1) corresponds to a flat space-time [1]. Another situation in 2+1 dimensions is to consider the cosmological constant Λ. In this case, outside sources, the exterior gravitational fields are spaces of constant curvature: de Sitter space for $\Lambda>0$ and anti–de Sitter space for $\Lambda<0$ [2].

The 2+1 local electromagnetic field is given by (''cutting out'' one of the spatial dimensions)

$$F=\tfrac{1}{2}F_{ab}dx^a\wedge dx^b=E_1dx^1\wedge dx^0+E_2dx^2\wedge dx^0+Bdx^1\wedge dx^2. \tag{2}$$

Thus the electromagnetic tensor has only three independent components [1,3], two for the vector electric field and one for the scalar magnetic field ($\vec{B}$ to be a vector needs the missing dimension).

On the other hand, since the tensor of Levi-Civitá in a three-dimensional space has the form $\varepsilon_{[abc]}$ with $\varepsilon_{012}=1$, the dual of Eq. (2) is a vector given by $^*F_a=\frac{1}{2}\varepsilon_{abc}F^{bc}$ (latin indices are assumed to take on the values 0, 1, 2). Therefore, the source free Maxwell equations in (2+1)-dimensional gravity lack invariance under dual transformation. This means that, for a given Einstein-Maxwell space-time without sources, it is not possible to transform an electric field into a magnetic field and vice versa. So that for the self-consistent problem we must solve the Einstein-Maxwell equations for an electric or for a magnetic fields separately.

The presence of electromagnetic fields can curve the space outside sources. Also, the curvature is present when the space-time is filled with a perfect fluid. The above picture is confirmed by the following electrovacuum static solution, found by Gott, Simon, and Alpert [1]:

$$ds^2=\frac{kQ^2}{2\pi}\ln\left(\frac{r_c}{r}\right)dt^2-\frac{2\pi}{kQ^2}\left[\ln\left(\frac{r_c}{r}\right)\right]^{-1}dr^2-r^2d\phi, \tag{3}$$

where r_c and Q are constants and the electric field is given by $E_r=Q/r$. This space-time has a horizon at $r=r_c$ and is the analogue to the Reissner-Nordström solution in 3+1 dimensions. Recently Bañados, Teitelboim, and Zanelli (BTZ) found the static electrically charged solution ($J=0$) [4], which is the 2+1 Kottler analogue [5,6]:

$$ds^2=h(r)dt^2-h^{-1}(r)dr^2-r^2d\phi^2, \tag{4}$$

where $h(r)=-M+r^2/l^2-\frac{1}{2}Q^2\ln r$ ($-\infty<t<\infty$, $0<r<\infty$, and $0\leqslant\phi\leqslant 2\pi$). The constant l is related to the cosmological constant Λ by $\Lambda=-l^{-2}$. In order that the horizon exists, one must have $M>0$. If $\Lambda=0$, we obtain the solution Eq. (3). The BTZ solution, however, does not completely describe the coupling between the gravitational and electromagnetic fields in 2+1 dimensions because the magnetic case must be considered separately.

In this work we find the exact Einstein-Maxwell solutions which give a complete description of the static superpositions of a perfect fluid and an electric or a magnetic field. The solutions are obtained with the help of the procedure used in Ref. [7], where the four-dimensional case was studied for a neutral perfect fluid filling a spherically symmetric space-time.

In 2+1 dimensions, the metric for an arbitrary static circularly symmetric space-time can be written in the form

$$ds^2=e^{2\alpha(r)}dt^2-e^{2\beta(r)}dr^2-e^{2\gamma(r)}d\theta^2. \tag{5}$$

The general form of the electromagnetic field tensor which shares the static circularly symmetric space-time is given by $F=E_rdr\wedge dt+Bdr\wedge d\theta$. The electromagnetic field tensor in terms of a four-potential,

$$A=A_a(r)dx^a, \tag{6}$$

is given by $F=dA=\frac{1}{2}F_{ab}dx^a\wedge dx^b$, where the functions $A_a(r)$ can be freely specified. So $dA=F=q_0A_0'dr\wedge dt+q_2A_2'dr\wedge d\theta$ where A_0 and A_2 are arbitrary functions of the r coordinate, the differentiation with respect to r is denoted by the prime and the constant coefficients q_0 and q_2 are introduced for switching off the electric and/or magnetic fields. This implies that the r component of the electric field is given by $q_0 A_0'$ and of the magnetic field by $q_2 A_2'$.

To write the Einstein's equations we will use the tetrad formalism and Cartan structure equations. A convenient orthonormal basis for the metric (5) is

$$\theta^{(0)} = e^{\alpha} dt, \quad \theta^{(1)} = e^{\beta} dr, \quad \theta^{(2)} = e^{\gamma} d\theta. \tag{7}$$

To construct Einstein-Maxwell fields, we must consider the Maxwell's equations and the stress-energy tensor of the electromagnetic field which, with respect to Eq. (7) in Gaussian units, is defined by [1]

$$T_{(a)(b)} = \frac{g_{(a)(b)}}{8\pi} F_{(c)(d)} F^{(c)(d)} - \frac{1}{2\pi} F_{(a)(c)} F^{(c)}_{(b)}. \tag{8}$$

In 2+1 dimensions the trace of Eq. (8) is $T = (1/8\pi) F_{ab} F^{ab}$. To get its components we must compute $F_{(a)(b)}$. In the coordinate basis

$$F_{ab} = \tfrac{1}{2} q_0 A_0' \delta^r_{[a} \delta^t_{b]} + \tfrac{1}{2} q_2 A_2' \delta^r_{[a} \delta^\theta_{b]} \tag{9}$$

or in the basis (7) $F_{(a)(b)} = q_0 I_e \delta^1_{[a} \delta^0_{b]} + q_2 I_m \delta^1_{[a} \delta^2_{b]}$, where

$$I_e = \tfrac{1}{2} A_0' e^{-\alpha - \beta} \tag{10}$$

and

$$I_m = \tfrac{1}{2} A_2' e^{-\beta - \gamma}. \tag{11}$$

Thus from Eq. (8) we get

$$\begin{aligned} T^{\mathrm{em}}_{(a)(b)} \theta^{(a)} \otimes \theta^{(b)} = {} & \left(\frac{q_0^2}{\pi} I_e^2 + \frac{q_2^2}{\pi} I_m^2 \right) \theta^{(0)} \otimes \theta^{(0)} \\ & - \left(-\frac{q_0^2}{\pi} I_e^2 + \frac{q_2^2}{\pi} I_m^2 \right) \theta^{(1)} \otimes \theta^{(1)} \\ & - \left(\frac{q_0^2}{\pi} I_e^2 + \frac{q_2^2}{\pi} I_m^2 \right) \theta^{(2)} \otimes \theta^{(2)} \\ & - \frac{2 q_0 q_2}{\pi} I_e I_m \left(\theta^{(0)} \otimes \theta^{(2)} + \theta^{(2)} \otimes \theta^{(0)} \right). \end{aligned}$$

From the tetra (7) and using Cartan exterior forms calculus the following nontrivial components of the Einstein's equations with the cosmological constant are obtained:

$$e^{-2\beta} (\gamma' \beta' - \gamma'' - \gamma'^2) = \Lambda + \frac{\kappa q_0^2 I_e^2}{\pi} + \frac{\kappa q_2^2 I_m^2}{\pi}, \tag{12}$$

$$-\alpha' \gamma' e^{-2\beta} = \Lambda + \frac{\kappa q_0^2 I_e^2}{\pi} - \frac{\kappa q_2^2 I_m^2}{\pi}, \tag{13}$$

$$e^{-2\beta} (\alpha' \beta' - \alpha'' - \alpha'^2) = \Lambda - \frac{\kappa q_0^2 I_e^2}{\pi} - \frac{\kappa q_2^2 I_m^2}{\pi}, \tag{14}$$

$$\frac{2 \kappa q_0 q_2}{\pi} I_e I_m = 0. \tag{15}$$

Now we must consider the Maxwell's equations. The contravariant density components of Eq. (9) are

$$\sqrt{-g} F^{ab} = e^{\alpha + \beta + \gamma} \left(\frac{2 q_0 I_e^2}{A_0'} \delta_r^{[a} \delta_t^{b]} + \frac{2 q_2 I_m^2}{A_2'} \delta_r^{[a} \delta_\theta^{b]} \right).$$

It is clear that the source-free Maxwell's equations are satisfied if

$$e^{\alpha + \beta - \gamma} = A_0', \quad e^{-\alpha + \beta + \gamma} = A_2', \tag{16}$$

where the constants of integration, without any loss of generality, have been made equal to 1. So the Einstein-Maxwell equations are given by Eqs. (12)–(15) and Eq. (16). To obtain the Einstein-Maxwell solutions it is useful to notice that Eq. (15) says to us that either the electric or the magnetic field must be zero.

It is easy to check that under the transformation

$$q_0 = i q_2, \quad q_2 = i q_0, \quad \alpha \rightleftharpoons \gamma \tag{17}$$

the Einstein-Maxwell equations are invariant. This means that if we have the magnetic solution, then one can obtain the electric analogue by making the formal transformation (17). In other words, if the Einstein-Maxwell solution is given in the form of Eq. (5), then we obtain an analogous metric making

$$dt = i d\theta, \quad d\theta = i dt, \quad q_0 = i q_2, \quad q_2 = i q_0. \tag{18}$$

Now, we will find a general magnetic solution (in analytical form). So we must consider Eq. (15). That means that in this case either the electric or the magnetic field must be zero, so that in the considered equations we must set $q_0 = 0$ (or $A_0 = 0$). This implies that, in order to solve the self-consistent equations, it is not necessarily to consider the first condition of Eq. (16). Subtracting Eqs. (13) and (14) and using the second condition of Eq. (16), we obtain $e^{2\alpha} = D e^{C A_2(r)}$, where A_2 is an arbitrary function of r, and C and D are constants of integration. On the other hand the combination of Eqs. (13) and (14) leads us to the equation

$$(\alpha' e^{\alpha - \beta + \gamma})' = -2 \Lambda A_2' D e^{CA} + \frac{\kappa}{2\pi} q_2^2 A_2' \tag{19}$$

which yields with the help of Eq. (16) the following expression for the function $e^{2\gamma}$:

$$e^{2\gamma} = \frac{\kappa}{\pi C} q_2^2 A_2 - \frac{4 \Lambda D}{C^2} e^{C A_2} + F, \tag{20}$$

where F is a new constant of integration. From Eq. (16) we obtain $e^{2\beta} = D A_2'^2 e^{C A_2} e^{-2\gamma}$. Introducing a new coordinate $\tilde{r}$ defined by $\tilde{r} = A_2(r)$, we have

$$ds^2 = D e^{C \tilde{r}} dt^2 - D e^{C \tilde{r}} e^{-2\gamma} d\tilde{r}^{\,2} - e^{2\gamma} d\theta^2, \tag{21}$$

where now $e^{2\gamma} = \kappa q_2^2 \tilde{r} / \pi C - 4 \Lambda D e^{C \tilde{r}} / C^2 + F$. In this coordinate gauge the magnetic field is constant. The solution Eq. (21) cannot be carried out in the spatial gauge $g_{22} = r^2$ with the help of the transformation

$$\tilde{r}^{\,2}=\frac{\kappa q_2^2}{\pi C}r-\frac{4\Lambda D}{C^2}e^{Cr}+F, \tag{22}$$

because the resulting metric does not take an analytical form. However it is possible to obtain the metrics for the gauge $g_{22}=r^2$ in exact form by setting $\Lambda=0$ or switching off the magnetic field $(q_2=0)$. This means that if $q_2=0$ then, from Eq. (22) one can obtain the BTZ noncharged three-dimensional black hole [4]; and that when $\Lambda=0$ we obtain the 3+1 magnetic Reissner-Nordström counterpart. In fact, in this case we obtain metric

$$ds^2=e^{ar^2}dt^2-e^{ar^2}dr^2-r^2d\theta^2, \tag{23}$$

with $a=\kappa q_2^2/\pi$. From Eq. (23) it follows that the 2+1 magnetic monopole counterpart is not a black hole, contrasting with the 2+1 electric analogue. If we switch off the magnetic field $(a=0)$, one obtains the flat three-dimensional space-time.

Finally, we note that application of transformation (18) on Eq. (21) leads to

$$ds^2=e^{2\alpha}dt^2-De^{C\tilde{r}}e^{-2\alpha}d\tilde{r}^{\,2}-De^{C\tilde{r}}d\theta^2. \tag{24}$$

where $e^{2\alpha}=(-\kappa q_2^2/\pi C)\tilde{r}-(4\Lambda D/C^2)e^{C\tilde{r}}+F$. In the gauge $g_{22}=r^2$ the metric (24) takes on the new form

$$ds^2=e^{2\alpha}dt^2-e^{-2\alpha}dr^2-r^2d\theta^2, \tag{25}$$

where now $e^{2\alpha}=(-2\kappa q_2^2/\pi)\ln r-\Lambda r^2+F$. Then the BTZ charged black hole Eq. (4) is obtained if $\Lambda=-l^{-2}$, $F=-M$, and $Q^2=4\kappa q_2^2/\pi$. When $\Lambda=0$ one gets the electrically charged solution (3). In this case the electromagnetic potential is given by $A=\ln r$. This means that $dA=q_2/r dr \wedge dt$ (or $E_r=q_2/r$). If one uses the transformation (18), then $dA=-q_0/r dr\wedge d\theta$, so that the electric field is replaced by a magnetic field and $B=-q_0/r$.

Now, in order to obtain the Einstein-Maxwell fields with a neutral fluid, we must consider the stress-energy tensor of the perfect fluid, which is given by

$$T^{\rm PF}_{(a)(b)}=(\mu+p)U_{(a)}U_{(b)}-pg_{(a)(b)}, \tag{26}$$

where μ and p are the mass-energy density and pressure of the fluid, respectively. $U_{(a)}$ is its timelike four-velocity. If we take the four velocity $\mathbf{U}=\theta^{(0)}$, then Eq. (26) becomes $T^{\rm PF}_{(0)(0)}=\mu$, $T^{\rm PF}_{(1)(1)}=T^{\rm PF}_{(2)(2)}=T^{\rm PF}_{(3)(3)}=p$. With the electromagnetic potential (6) and a neutral perfect fluid, the Einstein-Maxwell equations with cosmological constant are now given by

$$e^{-2\beta}(\gamma'\beta'-\gamma''-\gamma'^2)=\Lambda+\frac{\kappa q_0^2I_e^2}{\pi}+\frac{\kappa q_2^2I_m^2}{\pi}+\kappa\mu, \tag{27}$$

$$-\alpha'\gamma' e^{-2\beta}=\Lambda+\frac{\kappa q_0^2I_e^2}{\pi}-\frac{\kappa q_2^2I_m^2}{\pi}-\kappa p, \tag{28}$$

$$e^{-2\beta}(\alpha'\beta'-\alpha''-\alpha'^2)=\Lambda-\frac{\kappa q_0^2I_e^2}{\pi}-\frac{\kappa q_2^2I_m^2}{\pi}-\kappa p, \tag{29}$$

$$\frac{2\kappa q_0q_2}{\pi}I_eI_m=0, \tag{30}$$

and by conditions (16). It is easy to see that Eqs. (27)–(30) are not invariant under transformation (18). This means that one must solve the system (27)–(30) for an electric or a magnetic field separately [Eq. (30) implies that either the electric or the magnetic field must be zero].

First we consider the electric solution: In this case we cannot obtain the solution for the general electromagnetic potential (6) with an arbitrary function A_0 $(A_2=0)$. We must consider some concrete function, such as the four-potential in the form

$$A_0=\begin{cases}\dfrac{1}{n+1}r^{n+1}dt & \text{if } n\neq-1,\\ \ln r dt & \text{if } n=-1,\end{cases} \tag{31}$$

where n is an arbitrary constant, and solve the Einstein-maxwell equations. Thus the electric field takes the form $E=q_0r^n$. In the case of the circularly symmetric metric, one takes $e^{2\gamma}=r^2$, arriving at the following metrics [8].

For $n\neq 1,-1,-3$,

$$e^{2\alpha}=r^{2(n+1)}e^{-2\beta}=\frac{\kappa q^2r^{2(n+1)}}{4\pi(n-1)(n+1)}+\frac{Ar^{n+3}}{n+3}+B,$$

$$\kappa p=\frac{\kappa q^2(n+1)}{8\pi(n-1)r^2}+\frac{A}{2}r^{-(n+1)}+\Lambda,$$

$$\kappa\mu=\frac{B(n+1)}{r^{2(n+2)}}+\frac{A(n-1)}{2(n+3)}r^{-(n+1)}-\frac{\kappa q^2}{8\pi r^2}-\Lambda.$$

For $n=-1$,

$$e^{2\alpha}=e^{-2\beta}=-\frac{\kappa q^2}{4\pi}\ln r+Ar^2+B,$$

$$\kappa p=-\kappa\mu=A+\Lambda.$$

For $n=1$,

$$e^{2\alpha}=r^4e^{-2\beta}=\frac{\kappa q^2}{8\pi}r^4\ln r+Ar^4+B,$$

$$\kappa p=\frac{\kappa q^2}{16\pi r^2}(3+4\ln r)+\frac{2A}{r^2}+\Lambda,$$

$$\kappa\mu=-\frac{3\kappa q^2}{16\pi r^2}+\frac{2B}{r^6}-\Lambda.$$

For $n=-3$,

$$e^{2\alpha}=r^{-4}e^{-2\beta}=\frac{\kappa q^2}{32\pi r^4}+A\ln r+B,$$

$$\kappa p=\frac{\kappa q^2}{16\pi r^2}+\frac{A}{2}r^2+\Lambda,$$

$$\kappa\mu = -\frac{\kappa q^2}{8\pi r^2} - \frac{Ar^2}{2}(1+\ln r) - 2Br^2 - \Lambda.$$

From $n=-1$ we see that if $A=-\Lambda$ then we obtain the $2+1$ Kottler solution analogue (4). We remark that when $B=q=0$ and the fluid obeys a γ-law equation, i.e., μ and p are related by an equation of the form $p=(\gamma-1)\mu$ where γ is a constant (which, for physical reasons satisfies the inequality $1\leqslant\gamma\leqslant 2$), the constant γ may be expressed as $\gamma = 2(n+1)/(n-1)$, where the limits of n are $-\infty\leqslant n\leqslant -3$. In this case

$$\kappa\mu = \kappa\frac{n-1}{n+3}p = \frac{A(n-1)}{2(n+3)}r^{-(n+1)}. \tag{32}$$

Recently, Gürses [9] obtained a class of metrics of Einstein theory with perfect fluid sources in $2+1$ dimensions. However this class of solutions was found for the particular case $\mu=$const and $p=$const.

Finally, we present the circularly symmetric magnetic case ($A_0=0$) which takes the form

$$ds^2 = De^{CA_2(r)}(dt^2 - r^{-2}A_2'^2 dr^2) - r^2 d\theta^2,$$

where the mass-energy density is given by

$$\kappa\mu = \frac{e^{-CA_2}}{D}\left(\frac{Cr}{2A_2'} + \frac{A_2''}{A_2'^3} - \frac{1}{A_2'^2} - \frac{\kappa q_2^2}{4\pi}\right) - \Lambda,$$

and the pressure by

$$\kappa p = \frac{e^{-CA_2}}{D}\left(\frac{Cr}{2A_2'} - \frac{\kappa q_2^2}{4\pi}\right) + \Lambda.$$

Before we finish this paper we would like to make a few comments about our most important results. We have studied the static Einstein-Maxwell fields in $2+1$ dimensions and obtained new exact solutions with circular symmetry. All of them were found in the presence of the cosmological constant Λ. It is noteworthy that the $2+1$ magnetic Reissner-Nordström analogue is not a black hole in contrast with the $2+1$ electric Reissner-Nordström analogue, where a black hole is present. It is shown that the magnetic solution obtained with the help of the procedure used in [7], can be obtained from the static electrically charged BTZ metric using transformation (18). In this case the radial parameter satisfies $0<r<\infty$.

Superpositions of a perfect fluid and an electric or a magnetic field are separately studied and their corresponding solutions found.

Shortly after we completed this work Bañados informed us of the existence of Ref. [10] where a magnetic solution (HW solution) was also obtained by a different procedure. However, this solution uses a gauge which lacks a parameter. This could be seen by introducing the new coordinate (and taking a negative cosmological constant $\Lambda=-l^{-2}$) $\tilde{r} = \ln[(r^2/l^2 - M)/D]^{1/C}$.

Then Eq. (21) with this new coordinate becomes

$$ds^2 = \left(\frac{r^2}{l^2} - M\right)dt^2 - r^2\left(\frac{r^2}{l^2} - M\right)^{-1} e^{-2\gamma}dr^2 - e^{2\gamma}d\theta^2,$$

where $e^{2\gamma} = r^2 + (\kappa q_2^2 l^4/4\pi)\ln(r^2/l^2 - M) + F$. In this gauge then the lacking parameter is F.

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