sinh-Gordon, cosh-Gordon, and Liouville equations for strings and multistrings in constant curvature spacetimes

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We find that the fundamental quadratic form of classical string propagation in $(2+1)$-dimensional constant curvature spacetimes solves the sinh-Gordon equation, the cosh-Gordon equation, or the Liouville equation. We show that in both de Sitter and anti–de Sitter spacetimes (as well as in the $2+1$ black hole anti–de Sitter spacetime), *all* three equations must be included to cover the generic string dynamics. The generic properties of the string dynamics are directly extracted from the properties of these three equations and their associated potentials (irrespective of any solution). These results complete and generalize earlier discussions on this topic (until now, only the sinh-Gordon sector in de Sitter spacetime was known). We also construct new classes of multistring solutions, in terms of elliptic functions, to all three equations in both de Sitter and anti–de Sitter spacetimes. Our results can be straightforwardly generalized to constant curvature spacetimes of arbitrary dimension, by replacing the sinh-Gordon equation, the cosh-Gordon equation, and the Liouville equation by their higher dimensional generalizations. [S0556-2821(96)03616-8]

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I. INTRODUCTION AND RESULTS

In this paper we discuss the dynamics of a relativistic string in constant curvature spacetimes, using a combination of geometrical methods and physical insight. The kind of problems we are interested in here and the way of reasoning, historically, had the origin in investigations of the motion of vortices in a superfluid [1,2]. Interestingly enough, the latter problem, which is equivalent to a theory of dual strings interacting in a particular way through a scalar field [1,2], reduces to solving two coupled nonlinear partial differential equations, one of which being a generalized sine-Gordon equation. It was soon realized that exactly the same equations appear when considering a two-dimensional σ model corresponding to O(4) [3–5]. This model, on the other hand, describes a relativistic string in a (Euclidean signature) constant curvature space.

For the theory of fundamental strings, it is important to consider formulations in curved spacetimes, also, essentially as descriptions of one string in the background created by the others. The string equations of motion in curved spacetimes are generally highly nonlinear in any gauge, which in most cases means that the system is nonintegrable. Exceptional cases are, among others, strings in maximally symmetric spacetimes [6,7]. From the physical point of view, de Sitter spacetime plays a particular role in this family of spacetimes, since it describes an inflationary universe. String theory in de Sitter spacetime is therefore also of interest from the point of view of cosmic strings and cosmology and for the open question of string self-sustained inflation [8,9]. Specific problems concerning the integrability of the equations describing the dynamics of classical strings in de Sitter spacetime were discussed in [10,11]. The present work is a completion and generalization of the results presented in those papers.

In Sec. II we set up the general formalism for classical strings in de Sitter and anti–de Sitter spacetimes, and we derive the equations fulfilled by the fundamental quadratic form for a generic string configuration. The fundamental quadratic form $\alpha(\tau,\sigma)$ is a measure of the invariant string size. We show that it solves the sinh-Gordon equation, the cosh-Gordon equation, or the Liouville equation. We find that in order to cover the generic string dynamics, *all* three equations must be taken into account. Associated potentials ($\pm 2\cosh\alpha$, $\pm 2\sinh\alpha$, $\pm e^{\alpha}$) to these equations can be, respectively, defined [$(+)$ sign for anti–de Sitter spacetime and $(-)$ sign for de Sitter spacetime]. Generic properties of the string dynamics are then directly extracted at the level of the equations of motion from the properties of these potentials (irrespective of any solution). The three equations correspond to three different sectors of the string dynamics [until now only the sinh-Gordon sector (corresponding to the $\cosh\alpha$ potential) in de Sitter spacetime was known]. The differences between the three sectors in each spacetime appear mainly for small strings [strings with proper size $<1/(\sqrt{2}H)$].

In de Sitter spacetime, the sinh-Gordon sector characterizes the evolution in which small strings necessarily collapse into a point, while in the cosh-Gordon sector, strings never collapse but reach a minimal size. In anti–de Sitter spacetime, the situation is exactly the opposite: The cosh-Gordon sector characterizes the evolution in which strings necessarily collapse into a point, while in the sinh-Gordon sector, strings never collapse but reach a minimal size. On the other hand, the dynamics of large strings is rather similar in the three sectors in each spacetime [see Figs. 1(a) and 1(b), for instance].

The dynamics of small strings is rather similar in de Sitter

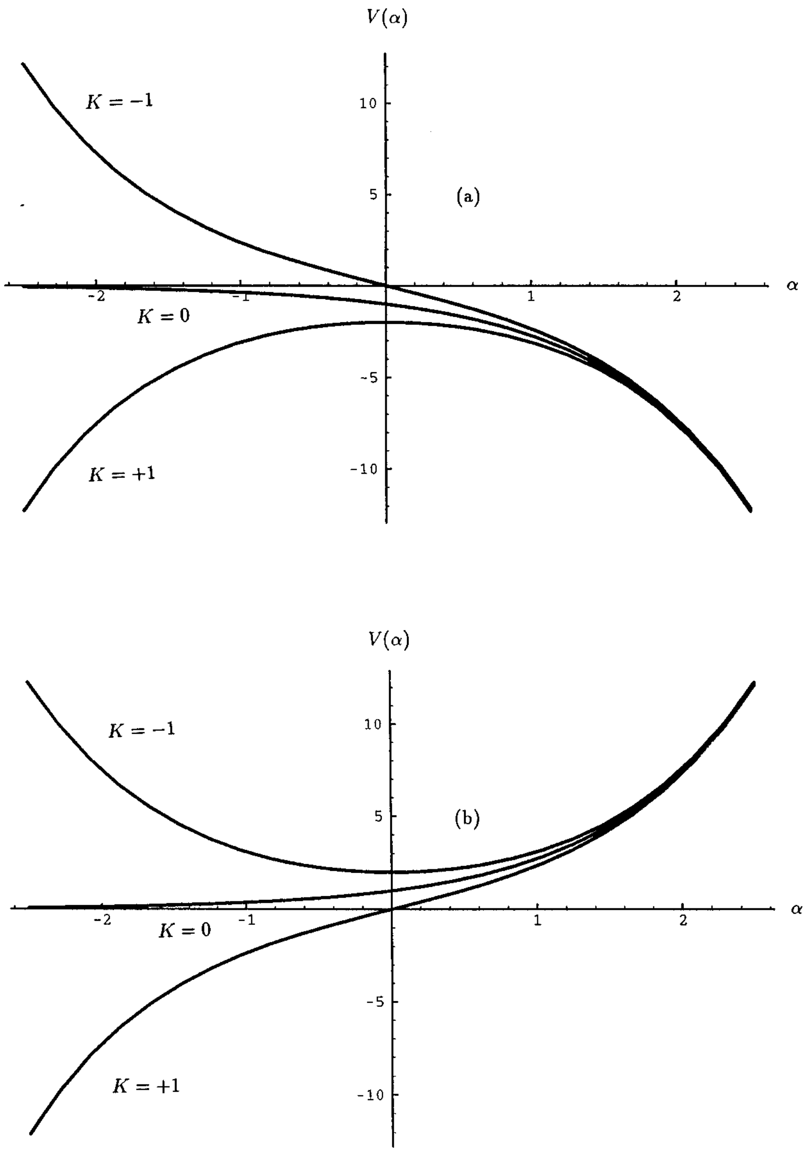

FIG. 1. The potentials (2.25) and (2.26) determining the dynamics of strings in (a) de Sitter spacetime and (b) anti–de Sitter spacetime, respectively. For each spacetime, the differences between the three sectors ($K=\pm 1$, 0) appear for negative α [i.e., for strings with proper size $<1/(\sqrt{2}H)$]. The differences between de Sitter and anti–de Sitter potentials are for positive α [i.e., for strings with proper size $>1/(\sqrt{2}H)$]. For small strings (large negative α) the dynamics is similar in both de Sitter and anti–de Sitter spacetimes, while for large strings (large positive α) it is completely different in the two spacetimes.

and anti–de Sitter spacetimes, while for large strings [strings with proper size $>1/(\sqrt{2}H)$] the dynamics is drastically different in the two spacetimes. In de Sitter spacetime, the presence of potentials unbounded from below for positive α, in all three sectors, makes string instability (indefinetely growing strings) unavoidable (in anti–de Sitter spacetime, the positive potential barriers for positive α prevents the strings from growing indefinetely).

In Sec. III we present new classes of explicit solutions in both de Sitter and anti–de Sitter spacetimes, which cover all the three sectors. These solutions exhibit the multistring property [12–15]; namely, one single world sheet describes a finite or infinite number of different and independent strings. The presence of multistrings is a characteristic feature in spacetimes with a cosmological constant (constant curvature or asymptotically constant curvature spacetimes).

In Sec. IV, we show that our results also hold for the $(2+1)$-dimensional black-hole anti–de Sitter spacetime [16], and we complete earlier investigation on the dynamics of circular string configurations in this spacetime [17].

Finally, in Sec. V we give our conclusions.

II. GENERAL FORMALISM

For simplicity, the following analysis is performed for $(2+1)$-dimensional spacetimes. However, it is straightforward to generalize the results to arbitrary dimensions, following the lines of [11].

We can treat de Sitter and anti–de Sitter spacetimes simultaneously by introducing the following notation. We consider a flat spacetime with line element $ds^2_{(\epsilon)}$ where $\epsilon=\pm 1$,

$$ds^2_{(\epsilon)}=-dt^2+\epsilon du^2+dx^2+dy^2, \tag{2.1}$$

and restrict ourselves to the submanifold

$$\eta^{(\epsilon)}_{\mu\nu}q^\mu q^\nu=\epsilon, \tag{2.2}$$

where

$$\eta^{(\epsilon)}_{\mu\nu}=\text{diag}(-1,\epsilon,1,1) \tag{2.3}$$

and q^μ is in the form of

$$q^{\mu}=H(t,u,x,y). \tag{2.4}$$

That is, $\epsilon=+1$ corresponds to de Sitter spacetime while $\epsilon=-1$ corresponds to anti–de Sitter spacetime and H is the Hubble constant of de Sitter (anti–de Sitter) spacetime.

Let us now consider a bosonic string embedded in the spacetime (2.1)–(2.4). In the conformal gauge, where the string world-sheet metric is diagonal, the classical string equations of motion and constraints take the form

$$\ddot{q}^{\mu}-q''^{\mu}+\epsilon\eta^{(\epsilon)}_{\rho\sigma}(\dot{q}^{\rho}\dot{q}^{\sigma}-q'^{\rho}q'^{\sigma})q^{\mu}=0, \tag{2.5}$$

$$\eta^{(\epsilon)}_{\mu\nu}\dot{q}^{\mu}q'^{\nu}=\eta^{(\epsilon)}_{\mu\nu}(\dot{q}^{\mu}\dot{q}^{\nu}+q'^{\mu}q'^{\nu})=0. \tag{2.6}$$

Here an overdot and a prime denote differentiation with respect to the world-sheet coordinates τ and σ, respectively.

The induced line element on the string world sheet is given by

$$dS^2_{(\epsilon)}=\frac{1}{H^2}\eta^{(\epsilon)}_{\mu\nu}dq^{\mu}dq^{\nu}$$
$$=-\frac{1}{2H^2}\eta^{(\epsilon)}_{\mu\nu}(\dot{q}^{\mu}\dot{q}^{\nu}-q'^{\mu}q'^{\nu})(-d\tau^2+d\sigma^2). \tag{2.7}$$

Since we consider only timelike world sheets, we can define a real function $\alpha^{(\epsilon)}$ by

$$e^{\alpha^{(\epsilon)}}\equiv-\eta^{(\epsilon)}_{\mu\nu}(\dot{q}^{\mu}\dot{q}^{\nu}-q'^{\mu}q'^{\nu})=-\eta^{(\epsilon)}_{\mu\nu}q^{\mu}_{+}q^{\nu}_{-}, \tag{2.8}$$

and we have introduced world-sheet light-cone coordinates $\sigma_{\pm}=(\tau\pm\sigma)/2$, that is to say, $q^{\mu}_{\pm}=\dot{q}^{\mu}\pm q'^{\mu}$, etc.

The fundamental quadratic form $\alpha^{(\epsilon)}$ is a measure of the invariant string size $S_{(\epsilon)}$, as follows from Eqs. (2.7) and (2.8):

$$S_{(\epsilon)}=\frac{1}{\sqrt{2}H}e^{\alpha^{(\epsilon)}/2}. \tag{2.9}$$

The string equations of motion and constraints, Eqs. (2.5) and (2.6), can now be written in the more compact form

$$q^{\mu}_{+-}=\epsilon e^{\alpha^{(\epsilon)}}q^{\mu}, \tag{2.10}$$

$$\eta^{(\epsilon)}_{\mu\nu}q^{\mu}_{\pm}q^{\nu}_{\pm}=0. \tag{2.11}$$

It is convenient to introduce the basis [10,11] (see also [1–5,18–20])

$$\mathcal{U}_{(\epsilon)}=\{q^{\mu},q^{\mu}_{+},q^{\mu}_{-},l^{\mu}_{(\epsilon)}\},\quad l^{\mu}_{(\epsilon)}\equiv e^{-\alpha^{(\epsilon)}}e^{\mu}_{(\epsilon)\rho\sigma\delta}q^{\rho}q^{\sigma}_{+}q^{\delta}_{-}, \tag{2.12}$$

$$\eta^{(\epsilon)}_{\mu\nu}l^{\mu}_{(\epsilon)}l^{\nu}_{(\epsilon)}=1, \tag{2.13}$$

and $e^{\mu}_{(\epsilon)\rho\sigma\delta}$ is the completely antisymmetric four-tensor in the spacetime (2.1). The second derivatives of q^{μ}, expressed in the basis $\mathcal{U}_{(\epsilon)}$, are given by

$$q^{\mu}_{++}=\alpha^{(\epsilon)}_{+}q^{\mu}_{+}+u^{(\epsilon)}l^{\mu}_{(\epsilon)},\quad q^{\mu}_{--}=\alpha^{(\epsilon)}_{-}q^{\mu}_{-}+v^{(\epsilon)}l^{\mu}_{(\epsilon)},$$
$$q^{\mu}_{+-}=e^{\alpha^{(\epsilon)}}q^{\mu}, \tag{2.14}$$

where the functions $u^{(\epsilon)}$ and $v^{(\epsilon)}$ are implicitly defined by

$$u^{(\epsilon)}\equiv\eta^{(\epsilon)}_{\mu\nu}q^{\mu}_{++}l^{\nu}_{(\epsilon)},\quad v^{(\epsilon)}\equiv\eta^{(\epsilon)}_{\mu\nu}q^{\mu}_{--}l^{\nu}_{(\epsilon)}, \tag{2.15}$$

and satisfy

$$u^{(\epsilon)}_{-}=v^{(\epsilon)}_{+}=0. \tag{2.16}$$

Then, by differentiating Eq. (2.8) twice, we get

$$\alpha^{(\epsilon)}_{+-}-\epsilon e^{\alpha^{(\epsilon)}}+u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})e^{-\alpha^{(\epsilon)}}=0. \tag{2.17}$$

In the previous discussions [10,11], it was implicitly assumed that the product $u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})$ is positive definite. In that case the conformal transformation on the world-sheet metric (2.7),

$$\alpha^{(\epsilon)}(\sigma_{+},\sigma_{-})=\hat{\alpha}^{(\epsilon)}(\hat{\sigma}_{+},\hat{\sigma}_{-})+\tfrac{1}{2}\ln|u^{(\epsilon)}(\sigma_{+})||v^{(\epsilon)}(\sigma_{-})|,$$
$$\hat{\sigma}_{+}=\int\sqrt{|u^{(\epsilon)}(\sigma_{+})|}d\sigma_{+},\quad\hat{\sigma}_{-}=\int\sqrt{|v^{(\epsilon)}(\sigma_{-})|}d\sigma_{-}, \tag{2.18}$$

reduces Eq. (2.17) to the equation

$$\alpha^{(\epsilon)}_{+-}-\epsilon e^{\alpha^{(\epsilon)}}+e^{-\alpha^{(\epsilon)}}=0. \tag{2.19}$$

This equation is the sinh-Gordon equation in the case of de Sitter spacetime $(\epsilon=+1)$ and the cosh-Gordon equation in the case of anti–de Sitter spacetime $(\epsilon=-1)$.

It must be noticed, however, that for a generic string world sheet, the product $u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})$ is neither positive nor negative definite. In fact, in the next section we shall construct explicit solutions to the string equations of motion and constraints (2.10) and (2.11) corresponding to $u^{(\epsilon)}\times(\sigma_{+})v^{(\epsilon)}(\sigma_{-})$ positive, $u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})$ negative, and $u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})$ identically zero, in *both* de Sitter and anti–de Sitter spacetimes.

In the case that $u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})$ is negative, the conformal transformation (2.18) reduces Eq. (2.17) to

$$\alpha^{(\epsilon)}_{+-}-\epsilon e^{\alpha^{(\epsilon)}}-e^{-\alpha^{(\epsilon)}}=0, \tag{2.20}$$

and including also the case when $u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})=0$, we conclude that the most general equation fulfilled by the fundamental quadratic form $\alpha^{(\epsilon)}$ is

$$\alpha^{(\epsilon)}_{+-}-\epsilon e^{\alpha^{(\epsilon)}}+Ke^{-\alpha^{(\epsilon)}}=0, \tag{2.21}$$

where

$$K=\begin{cases}+1, & u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})>0,\\ -1, & u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})<0,\\ 0, & u^{(\epsilon)}(\sigma_{+})v^{(\epsilon)}(\sigma_{-})=0,\end{cases} \tag{2.22}$$

and

$$\epsilon=\begin{cases}+1 & \text{de Sitter spactime,}\\ -1 & \text{anti-de Sitter spacetime.}\end{cases} \tag{2.23}$$

Equation (2.21) is either the sinh-Gordon equation ($\epsilon=K=\pm 1$), the cosh-Gordon equation ($\epsilon=-K=\pm 1$), or the Liouville equation ($K=0$), and all three equations appear in both de Sitter and anti–de Sitter spacetimes. This does not mean, of course, that the string dynamics is the same in de Sitter and anti–de Sitter spacetimes.

Let us define a potential $V^{(\epsilon)}(\alpha^{(\epsilon)})$ by

$$\alpha^{(\epsilon)}_{+-}+\frac{dV^{(\epsilon)}(\alpha^{(\epsilon)})}{d\alpha^{(\epsilon)}}=0 \tag{2.24}$$

[so that if $\alpha^{(\epsilon)}=\alpha^{(\epsilon)}(\tau)$, then $\frac{1}{2}(\dot{\alpha}^{(\epsilon)})^2+V^{(\epsilon)}(\alpha^{(\epsilon)})=$ const]. Then, it follows that, in the case of de Sitter spacetime,

$$V^{(+1)}(\alpha)=\begin{cases}-2\cosh\alpha, & K=+1,\\ -2\sinh\alpha, & K=-1,\\ -e^{\alpha}, & K=0,\end{cases} \tag{2.25}$$

while, in the case of anti–de Sitter spacetime,

$$V^{(-1)}(\alpha)=\begin{cases}2\sinh\alpha, & K=+1,\\ 2\cosh\alpha, & K=-1,\\ e^{\alpha}, & K=0,\end{cases} \tag{2.26}$$

and we have skipped the ($\pm$) index on α. Notice that the cosh α potential corresponds to the sinh-Gordon equation and vice versa.

The results (2.24)–(2.26) are represented in Fig. 1, showing the different potentials in the cases of de Sitter and anti–de Sitter spacetimes, respectively. Until now only the $K=+1$ sector in de Sitter spacetime was known. The new features introduced by the new sectors $K=0$ (corresponding to the Liouville equation) and $K=-1$ (corresponding to the cosh-Gordon equation in the case of de Sitter spacetime and to the sinh-Gordon equation in the case of anti–de Sitter spacetime) appear for negative α (‘‘small’’ strings). Small strings with proper size $<1/(\sqrt{2}H)$ in the $K=-1$ sector (inside the horizon in the case of de Sitter spacetime) do not collapse into a point (as is the case in the $K=+1$ sector) but have a minimal size.

The main differences between de Sitter and anti–de Sitter potentials are for positive α [strings with proper size $>1/(\sqrt{2}H)$]. In the case of de Sitter spacetime [Fig. 1(a)], the potentials are unbounded from below for large strings (large positive α), while for small strings (large negative α) they are either growing indefinetely, flat, or unbounded from below. In the case of anti–de Sitter spacetime [Fig. 1(b)], on the other hand, the potentials grow indefinitely for large strings (large positive α), while for small strings (large negative α) they are either growing indefinitely, flat, or unbounded from below.

From these results we can deduce the generic features of strings propagating in de Sitter and anti–de Sitter spacetimes: Large strings (large positive α) in de Sitter spacetime generically expand indefinitely, while small strings (large negative α) either bounce or collapse. In anti–de Sitter spacetime, large strings generically contract, while small strings either bounce or collapse. For small strings (large negative α) the dynamics is similar in de Sitter and anti–de Sitter spacetimes, while for large strings (large positive α) it is completely different in the two spacetimes.

Notice that the ϵ in Eq. (2.21), which distinguishes between de Sitter and anti–de Sitter spacetimes, corresponds to the ‘‘K’’ in the notation of Ref. [10]. Our K in Eq. (2.21) was missed in Refs. [10,11]; only the solutions corresponding to $K=+1$ were found there.

III. EXPLICIT EXAMPLES

The exact (‘‘global,’’ i.e., the whole world sheet) solutions to the string equations of motion and constraints in de Sitter and anti–de Sitter spacetimes considered in the literature until now [12–15,17,21–23] describe different classes of string solutions of generic shape, circular strings, and stationary strings. These solutions exhibit the multistring property [12–15]; namely, one single world sheet describes a finite or infinite number of different and independent strings. The presence of multistrings is a characteristic feature in spacetimes with a cosmological constant (constant curvature or asymptotically constant curvature spacetimes). All these solutions fall in the $K=+1$ sector, i.e., are solutions to the sinh-Gordon equation in the case of de Sitter spacetime and to the cosh-Gordon equation in the case of anti–de Sitter spacetime. We shall now construct larger families of exact solutions which fall into *all* three sectors $K=\pm 1, 0$.

Consider first the following algebraic problem: to find the most general ansatz which reduces the string equations of motion and constraints to *ordinary* differential equations in spacetimes of the form

$$ds^2=-a(r)dt^2+\frac{dr^2}{a(r)}+r^2d\phi^2. \tag{3.1}$$

The string equations of motion are given by

$$\ddot{t}-t''+\frac{a_{,r}}{a}(\dot{t}\dot{r}-t'r')=0,$$

$$\ddot{r}-r''-\frac{a_{,r}}{2a}(\dot{r}^2-r'^2)+\frac{aa_{,r}}{2}(\dot{t}^2-t'^2)-ar(\dot{\phi}^2-\phi'^2)=0,$$

$$\ddot{\phi}-\phi''+\frac{2}{r}(\dot{\phi}\dot{r}-\phi'r')=0, \tag{3.2}$$

while the constraints take the form

$$-a(\dot{t}^2+t'^2)+\frac{1}{a}(\dot{r}^2+r'^2)+r^2(\dot{\phi}^2+\phi'^2)=0,$$

$$-a\dot{t}t'+\frac{1}{a}\dot{r}r'+r^2\dot{\phi}\phi'=0. \tag{3.3}$$

Since the Christoffel symbols depend only on r, the desired ansatz is

$$r=r(\xi^1),\quad t=t(\xi^1)+c_1\xi^2,\quad \phi=\phi(\xi^1)+c_2\xi^2, \tag{3.4}$$

where (ξ^1,ξ^2) are the two world-sheet coordinates (one of which is timelike, the other spacelike), and (c_1,c_2) are two arbitrary constants. With this ansatz, Eqs. (3.2) are solved by

$$\frac{dt}{d\xi^1}=\frac{k_1}{a(r)}, \quad \frac{d\phi}{d\xi^1}=\frac{k_2}{r^2}, \tag{3.5}$$

$$\left(\frac{dr}{d\xi^1}\right)^2=-a(r)r^2c_2^2-\frac{a(r)}{r^2}k_2^2+k_1^2+a^2(r)c_1^2, \tag{3.6}$$

where (k_1,k_2) are two integration constants. For the constraints, Eqs. (3.3), to be satisfied we must have

$$k_1c_1=k_2c_2. \tag{3.7}$$

In particular, circular string dynamics as considered in [12–15,17,21,23] corresponds to $c_1=k_2=0$ and $(\xi^1,\xi^2)=(\tau,\sigma)$, while the infinitely long stationary strings considered in [15] correspond to the ''dual'' choice $c_2=k_1=0$ and $(\xi^1,\xi^2)=(\sigma,\tau)$.

The induced line element on the string world sheet is

$$dS^2=[r^2c_2^2-a(r)c_1^2][-(d\xi^1)^2+(d\xi^2)^2], \tag{3.8}$$

such that the fundamental quadratic form is given by

$$e^{\alpha}=2|r^2c_2^2-a(r)c_1^2|. \tag{3.9}$$

Let us now return to our main interest here: strings in de Sitter and anti–de Sitter spacetimes. In this case, the function $a(r)$ is given by

$$a_{(\epsilon)}=1-\epsilon H^2r^2. \tag{3.10}$$

In the case of anti–de Sitter spacetime $(\epsilon=-1)$, the static coordinates (t,r,ϕ) cover the complete manifold, while for de Sitter spacetime $(\epsilon=+1)$, they cover only the region inside the horizon; the complete de Sitter manifold can, however, be covered by four coordinate patches of the form (3.1) and (3.10); see, for instance [24],. Notice that Eq. (3.6) for the radial coordinate can be solved explicitly in terms of the Weierstrass elliptic $\wp$ function [25]. The other two equations (3.5) can then be integrated, the results being expressed in terms of the Weierstrass elliptic σ and ζ functions [25]. We have thus solved completely the string equations of motion and constraints using the ansatz (3.4) in both de Sitter and anti–de Sitter spacetimes, but the explicit expressions of the solutions are not important here. It should be also stressed that in general the ansatz (3.4) does not lead to solutions automatically satisfying the standard closed or open string boundary conditions; see, for instance, [26]. However, imposing the boundary conditions does not arise any problem. In some cases the ansatz (3.4) actually *does* lead to solutions satisfying the standard boundary conditions; an example is $c_1=k_2=0$, in which case the solution describes dynamical circular strings [12–15,17,21,23]. Finally, we are often interested in string solutions that do not satisfy the standard closed or open string boundary conditions; this is, for instance, the case for infinitely long strings [15,27] or finite open strings with external forces acting on the end points of the strings [28,29].

Let us consider the spacetime region where $(c_2^2+\epsilon H^2c_1^2)r^2\geqslant c_1^2$ (similar conclusions are reached in the other region). In this case ξ^1 is the timelike world-sheet coordinate, $\xi^1\equiv\tau/H$. Then, Eqs. (3.6) and (3.9) lead to

$$\begin{aligned}\left(\frac{d\alpha^{(\epsilon)}}{d\tau}\right)^2&-2\epsilon e^{\alpha^{(\epsilon)}}+\frac{8}{H^2}[c_1^2c_2^2-(c_2^2+\epsilon H^2c_1^2)\\&\times(k_1^2+\epsilon H^2k_2^2)]e^{-\alpha^{(\epsilon)}}\\&=-\frac{4}{H^2}(c_2^2-\epsilon H^2c_1^2).\end{aligned} \tag{3.11}$$

Now, by tuning the constants of motion to fix the sign of the square brackets, and by performing conformal transformations of the form (2.18), we can, after differentiation with respect to τ, reduce this equation to either the sinh-Gordon equation, the cosh-Gordon equation, or the Liouville equation:

$$\epsilon[c_1^2c_2^2-(c_2^2+\epsilon H^2c_1^2)(k_1^2+\epsilon H^2k_2^2)]<0\Rightarrow \text{sinh-Gordon},$$

$$\epsilon[c_1^2c_2^2-(c_2^2+\epsilon H^2c_1^2)(k_1^2+\epsilon H^2k_2^2)]>0\Rightarrow \text{cosh-Gordon},$$

$$[c_1^2c_2^2-(c_2^2+\epsilon H^2c_1^2)(k_1^2+\epsilon H^2k_2^2)]=0\Rightarrow \text{Liouville}.$$

Thus, we have constructed explicit solutions to the string equations of motion and constraints associated to the sinh-Gordon equation, the cosh-Gordon equation, or the Liouville equation and all three equations appear in both de Sitter and anti–de Sitter spacetimes.

We close this section with the following remark. The ansatz (3.4) is a generalization of both the circular string ansatz $[c_1=0,\ \phi(\xi^1)=\text{const},\ \xi^1\ \text{timelike}]$ and the stationary string ansatz $[c_2=0,\ t(\xi^1)=\text{const},\ \xi^1\ \text{spacelike}]$. In both these cases, it was shown in Refs. [12–15] that the resulting solutions in de Sitter and anti–de Sitter spacetimes should be interpreted as multistring solutions, that is to say, string solutions where one single world sheet describes finitely or infinitely many different and independent strings. The existence of such multistring solutions appears to be a quite general feature in constant curvature (and asymptotically constant curvature) spacetimes.

IV. 2+1 BH-ADS SPACETIME

As another example to illustrate our general results of Sec. II, we now consider the $(2+1)$-dimensional black hole anti–de Sitter (BH-AdS) spacetime.

The metric of the $2+1$ dimensional BH-ADS spacetime in its standard form is given by [16]

$$ds^2=\left(\frac{J^2}{4r^2}-\Delta\right)dt^2+\frac{dr^2}{\Delta}-Jdtd\phi+r^2d\phi^2, \tag{4.1}$$

where

$$\Delta=\frac{r^2}{l^2}-M+\frac{J^2}{4r^2}. \tag{4.2}$$

Here M represents the mass, J is the angular momentum, and the cosmological constant is $\Lambda=-1/l^2$. The causal structure is similar to that of the four-dimensional Kerr spacetime. There is no strong curvature singularity at $r=0$; however, $R_{\mu\nu}=2\Lambda g_{\mu\nu}$. This is a constant curvature spacetime locally and asymptotically isometric to $(2+1)$-dimensional anti–de Sitter spacetime; this is of course why it is also relevant for our purposes here. For more details on the local and global geometry of the BH-ADS spacetime, see, for instance, Refs. [16,30–32].

The problem of the string propagation in the BH-ADS spacetime was completely analyzed and the circular string motion was exactly solved, in terms of elliptic functions, by the present authors in [17]. The equation determining the string loop radius as a function of time is

$$\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{r^2}{l^2} - M\right) + \frac{J^2}{4} - E^2 = 0, \tag{4.3}$$

where E^2 is a non-negative integration constant, while the fundamental quadratic form α, which determines the invariant size of the string, is given by

$$e^{\alpha} = 2r^2/l^2. \tag{4.4}$$

It is then straightforward to show that Eq. (4.3) becomes

$$\left(\frac{d\alpha}{d\tau}\right)^2 + 2e^{\alpha} - \frac{8}{l^2}\left(E^2 - \frac{J^2}{4}\right)e^{-\alpha} = 4M. \tag{4.5}$$

After performing a conformal transformation of the form (2.18) and differentiating with respect to τ, this equation reduces to the (i) sinh-Gordon equation if $E^2 < J^2/4$, (ii) to the cosh-Gordon equation if $E^2 > J^2/4$, and (iii) to the Liouville equation if $E^2 = J^2/4$; thus, all three equations are present. Notice finally that the three different types of allowed dynamics as reported in [17], essentially whether the circular string collapses into $r=0$ [case (ii)] or not [case (i)], precisely correspond to these different equations [in the limiting case (iii), the string contracts from the static limit to $r=0$].

V. CONCLUDING REMARKS

In conclusion, we have shown that the fundamental quadratic form of classical string propagation in $(2+1)$-dimensional constant curvature spacetimes solves the sinh-Gordon equation, the cosh-Gordon equation, or the Liouville equation. We have shown that in both de Sitter and anti–de Sitter spacetimes (as well as in the $2+1$ BH-ADS spacetime), all three equations must be included to cover the generic string dynamics. This is particularly enlightening, since generic features of the string propagation in these spacetimes can be read off directly at the level of the equations of motion from the properties of the sinh, cosh, and Liouville potentials, without need of solving the equations. We also constructed new classes of explicit solutions to *all* three equations in both de Sitter and anti–de Sitter spacetimes, exhibiting the multistring property.

Finally it is worth observing that our results suggest the existence of various kinds of dualities relating the different string solutions in de Sitter and anti–de Sitter spacetimes. From the potentials, Eqs. (2.25) and (2.26), it follows, in particular, that small strings are dual ($\alpha \to -\alpha$) to large strings in the $K=+1$ ($K=-1$) sector of de Sitter (anti–de Sitter) spacetime. Furthermore, small (large) strings in the $K=-1$ sector in de Sitter spacetime are dual ($\alpha \to -\alpha$, $\epsilon \to -\epsilon$) to large (small) strings in the $K=+1$ sector in anti–de Sitter spacetime.

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