Chaotic amplification of neutrino chemical potentials by neutrino oscillations in big bang nucleosynthesis

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We investigate in detail the parameter space of active-sterile neutrino oscillations that amplifies neutrino chemical potentials at the epoch of big bang nucleosynthesis. We calculate the magnitude of the amplification and show evidence of chaos in the amplification process. We also discuss the implications of the neutrino chemical potential amplification in big bang nucleosynthesis. It is shown that with a ~ 1 eV ν_e, the amplification of its chemical potential by active-sterile neutrino oscillations can lower the effective number of neutrino species at big bang nucleosynthesis to significantly below three. [S0556-2821(96)05616-0]

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I. INTRODUCTION

Neutrino oscillations have been suggested to explain several experimental results and, if proven true, they will represent a significant step toward physics beyond the standard model of particle physics [1]. Mixings between active neutrinos (ν_e, ν_μ, or ν_τ) and sterile neutrinos (hypothetical neutrinos that do not interact with known particles via the strong, weak or electromagnetic interactions) are one possible source of neutrino oscillations. The most stringent constraints on the parameters of active-sterile neutrino mixings come from cosmological and astrophysical considerations [2–6]. In particular, the active-sterile neutrino oscillations at the epoch of big bang nucleosynthesis (BBN) have been investigated extensively and tight constraints have been obtained based on the primordial ^{4}He abundance in our universe. Interestingly, it was recently pointed out by Foot, Thomson, and Volkas [7] that some parameter space of the active-sterile neutrino mixings can amplify neutrino asymmetries (neutrino chemical potentials) so that the previous constraints on active-sterile neutrino mixings based on BBN can be alleviated [7,8]. Also an asymmetry in the electron neutrino sector as large as ~ 0.1 at the time of BBN can change significantly the BBN prediction of the primordial ^{4}He abundance.

In Sec. II of this paper we expand the original investigation of Foot, Thomson, and Volkas [7] by calculating the parameter space that amplifies neutrino chemical potentials and the magnitude of the amplification. But instead of relying on a simplified equation that only applies outside the resonant regime and when neutrinos are incoherent, we analyze the problem based on the original equations in the density matrix formalism, both analytically and numerically. Our analyses reveal many interesting features of the amplification process that cannot be revealed by the simplified approach. For example, the neutrino asymmetry can be oscillatory long after the initial resonant crossing. There is evidence which suggests that the oscillatory asymmetry is chaotic. As a result, although the order of magnitude of the final neutrino chemical potential is readily predictable, the sign of the chemical potential is very sensitive to the mixing parameters and the input parameters of numerical calculations. This oscillatory behavior and evidence of a chaotic amplification are probed in Sec. II.

In Sec. III, we discuss two implications of our results in Sec. II and show how active neutrinos as dark matter candidates—having ~ 1 eV mass—can lower the effective number of neutrino species in BBN to significantly below 3 by mixing with a lighter sterile neutrino.

II. FORMALISM AND CALCULATIONS

Throughout the paper, we adopt a unit in which $\hbar = c = k = 1$. We also use a convention to denote the number density of a particle i by N_i, and the number density relative to its equilibrium value [$2\zeta(3)T^3/\pi^2$ for photons, 3/4 of that for electron positrons, and 3/8 of that for neutrinos] by n_i. The Hubble expansion rate is $H = 7T^2/M_P$ where M_P is the Planck mass and T the temperature of the universe. Finally, T_6 denotes T in the unit of MeV. Since we are only concerned with the era of BBN, we limit our discussion to $1 \lesssim T_6 \lesssim 100$.

Mixtures of an active neutrino ν_α and a sterile neutrino ν_s can be described by a density matrix [3]

$$\rho_\nu = \begin{pmatrix} \rho_{\alpha\alpha} & \rho_{\alpha s} \\ \rho_{s\alpha} & \rho_{ss} \end{pmatrix} = \frac{P_0 I + \mathbf{P}\cdot\vec{\sigma}}{2}, \quad (1)$$

where $\vec{\sigma}$ are the Pauli matrices. The number densities of ν_α and ν_s in the mixture, relative to their equilibrium values, are, respectively,

$$n_{\nu_\alpha} = \frac{P_0 + P_z}{2}, \quad n_{\nu_s} = \frac{P_0 - P_z}{2}. \quad (2)$$

The evolution of the total relative number density of the neutrino mixture at the epoch of BBN is [3]

$$\dot{P}_0 = \sum_{i=e,\nu_\beta;\beta\neq\alpha} \langle \Gamma(\nu_\alpha \overline{\nu_\alpha} \rightarrow i\overline{i}) \rangle (n_i n_{\overline{i}} - n_{\nu_\alpha} n_{\overline{\nu_\alpha}}), \quad (3)$$

where $\langle \Gamma \rangle$ are reaction rates averaged over a thermal spectrum. Values of $\langle \Gamma \rangle$ are listed in Table I of Ref. [3] or Ref. [5]. The evolution of $\mathbf{P}$ is [3]

$$\dot{\mathbf{P}}=\mathbf{V}\times\mathbf{P}+\dot{P}_0\hat{\mathbf{z}}-D\mathbf{P}_\perp, \tag{4}$$

where $\mathbf{V}$ represents the frequency and the axis of the oscillation in the $\mathbf{P}$ space, and $\mathbf{P}_\perp=P_x\hat{\mathbf{x}}+P_y\hat{\mathbf{y}}$. The D term represents the damping of $\mathbf{P}_\perp$ due to neutrino interactions which constantly reduce a mixed neutrino state into an eigenstate of either ν_α or ν_s.

At the epoch of BBN,

$$V_x=\frac{\delta m^2}{2E}\sin 2\theta,\quad V_y=0,\quad V_z=-\frac{\delta m^2}{2E}\cos 2\theta+V_\alpha^L+V_\alpha^T, \tag{5}$$

where δm^2 and θ are the usual vacuum mixing parameters, and E is the energy of the neutrinos. V_α^L is the contribution of the matter effect from asymmetries in the background plasma [9]:

$$\begin{aligned}V_\alpha^L&=\sqrt{2}G_FN_\gamma\left\{L_0+0.375\left[2(n_{\nu_\alpha}-n_{\overline{\nu_\alpha}})+\sum_{\nu_\beta\neq\nu_\alpha}(n_{\nu_\beta}-\eta_{\overline{\nu_\beta}})\right]\right\}\\&\approx 0.13G_FT^3\left[8L_0/3+2(n_{\nu_\alpha}-n_{\overline{\nu_\alpha}})+\sum_{\nu_\beta\neq\nu_\alpha}(n_{\nu_\beta}-n_{\overline{\nu_\beta}})\right],\end{aligned} \tag{6}$$

where L_0 represents the contribution from the baryonic asymmetry as well as the asymmetry in electron positrons, and is $\sim 10^{-9}$. N_γ is the photon number density. The $n_\nu-n_{\overline{\nu}}$ terms represent the asymmetries in active neutrinos and thus their nonzero chemical potentials. If ξ_ν, the chemical potential of ν divided by kT, is much smaller than 1, $n_\nu-n_{\overline{\nu}}\approx 1.8\xi_\nu$.

V_α^T is the contribution of the matter effect due to a finite temperature [9]:

$$\begin{aligned}V_\alpha^T&=-\sqrt{2}G_FN_\gamma[12.61ET(n_{\nu_\alpha}+n_{\overline{\nu_\alpha}})/4M_Z^2\\&\quad+12.61ET/M_W^2],\quad \alpha=e,\\&=-\sqrt{2}G_FN_\gamma[12.61ET(n_{\nu_\alpha}+n_{\overline{\nu_\alpha}})/4M_Z^2],\quad \alpha=\mu,\tau.\end{aligned} \tag{7}$$

It has been shown that Eqs. (3)–(7) give a good description of neutrino oscillations in BBN if the average neutrino energy $E\approx 3.151T$ is inserted in $\mathbf{V}$ and if D is thermally averaged [3]. Therefore, numerically,

$$V_\alpha^T\approx\begin{cases}-250G_F^2T^5, & \alpha=e\\ -70G_F^2T^5, & \alpha=\mu,\tau.\end{cases} \tag{8}$$

The damping coefficient D, consisting of contributions from both elastic scattering and inelastic scattering of ν_α, is [3,5]

$$D\approx\begin{cases}(1.3+0.4n_{\nu_\alpha}+0.5n_{\overline{\nu_\alpha}})G_F^2T^5, & \alpha=e\\ (0.8+0.4n_{\nu_\alpha}+0.5n_{\overline{\nu_\alpha}})G_F^2T^5, & \alpha=\mu,\tau.\end{cases} \tag{9}$$

The reason we display n_{ν_α} and $n_{\overline{\nu_\alpha}}$ explicitly instead of approximating them to 1 is for the convenience of calculating the difference in the coefficient between the $\nu_\alpha-\nu_s$ and $\overline{\nu_\alpha}-\overline{\nu_s}$ oscillations. Since we often compare D to the Hubble expansion rate H, we note

$$D\approx 0.5T_6^3H. \tag{10}$$

The initial condition for Eqs. (3) and (4) is usually chosen to be $P_0=P_z=1$ and $P_x=P_y=0$, at $T_{\rm init}$ when V_α^T dominates over $\delta m^2/2E$: i.e.,

$$T_{\rm init}\gg 15\left|\frac{\delta m^2\cos 2\theta}{\rm eV^2}\right|^{1/6} \tag{11}$$

(for the moment we assume any neutrino asymmetry is negligible). That is, the neutrino ensemble consists purely of ν_α, which is a good approximation because $V_z\gg V_x$ so that $\mathbf{V}$ is almost aligned with the $\hat{\mathbf{z}}$ axis.

As the universe expands and its temperature drops, $|V_x/V_z|$ becomes larger, the amplitude of the oscillation consequently increases. Eventually, if there is no amplification of neutrino asymmetries, V_α becomes negligible, and $\mathbf{V}$ settles down into its vacuum value. During the process, if the mixing has $\delta m^2<0$ (ν_α heavier than ν_s), a resonance can occur when $\mathbf{V}$ crosses the $\hat{\mathbf{x}}$ axis at a temperature

$$T_{\rm res}\approx 13(16)\times\left|\frac{\delta m^2\cos 2\theta}{1\ \ \rm eV^2}\right|^{1/6}\ \ {\rm MeV}\quad \text{for}\ \ \alpha=e(\mu,\tau). \tag{12}$$

$\mathbf{P}$ and $\mathbf{V}$ before and after the resonance are illustrated in Fig. 1.

During the oscillation, the interactions between ν_α and the background plasma play two roles. First, the interactions (including both elastic and inelastic ones, represented by the D term) reduce mixed neutrino states into either ν_α or ν_s, which effectively damp the amplitude of $\mathbf{P}_\perp$ and at the same time randomize the phase of the neutrino oscillation. When the regenerated ν_α (and ν_s but mostly ν_α) oscillate into ν_s (and ν_α) again, the portion of ν_α in excess of ν_s, $\underline{P_z}$, decreases toward 0. Secondly, the inelastic process—$\nu_\alpha\overline{\nu_\alpha}$ pair productions (the $\dot{P}_0\hat{\mathbf{z}}$ term)—constantly replenishes the number of ν_α that is being depleted by oscillation, maintaining its population as a full relativistic species as long as such pair productions are potent ($T\gtrsim 3$ MeV for ν_e and $\gtrsim 5$ MeV for ν_μ or ν_τ).

Equations (1)–(11) can be equally applied to the antineutrino sector, with notations for particles and antiparticles switched and L_0 replaced by $-L_0$. Apparently, since ν_α and $\overline{\nu_\alpha}$ can only be produced as pairs, $\dot{P}_0=\dot{\overline{P}}_0$.

The relevance to BBN comes at $T\sim 1$ MeV when the neutron to proton ratio freezes out. If a significant population of ν_s is produced, or a significant asymmetry in $\nu_e\overline{\nu_e}$ is generated through the active-sterile neutrino oscillations, the neutron to proton ratio can be affected and the resultant ^{4}He primordial abundance altered from the standard BBN predictions. When neutrino asymmetries are negligible, to be consistent with the observed primordial ^{4}He abundance requires [3,5]

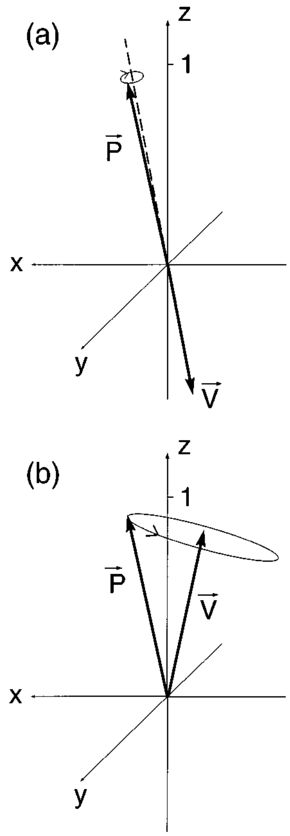

FIG. 1. Illustrations of evolution of **P** and **V** for $\delta m^2<0$. (a) Before resonance; (b) right after resonance.

$$\delta m^2 \sin^4 2\theta \lesssim 10^{-9} \ \ \mathrm{eV}^2. \tag{13}$$

(The bound on the $\nu_e-\nu_s$ mixing is tighter on the low δm^2 end. See Refs. [3] and [5] for precise constraints.)

Under conditions that

$$|\mathbf{V}| \gg D \gg |\dot{\mathbf{V}}|/|\mathbf{V}|, \tag{14}$$

i.e., the damping of **P** and the change in **V** are negligible within one cycle of oscillation of **P**, Eq. (4) can be simplified to the lowest order to

$$P_x = V_x P_z/V_z, \quad P_y=0, \quad \dot{P}_z = -DV_x^2 P_z/(V_x^2+V_z^2)+\dot{P}_0. \tag{15}$$

In the epoch of our concern, and in the absence of an amplification of neutrino asymmetries, Eq. (14) is satisfied except near the resonance region where $|V_z|\sim|V_x|$. This is because $V_\alpha^T = O(10^2)D$ and $|\dot{V}_\alpha/V_\alpha|\sim H \ll D$ at $T_6 \gg 1$. We will discuss the case of amplified asymmetries later in the section.

Similarly for antineutrinos, the approximate equations are

$$\overline{P_x} = V_x \overline{P_z}/\overline{V_z}, \quad \overline{P_y}=0, \quad \dot{\overline{P_z}} = -\overline{D}V_x^2 \overline{P_z}/(V_x^2+\overline{V_z^2})+\dot{\overline{P_0}} \tag{16}$$

under conditions that $|\mathbf{V}| \gg \overline{D} \gg |\dot{\overline{\mathbf{V}}}|/|\overline{\mathbf{V}}|$. Assuming there is no asymmetries in neutrinos other than the oscillating $\nu_\alpha \overline{\nu_\alpha}$ sector,

$$\overline{V_z} = V_z - 2\beta(\Delta P_z + 8L_0/3) = V_0 - \beta(\Delta P_z + 8L_0/3), \tag{17}$$

where $\beta = 0.375\sqrt{2}G_F \underline{N_\gamma} \approx 0.13 G_F T^3$, $V_0 = -\delta m^2 \cos 2\theta/2E + V_\alpha^T$, and $\Delta P_z = P_z - \overline{P_z} = 2(n_{\nu_\alpha} - n_{\overline{\nu_\alpha}})$. It is also noted that $\overline{D}-D \approx 0.05 \Delta P_z G_F^2 T^5 \ll D\Delta \underline{P_z}$.

The asymmetry in the $\nu_\alpha \overline{\nu_\alpha}$ sector can then be described by ΔP_z which to its lowest order satisfies

$$\begin{aligned}
\Delta\dot{P}_z &= DP_z V_x^2 \left(\frac{1}{V_x^2+\overline{V_z^2}} - \frac{1}{V_x^2+V_z^2} \right) \\
&\quad - D\frac{V_x^2}{V_x^2+\overline{V_z^2}}\Delta P_z + (\overline{D}-D)\frac{V_x^2}{V_x^2+\overline{V_z^2}}\overline{P_z} \\
&\approx \frac{DV_x^2}{V_x^2+[V_0-\beta(\Delta P_z+8L_0/3)]^2} \\
&\quad \times \left\{ \frac{4V_0\beta(\Delta P_z+8L_0/3)P_z}{\{V_x^2+[V_0+\beta(\Delta P_z+8L_0/3)]^2\}\Delta P_z} - 1 \right\} \Delta P_z .
\end{aligned} \tag{18}$$

The equation resembles Eq. (15) of Foot, Thomson, and Volkas [7] except that their equation omitted the second term and L_0, and assumed P_z to be 1.

When $V_0<0$, Eq. (18) is a damping equation for $|\Delta P_z|>8L_0/3$, and no amplification of $|\Delta P_z|$ to $\gg 10^{-9}$ can occur. This rules out ν_α-ν_s mixings with $\delta m^2>0$ (ν_s heavier than ν_α). Only when $\delta m^2<0$ and V_0 switchs to a positive value (resonance crossing) before $T\sim 1$ MeV does an amplification of ΔP_z become plausible.

For mixings with $\delta m^2<0$, V_0 still starts out negative at high temperatures $\gg T_{\mathrm{res}}$. Any $|\Delta P_z|>8L_0/3$ will be damped toward an asymptotic value such that $\Delta\dot{P}_z \to 0$. Thus,

$$\Delta P_z \to -\frac{8}{3}L_0 + \left| \frac{8V_x^2 L_0/3}{4V_0 \beta P_z} \right| \approx -\frac{8}{3}L_0 \tag{19}$$

when $|V_0| \gg |V_x|$.

When the system enters the resonant regime, Eq. (18) does not apply. We have to go back to the original equation (4) and its antineutrino counterpart, which give

$$\begin{aligned}
\Delta\dot{P}_x &= -V_0\Delta P_y - \beta(\Delta P_z + 8L_0/3)(P_y+\overline{P_y}) - D\Delta P_x , \\
\Delta\dot{P}_y &= V_0\Delta P_x + \beta(\Delta P_z + 8L_0/3)(P_x+\overline{P_x}) - V_x \Delta P_z \\
&\quad - D\Delta P_y ,
\end{aligned} \tag{20}$$

$$\Delta\dot{P}_y = V_x \Delta P_y ,$$

where $\Delta P_x = P_x - \overline{P_x}$ and $\Delta P_y = P_y - \overline{P_y}$. If we do not want ν_s to be brought into equilibrium, we have to restrict our discussions to the parameter space that satisfies Eq. (13). Then (1) the resonance crossing is nonadiabatic, and (2) D has to be small enough compared to the time scale of the resonance crossing so that most of the neutrinos do not scatter during the crossing. The nonadiabatic change in **V** leads to a coherent oscillation of **P** around the new **V** [see Fig.

1(b)]. The impotency of scatterings enables this coherency to be maintained throughout the resonant regime and beyond until $D \gtrsim |\dot{\mathbf{V}}|/|\mathbf{V}|$ so that the interactions have enough time to randomize the phases of neutrinos again. It also implies that the D term in Eq. (20) can be dropped in resonance.

When the temperature was high above $T_{\rm res}$ so that Eq. (18) applies, $\Delta P_x \approx V_x \Delta P_z / V_z$, ΔP_y, and $\Delta\dot{P}_y$ are approximately 0. But in and right after resonance, the approximation breaks down because of the rapid change of V_z and $\dot{V}_z$. Instead ΔP_y becomes oscillatory with a frequency of $\sim V_z$ [Fig. 1(b)]. $|\Delta P_z + 8L_0/3|$ will quickly be of order 10^{-9}, but a more important question is whether $|\Delta P_z + 8L_0/3|$ (and thus $|\Delta P_z|$) can be amplified to $\gg 10^{-9}$. Assuming that $|\beta \Delta P_z| \ll V_0$, the amplitude of the oscillating ΔP_y will be of order $\sim |P_z V_x (V_z^{-1} - \dot{V}_z^{-1})| \approx |2V_x \beta \Delta P_z / V_0^2|$ (from now on $P_z \sim 1$ and L_0 is dropped for simplicity because $|\Delta P_z + 8L_0/3| \sim |\Delta P_z|$). The amplification of $|\Delta P_z|$ depends on whether $\Delta\dot{P}_z$ has enough time to change ΔP_z by a factor of more than 1: i.e.,

$$\left| \frac{2V_x^2 \beta}{V_0^2} \right| \gtrsim V_0. \tag{21}$$

Since V_0 is a changing quantity, the condition of amplification depends on which V_0 to choose. For a crude estimate, a reasonable choice is the V_0 at the time when ΔP_y oscillates one cycle since the resonance [so that Eq. (21) is meaningful]. So

$$V_0 \sim \dot{V}_0 V_0^{-1} \sim \frac{H}{V_0} \frac{|\delta m^2|}{2E}. \tag{22}$$

Solving the equation assuming $E = 3.151T$ yields $V_0 \sim 10^{-2} |\delta m^2|^{-0.25} |\delta m^2/2E|$. Thus the condition of amplifying ΔP_z, i.e., Eq. (21), is

$$|\delta m^2|^{-5/12} \sin^2 2\theta \gtrsim 10^{-12}. \tag{23}$$

The growth of $|\Delta P_z|$ is limited once $|\beta \Delta P_z| \gg V_0$ (when $|\Delta P_z| \gg |\delta m^2/9 \text{ eV}^2| T_6^{-4}$). Because at this moment, the amplitude of ΔP_y becomes $\sim |2V_x/\beta \Delta P_z|$, and the amplitude of $\Delta\dot{P}_z/\Delta P_z$ becomes $\sim 2V_x^2/\beta \Delta P_z^2$, proportional to ΔP_z^{-2}.

An interesting feature of ΔP_z, shown both from Eq. (20) and numerical calculations, is that it keeps oscillating below the resonant temperature (Fig. 2). This is a direct consequence of the coherent oscillation of $\mathbf{P}$ and $\dot{\mathbf{P}}$. As a result of the oscillation, the sign of ΔP_z flips (so does V_z and $\dot{V}_z$) unless the change of ΔP_z in each cycle is smaller than the amplitude of ΔP_z itself; i.e.,

$$|\Delta P_z| > \frac{V_x^2}{|\beta \Delta P_z|} |\beta \Delta P_z|^{-1}. \tag{24}$$

Since for parameters that satisfy Eq. (23), the amplitude of $\Delta P_z \gtrsim (|\delta m^2|/9 \text{ eV}^2) T_6^{-4}$, a rough realization of the condition that ΔP_z will not be oscillatory between positive and negative values is

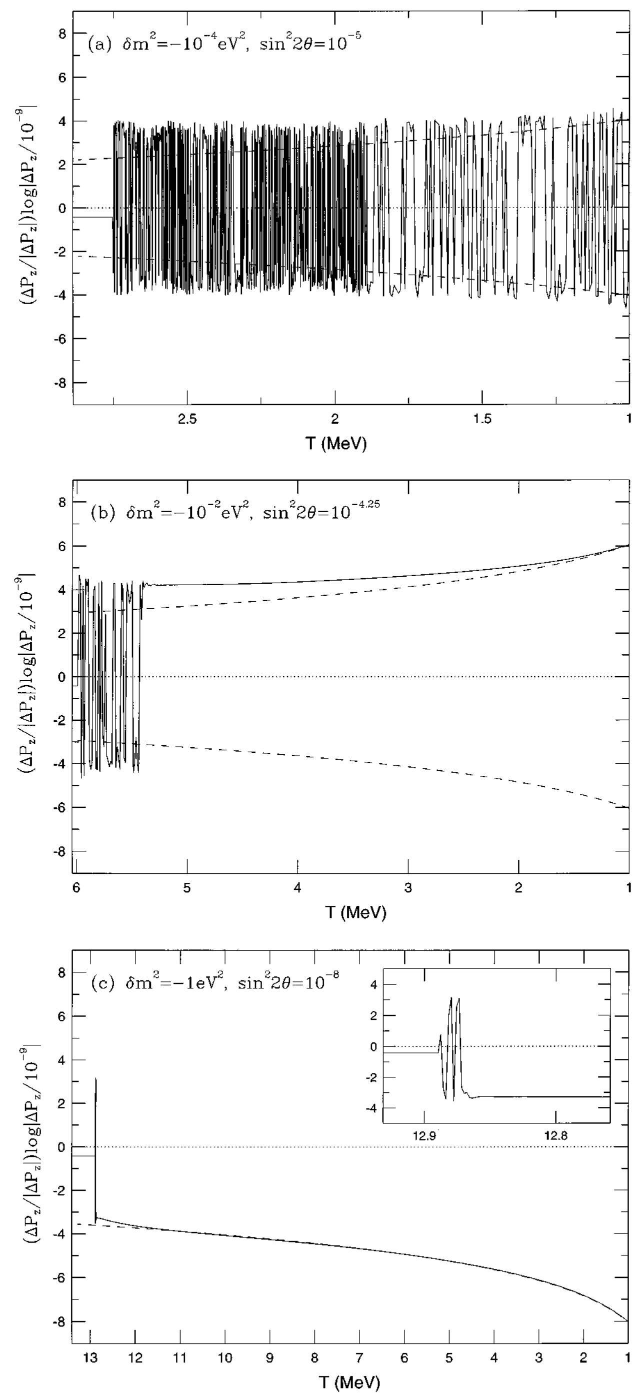

FIG. 2. The solid lines show the evolution of ΔP_z vs the temperature of the universe, for ν_e-ν_s mixing. Asymmetries less than 10^{-9} are ignored. The dash line shows $|\Delta P_z| = |\delta m^2/9 \text{ eV}^2| T_6^{-4}$. $L_0 = 10^{-9}$. (a) $\delta m^2 = -10^{-4}$ eV^2, $\sin^2 2\theta = 10^{-5}$; (b) $\delta m^2 = -10^{-2}$ eV^2, $\sin^2 2\theta = 10^{-4.25}$; (c) $\delta m^2 = -1$ eV^2, $\sin^2 2\theta = 10^{-8}$.

$$\frac{|\delta m^2|}{\sin^2 2\theta} \gtrsim 9 T_6^4 \text{ eV}^2. \tag{25}$$

At $T_6 = 1$, ΔP_z will not be oscillatory if $|\delta m^2|/\sin^2 2\theta \gtrsim 9$ eV^2.

Once ΔP_z stops flipping its sign, $|\dot{\mathbf{V}}|/|\mathbf{V}|$ and $|\ddot{\mathbf{V}}|/|\dot{\mathbf{V}}|$ decrease dramatically so that the damping D term becomes more and more important. Eventually Eq. (18) reapplies. Assuming $P_z \sim 1$ and neglecting L_0 and V_x in the denominator Eq. (18) yields

$$\dot{\Delta P_z} \approx D \frac{V_x^2}{(V_0-\beta\Delta P_z)^2}\left[\frac{4V_0\beta}{(V_0+\beta\Delta P_z)^2}-1\right]\Delta P_z . \quad (26)$$

Since $V_0>0$ after the initial resonance, this is an amplification equation if

$$\frac{4V_0\beta}{(V_0+\beta\Delta P_z)^2}>1 \quad \text{or} \quad \delta m^2 \lesssim 10^2 \ \text{eV}^2. \quad (27)$$

The amplification will not stop until

$$D\frac{V_x^2}{(V_0+\beta\Delta P_z)^2}\,\frac{4V_0\beta}{(V_0-\beta\Delta P_z)^2}\lesssim H, \quad (28)$$

which occurs at $|\beta\Delta P_z| \gtrsim V_0 \sim |\delta m^2/2E|$ and

$$\left(\frac{18T_6^7\sin^2 2\theta}{|\delta m^2|}\right)\left(\frac{\delta m^2/2E}{\beta\Delta P_z}\right)^4 \lesssim 1. \quad (29)$$

At $T_6 \sim 1$, nearly all the mixing parameters that show no oscillatory ΔP_z have $18T_6^7\sin^2 2\theta/|\delta m^2|<1$, so $|\beta\Delta P_z|$ is limited to $\sim|\delta m^2/2E|$ and

$$|\Delta P_z| \sim \frac{|\delta m^2|}{9T_6^4} \quad (30)$$

at a temperature of $\sim$1 MeV. This limit is confirmed by our numerical calculations and is similar to that of Foot, Thomson, and Volkas [7] based on their simplified equation. For $|\delta m^2| \gtrsim 10$ eV^2, however, since ΔP_z has to be much smaller than $P_z \sim 1$, $\Delta P_z \sim 0.1$.

The oscillatory behavior of ΔP_z after resonance is illustrated in Figs. 2(a)–2(c), for three different ν_e-ν_s mixing parameters. Each graph is the result of several million steps of integrations of Eq. (3) and Eq. (4) by adaptive Runge-Kutta method, with an error of less than 10^{-10} in each step. In Fig. 2(a), the mixing parameters do not satisfy Eq. (25) at $T_6=1$, so ΔP_z is still oscillatory at 1 MeV. In Figs. 2(b) and 2(c), $|\Delta P_z|$ settles down to $\sim|\delta m^2/9 \text{ eV}^2|T_6^{-4}$ at $T_6 \gtrsim 1$, in line with our estimate Eq. (30). Our numerical calculations also show that although the final settle-down value of $|\Delta P_z|$ is predictable, the sign of ΔP_z seems random among different parameter choices. For example, in case 2(b), $\delta m^2=-10^{-2}$ eV^2 and $\sin^2 2\theta = 10^{-4.25}$, a small change of $\sin^2 2\theta$ to $10^{-4.1875}$ yields an opposite sign of the final ΔP_z. The sign of ΔP_z can also be flipped by slight changes in the initial L_0 (as tiny as 0.01%) and calculational parameters, such as a different error control (from 10^{-10} to 5×10^{-10} in our example), or a step size, or even a slightly different relation between the average neutrino energy and the temperature (from $E=3.151T$ to $E=3.150T$ in our example). This is due to the large number of oscillations of ΔP_z before it approaches one of the two possible values, so that the final ΔP_z is very sensitive to the input and calculational parameters. Such behavior is not so significant in the case of $\delta m^2=-1$ eV^2 and $\sin^2 2\theta=10^{-8}$, because the number of oscillations of ΔP_z is small [Fig. 2(c)]. In this case, using $E=3.150T$ the evolution of **P** and $\overline{\mathbf{P}}$ traces the evolution of **P** and $\overline{\mathbf{P}}$ using $E=3.151T$ very well (in a sense that their difference is obviously still a perturbation at lower temperatures). This is also true if we change the initial L_0 by 0.01% (although the resultant perturbation can be several percent in the oscillatory epoch of ΔP_z). Nevertheless, a small change of $\sin^2 2\theta$ to $10^{-8.0625}$ still flips the sign of the final ΔP_z.

Figures 2(a) and 2(b) suggest a chaotic behavior in the epoch of oscillating ΔP_z. Figure 2(c) might be intrinsically chaotic too but the time scale of the oscillatory epoch may be too short for such behavior to show up. The sign switching of the final ΔP_z due to small deviations in mixing parameters also suggest that chaotic behaviors exist at least in certain areas of the parameter space. To find mathematical bases for chaos, we try to determining the Lyapunov exponents of the system [10]. Assuming $\vec{\phi}(t)=(\Delta P_x,\Delta P_y,\Delta P_z)$ represents a solution of Eq. (20), we investigate the behavior of a nearby solution $\vec{\phi}(t)+\overrightarrow{\delta\phi}(t)=(\Delta P_x,\Delta P_y,\Delta P_z)+(\delta P_x,\delta P_y,\delta P_z)$ (δP_x and δP_y are assumed to arise from P_x and P_y only so that we can carry out our analysis). The evolution of the $\overrightarrow{\delta\phi}(t)$ is obtained by linearizing Eq. (20):

$$\dot{\overrightarrow{\delta\phi}} = M\overrightarrow{\delta\phi}, \quad (31)$$

where

$$M=\begin{pmatrix} -D & -V_0-\beta\Delta P_z & -\beta(P_y+\overline{P_y}) \\ -V_0-\beta\Delta P_z & -D & -V_x+\beta(P_y+\overline{P_y}) \\ 0 & V_x & 0 \end{pmatrix}. \quad (32)$$

The eigenvalue of M, called the Lyapunov exponent, is

$$\lambda = \frac{\beta(P_y+\overline{P_y})(V_0+\beta\Delta P_z)}{\beta(P_x+\overline{P_x})-V_x} - D. \quad (33)$$

If λ is positive, nearby solutions will depart exponentially in phase space within a time scale of λ^{-1}. In our problem, since $P_x+\overline{P_x}$, $P_y+\overline{P_y}$ and ΔP_z are all oscillatory, so is λ. A crude analysis is to plug in the amplitudes of $P_x+\overline{P_x}$, $P_y+\overline{P_y}$, and ΔP_z, to see whether λ can be sometimes positive. In the limiting case of $|\beta\Delta P_z| \ll V_0$, $P_x+\overline{P_x} \sim P_y+\overline{P_y} \sim V_x/V_0$, the resultant λ is $\sim\beta V_0/(\beta-V_0)-D$ which can certainly be positive if $V_0>0$ and $|\delta m^2| \gg 9T_6^4$ eV^2, conditions satisfied after the initial resonance. In the limiting case of $|\beta\Delta P_z| \gg V_0$, $P_x+\overline{P_x} \sim P_y+\overline{P_y} \sim V_x/\beta\Delta P_z$, the resultant λ is $\sim\beta\Delta P_z/(1-\Delta P_z)-D$, which can again be positive (remember $\Delta P_z \ll 1$ and $\beta\Delta P_z \gg V_0 \gg D$). Therefore, our crude analysis shows that λ can at least be positive within a time scale of order λ^{-1} intermittently.[1] In other words, the system is not a classical textbook example of a chaotic system.

[1]In the analysis, we have chosen a particular set of nearby solutions, namely those having deviations in P_x and P_y but not in $\overline{P_x}$ and $\overline{P_y}$. The other extreme choices, in which δP_x or δP_y arises only in one particle population but not in its antiparticles, merely change the sign of $\beta\Delta P_z$ in the nominator of Eq. (33), thus do not

There is a concern that whether because of the chaotic behavior, ΔP_z can be averaged to zero by inhomogeneities of the background plasma. Even in the standard big bang picture based on the homogeneous and isotropic Friedmann-Robertson-Walker metric, there are thermal fluctuations in the local density at the primordial nucleosynthesis epoch [11]. Given the Harrison-Zol'dovich power spectrum and general relativity, the fluctuations at this epoch are constrained by cosmic background radiation experiments to be $<10^{-5}(l/ct)^{-2}$ at super-horizon scale $l>ct$ [11]. Fluctuations at subhorizon scales are damped by radiation to be much less than 10^{-5}. Our calculations show that at least for large $|\delta m^2|$, small $\sin^2 2\theta$ mixings, such as $\delta m^2=-1$ eV^2 and $\sin^2 2\theta=10^{-8}$, the final ΔP_z is sufficiently stable when inputs are varied by 10^{-5}. For small $|\delta m^2|$, large $\sin^2 2\theta$ mixings where the chaotic epoch is significant, such as when $\delta m^2=-10^{-2}$ eV^2 and $\sin^2 2\theta=10^{-4.25}$, the concern about ΔP_z being averaged to zero may become valid, depending on the mixing parameters and the details of the underlying cosmological model. In the discussions in the following section, we are mostly concerned with mixings with large $|\delta m^2|$ and small $\sin^2 2\theta$, so we do not attempt to consider ΔP_z averaged by inhomogeneities.

We are ultimately interested in the region of mixing parameters that amplifies neutrino chemical potentials, and the size of the amplification. In Figs. 3(a) and 3(b), we plot the parameter space allowed by BBN that amplifies neutrino chemical potentials. The boundary to the right which excludes parameters that bring ν_s into equilibrium is adopted from Shi, Schramm, and Fields [5]. The boundary to the left that distinguishes parameters that amplify neutrino chemical potentials from those that do not is based on our numerical calculations (smoothed), and agrees with our analytical estimate Eq. (23) within an order of magnitude. The lower cut on $|\delta m^2|$ is determined by requiring $T_{\rm res}\gtrsim 1$ MeV. The upper cut on $|\delta m^2|$ is dictated by laboratory bounds on ν_e mass in the ν_e-ν_s mixing case, and by Eq. (27) as well as cosmological considerations [12] in the $\nu_\mu(\nu_\tau)$-ν_s mixing case. The boundary that singles out parameters that have oscillatory ΔP_z at 1 MeV is plotted according to Eq. (25) which is confirmed by our numerical calculations.

We note that our numerical calculation of $\delta m^2=-1$ eV^2, $\sin^2 2\theta=10^{-8}$ ν_e-ν_s mixing yields an opposite sign of ΔP_z from that of Foot, Thomson, and Volkas [7]. But this may not be surprising due to the chaotic feature of the system, that different signs of ΔP_z may arise from different initial L_0, or even different choices of integrators, different errors or step sizes. It is also noted that the simplified equation in Ref. [7] [corresponding to Eq. (18) without the second term and L_0] is not suited for investigating the behavior of ΔP_z in the resonant regime and in the epoch of oscillatory ΔP_z thereafter. Finally, we note that the first calculation of the neutrino asymmetry done by Enqvist *et al.* [13] shows an oscillatory asymmetry down to $T_6\sim 1$, because their parameter choice, $\Delta m^2=-10^{-5}$ eV^2 and $\sin^2 2\theta=10^{-2}$, does not satisfy Eq. (25).

affect the general behavior of λ.

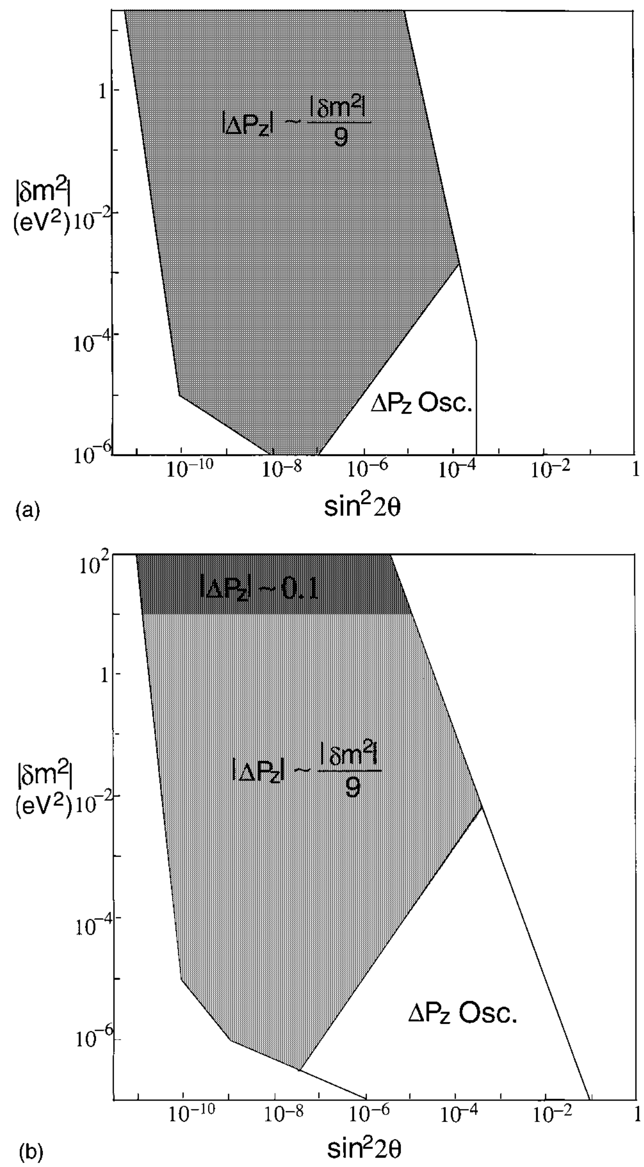

FIG. 3. This figure shows the allowed mixing parameters that amplify neutrino chemical potentials. The magnitude of ΔP_z at 1 MeV is shown. The region at the lower right noted with "ΔP_z Osc." has an oscillating ΔP_z at 1 MeV. For $|\delta m^2|\gtrsim 10$ eV^2, ΔP_z is limited to ~ 0.1. (a) ν_e-ν_s mixing (ν_e heavier than ν_s); (b) $\nu_\mu(\nu_\tau)$-ν_s mixing (ν_μ or ν_τ heavier than ν_s).

III. IMPLICATIONS

We concentrate on two implications of a neutrino asymmetry as large as Eq. (30) in BBN.

The first is on other active-sterile neutrino oscillations in BBN [7,8]. When neglecting neutrino asymmetries, large areas of parameter space of active-sterile neutrino oscillation are ruled out based on the argument that the sterile neutrino cannot be significantly populated so as to violate the primordial ^{4}He abundance observation. The forbidden areas include the large angle ν_μ-ν_s mixing with $\delta m^2\sim 10^{-2}$ eV^2 which can solve the atmospheric neutrino problem, and the large angle ν_e-ν_s mixing with $\delta m^2\sim 10^{-5}$ eV^2 which can solve the solar neutrino problem. This argument, however, no longer stands when a neutrino asymmetry as large as in Eq.

(30) is in place. For example, if ν_τ mixes with a lighter ν_s with $|\delta m^2|\geqslant 10^{-2}$ eV^2 and an angle in the shaded region of Fig. 3(b), the $\nu_\tau\overline{\nu_\tau}$ asymmetry amplified by the ν_τ-ν_s oscillation can be large enough to suppress the ν_μ-ν_s oscillation from the ν_μ-ν_s mixing solution to the atmospheric problem [8]. Similarly, if ν_τ or ν_μ mixes with a lighter ν_s with $|\delta m^2|\geqslant 10^{-5}$ eV^2 and an angle in the shaded region of Fig. 3(b), the amplified neutrino asymmetry can be large enough to suppress the ν_e-ν_s oscillation originating from the large angle ν_e-ν_s mixing solution to the solar neutrino problem. The suppressions in place even in the epoch of oscillating ΔP_z, because although an oscillating ΔP_z constantly drives the other active-sterile neutrino oscillations through resonance, the time of resonance crossing is too short to allow any significant oscillation. Thus both these two solutions to the atmospheric neutrino problem and the solar neutrino problem ruled out previously may still be viable if an active neutrino (more massive than ν_μ or ν_e respectively) mixes with a lighter sterile neutrino with parameters in the shaded region of Fig. 3(b).

The second implication is on the primordial ^{4}He abundance itself. Besides the number of neutrino species, the primordial ^{4}He abundance is also affected by a nonzero chemical potential in the $\nu_e\overline{\nu_e}$ sector at $T\sim 1$ MeV. The mechanism is that the asymmetry in $\nu_e\overline{\nu_e}$ changes the neutron/proton conversion rates, thereby changes the freeze-out time of the neutron to protron ratio as well as the ratio itself. A ξ_{ν_e} (the ν_e chemical potential divided by kT) or order 0.1 can induce an appreciable change in the prediction of the primordial ^{4}He abundance [14]. When $\xi_{\nu_e}\ll 1$, $Y\approx Y(\xi_{\nu_e}=0)-0.234\xi_{\nu_e}$, [15] and $\Delta P_z=2(n_{\nu_e}-n_{\overline{\nu_e}})\approx 3.6\xi_{\nu_e}$. So

$$Y=Y_0-0.065\Delta P_z. \tag{34}$$

The comparison with equation $Y=Y(N_\nu=3)+0.012(N_\nu-3)$ (where N_ν is the effective number of neutrino species in BBN) [16] indicates that $\Delta P_z\sim 0.1$ in the $\nu_e\overline{\nu_e}$ sector corresponds to roughly -0.55 neutrino species, and therefore has a significant impact on the predicted ^{4}He abundance.

There are two ways to generate a ξ_{ν_e} of order ± 0.1 by active-sterile neutrino oscillations. The direct way is to have a ~ 1 eV ν_e mix with a lighter ν_s [Fig. 3(a)]. If the atmospheric neutrino problem and the solar neutrino problem are to be solved by active neutrino oscillations, this implies that all three active neutrinos are almost degenerate with a mass of order 1 eV. This will be consistent with supernovae nucleosynthesis constraints [17,18] and compatible with the controversial Liquid Scintillation Neutrino Detector (LSND) result [19,20] if the claimed detection of ν_μ-ν_e oscillation solves the atmospheric neutrino problem [21]. Laboratory experiments limit the mass of ν_e to less than 5 eV [22]. The majorana mass of ν_e is further limited to less than 0.68 eV at 90% C.L. [23].

The indirect way of generating a significant ξ_{ν_e} is to have ν_τ (or ν_μ) mix with a lighter ν_s with 10^2 eV^2 $\gtrsim|\delta m^2|\gtrsim 1$ eV^2 and a desired angle, and transfer the asymmetry in the $\nu_\tau\overline{\nu_\tau}$ ($\nu_\mu\overline{\nu_\mu}$) sector into $\nu_e\overline{\nu_e}$ by a ν_e-ν_τ (ν_μ) mixing. But to yield an asymmetry of order 0.1 as well in $\nu_e\overline{\nu_e}$, the transfer has to be efficient. Take the ν_e-ν_τ oscillation as an example; the mixing has to satisfy

$$D'\left(\frac{V_x'}{V_z'}\right)^2\gtrsim H, \tag{35}$$

where D', V_x', and V_z' denote the counterparts of D, V_x, and V_z in the ν_τ-ν_s oscillation. Approximately

$$D'\sim 5G_F^2T^5\approx T_6^3H,$$

$$V_x'\approx\frac{\delta M^2}{6.3T}\sin 2\theta',$$

$$V_z'\approx-\frac{\delta m^2}{6.3T}\cos 2\theta'-180G_F^2T^5+0.13G_FT^3(n_{\nu_e}-n_{\overline{\nu_e}})$$
$$-0.13G_FT^3(n_{\nu_\tau}-n_{\overline{\nu_\tau}}), \tag{36}$$

where δM^2 and θ' are the vacuum mixing parameters of the ν_e-ν_τ mixing. An efficient transfer of asymmetry means that $(n_{\nu_e}-n_{\overline{\nu_e}})\sim(n_{\nu_\tau}-n_{\overline{\nu_\tau}})\sim|\delta m^2|/18T_6^4$. If the mass of ν_e is much lighter than ν_τ, $|\delta M^2|\approx|\delta m^2|$. Then $(D/H)(V_x'/V_z')^2\sim T_6^3\sin^2 2\theta'$ at temperatures approaching 1 MeV. So if ν_τ has a cosmologically interesting mass, $\Delta m^2\sim 20$–100 eV^2, $|\Delta P_z|\approx 0.1$ in $\nu_\tau\overline{\nu_\tau}$ can be reached at $T\approx 2$–3 MeV by the ν_τ-ν_s mixing according to Eq. (30), which can efficiently transfer into an asymmetry of similar order in $\nu_e\overline{\nu_e}$ if ν_τ mixes with ν_e with $\sin^2 2\theta'\sim 0.1$ and $|\delta M^2|\sim|\delta m^2|$. This required mixing between ν_e and ν_τ lies near the edge of current lab limits on the ν_e-ν_x mixing [24], and may be testable in the near future. Based on supernovae nucleosynthesis arguments, however, the required ν_e-ν_τ mixing is ruled out (although supernova models are uncertain to some extent) [17,18]. The above analysis applies similarly to the ν_μ-ν_s and ν_μ-ν_e mixings, but the required mixing between ν_e and ν_μ to transfer asymmetries efficiently has already been ruled out by laboratory experiments [22].

Of course, the indirect way of transferring asymmetries from $\nu_\tau\overline{\nu_\tau}$ (or $\nu_\mu\overline{\nu_\mu}$) into $\nu_e\overline{\nu_e}$ works if they have almost degenerate masses, ~ 1 eV. In this case the required ν_e-ν_μ (ν_τ) mixing will not be ruled out by laboratory experiments or astrophysical considerations.

Recently, Hata *et al.* [25] suggested a possible ''crisis'' in BBN because the prediction ^{4}He abundance from the standard BBN, coupled with predictions of ^{3}He and D from generic chemical evolution models, is too high to be consistent with the observed abundance, unless the number of effective neutrino species N_ν at the time of BBN is less than 2.6 (however, see [26]). $N<3$ is at odds with popular beliefs and most theoretical models which assume all ν_e, ν_μ, and ν_τ to be lighter than ~ 1 MeV and therefore $N_\nu\geqslant 3$. But as seen above, if an asymmetry of order $\Delta P_z\sim 0.1$ arises in the $\nu_e\overline{\nu_e}$ sector from active-sterile neutrino oscillations, this potential ''crisis'' can be solved. Of course, the mixing parameters of the active-sterile oscillations have to be right to yield a positive ΔP_z instead of a negative one. An interesting note is that the neutrino mass required to solve this ''crisis'' is ~ 1 eV, which qualifies neutrinos as dark matter candidates. A ν_e mass of ~ 1 eV is within a factor of 5 below the current limit on ν_e mass, and right on the edge of detection limit if ν_e are majorana neutrinos. If not introducing more sterile neutrinos, the solar neutrino problem and the atmospheric neutrinos have to be solved by mixings among the three active neutrinos, therefore requiring their masses to be almost degenerate. A model of three neutrinos almost degenerate in mass is favorable in forming structures in the universe [27], but may not be the most natural theoretical model so far.

IV. SUMMARY

In summary, we have calculated the parameter space of active-sterile neutrino mixings that amplifies neutrino chemical potentials, and the size of the amplification. Results are summarized in Fig. 3. By exploring the sensitivity of the amplification to the initial condition, the mixing parameters and the calculational parameters, and by analyzing the Lyapunov exponent of the system, we showed evidences that the amplification process is chaotic. We have also discussed the implications of our results on BBN. It was shown that a ν_e chemical potential of order $0.1kT$ could be achieved by either a mixing between ~ 1 eV ν_e and a lighter sterile neutrino, or a mixing between a ~ 1 eV ν_μ (or ν_τ) and a lighter sterile neutrino coupled with a mixing between almost degenerate ν_e and ν_μ (or ν_τ). Such a chemical potential in ν_e can lower the effective number of neutrino species to significantly below 3 at the epoch of BBN.

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