Inner structure of an evaporating charged black hole with ingoing charged null fluid

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We consider a charged black hole irradiated by a charged null fluid with negative energy density. As a consequence of this influx, the black hole's mass and charge shrink to zero after some finite retarded time. This model was previously proposed as a toy model for an evaporating charged black hole. Here we modify the model by changing one of the assumptions made concerning the orbits of the null fluid. This leads to a very different internal structure of the black hole. [S0556-2821(96)03116-5]

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I. INTRODUCTION

It is widely believed today that asymptotically flat black holes eventually evaporate due to the emission of Hawking radiation [1]. In spite of much effort, however, little is known about the back reaction effects of the quantum outflux. Of special interest is the following question: In what way does the evaporation process affect the inner structure of the black hole?

As is usual in the investigation of black holes' interiors, we shall consider here a charged black hole, and regard it as a toy model for a realistic, spinning black hole. The inner structure of a charged black hole is very similar to that of Kerr, which suggests that it could serve as a useful toy model.

It has been suggested by Kaminaga [2] that the evaporation process (or, more particularly, the internal aspects of this process) may be modeled by a flux of null fluid that flows into the black hole. (A null fluid can be thought of as a stream of massless particles that flows along null orbits.) This null fluid carries negative-energy density, so it causes the event horizon's area (and the black hole's mass) to shrink to zero. (The negativity of the energy influx follows from simple energy-conservation considerations, as the Hawking radiation carries away positive energy to the external universe.) Also, in order to model the complete evaporation of the charged black hole, we must assume that the null fluid is charged too (otherwise the black hole will not be able to get rid of its electric charge) [3]. Motivated by these considerations, Kaminaga [2] considered the following simple model: Initially, we have a Reissner-Nordström black hole. Then, at some particular retarded moment, a spherically symmetric flux of charged null fluid starts flowing into the black hole along ingoing radial null orbits. Consequently, the black hole shrinks (''evaporates''), until the mass and charge vanish at some finite retarded time. This model is described by an exact solution of the Maxwell-Einstein equations—the so-called charged Vaidya solution [4]. Kaminaga then used this model as a background geometry, in an attempt to study the quantum-field effects (Hawking radiation) on a dynamical (classical) background corresponding to an evaporating black hole.

The inner structure of the evaporating black hole in the above model differs from that of the nonevaporating charged black hole (the Reissner-Nordström geometry) in a very important respect: In the Reissner-Nordström black hole, the singularities are timelike, and there is a ''tunnel'' leading to other asymptotically flat universes. In the above model of an evaporating charged black hole, the ''tunnel'' is sealed by a null singularity (see Fig. 1).[1]

One of the basic assumptions in Kaminaga's model is that the ingoing charged particles go straight on radial ingoing null geodesics, all the way up to the central singularity at $r=0$. This assumption, however, is not so obvious: The self-consistent equation of motion for a charged null fluid (on a background with a nonvanishing electric field) must include a Lorentz-force term [5]. When this term is taken into account, the solutions describing radial orbits will typically have a branching point at some critical radius r_c. Beyond this critical point, there exist two possible continuations of the orbit (both are consistent with the equation of motion): (A) the simple continuation, according to which the massless particles continue along the same *ingoing* radial null direction, and (B) the nontrivial continuation, according to which the particles switch to the *outgoing* radial null direction. From a naive point of view, possibility A looks like the more natural one. In Ref. [5], however, a stability analysis is carried out, with respect to small deviations from the strict radial direction of motion (and also with respect to the presence of an arbitrarily small nonvanishing rest mass). In the perturbed orbits, no branching points exist, and it is therefore possible to study the limit of vanishing perturbations without any ambiguity. This analysis gives a very definitive result: In the limit in which the small perturbations are taken to vanish, one obtains continuation B uniquely. This analysis, therefore, signifies continuation B as the more physical one.

In this paper, we modify Kaminaga's model by considering continuation B (instead of A). That is, we assume that at the critical radius, the particles switch to outgoing null orbits. Since the null fluid is self-gravitating, this change in the flow pattern results in a modification of the spacetime geometry. As we shall show below, this leads to a very different causal structure. In particular, there is no null singularity (the only singularity is timelike), and the Reissner-Nordström tunnel, leading to other asymptotically flat universes, remains open.

[1]In Ref. [2] Kaminaga introduced a cutoff just before the null curvature singularity. In our discussion here we refer to the causal structure corresponding to the full geometry, without the cutoff.

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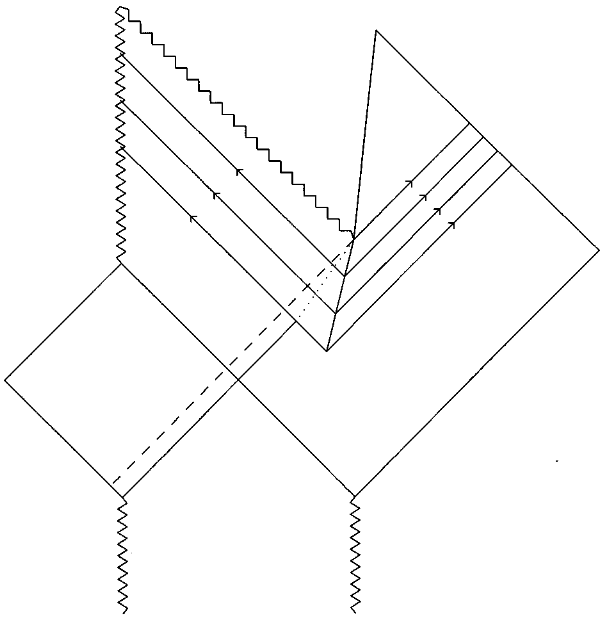

FIG. 1. The global structure of the spacetime corresponding to extension A. The broken line is the event horizon, and the dotted line is the apparent horizon. The outgoing lines with arrows outside the black hole represent the Hawking radiation emitted to the external universe. The ingoing lines with arrows represent the orbits of the ingoing null fluid.

We emphasize that no attempt is made here to analyze the true self-consistent geometry of a realistic black hole that evaporates due to the Hawking effect. Rather, our goal here is to explore the causal structure corresponding to some particular toy model, the Kaminaga's model, if one replaces continuation A by continuation B (which we regard as the more physical one). The investigation of the true inner structure of a realistic evaporating black hole will probably await the formulation of the self-consistent theory of quantum gravity.

In Sec. II we construct the metric of the spacetime corresponding to continuation B. Then, in Sec. III, we analyze its causal structure.

II. THE LINE ELEMENT

The spacetime of a charged black hole is described by the Reissner-Nordström geometry. In Schwarzschild coordinates, it takes the form

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{1}$$

where

$$F(r) \equiv 1 - \frac{2M_0}{r} + \frac{Q_0^2}{r^2}, \tag{2}$$

and $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$. Here, M_0 and Q_0 are the mass and the charge of the black hole, respectively. Using an ingoing Eddington coordinate, the line element becomes

$$ds^2 = -F(r)dv^2 + 2dv\,dr + r^2 d\Omega^2. \tag{3}$$

Assume now that at a certain retarded moment $v = v_0$ a charged null fluid of negative-energy-density starts falling into the black hole (corresponding to the beginning of the evaporation process). This situation is described by the charged Vaidya solution [4], a generalization of the line element (3), in which the mass and charged are allowed to depend on v:

$$ds^2 = 2dv\,dr - \left(1 - \frac{2M(v)}{r} + \frac{Q^2(v)}{r^2}\right)dv^2 + r^2 d\Omega^2. \tag{4}$$

Let $a \equiv M_0/Q_0$, and assume $a > 1$ (otherwise there would be no black hole). Following Ref. [2], we take M and Q to depend linearly on v, and we also assume that the mass to charge ratio remains fixed during the evaporation process:

$$M(v) = M_0 - \mu(v - v_0),$$

$$Q(v) = Q_0 - \mu(v - v_0)/a = M(v)/a,$$

where μ is a constant parameter. This simplifies the analysis, as it makes the geometry self-similar [6,7]. The parameter μ represents the evaporation rate, and we assume $0 < \mu \ll 1$. (In realistic evaporation scenarios, this is indeed the case as long as the black hole's mass is very large compared to the Planck mass.) As for the parameter a, in addition to the demand $a > 1$, we assume that a is of order unity, but not too close to 1.

For convenience, we shall transform v to a new null coordinate:

$$v' \equiv v - v_0 - M_0/\mu.$$

In this new coordinate we have $M = \mu' v'$ and $Q = \mu' v'/a$, where $\mu' \equiv -\mu$. The line element is now given explicitly by

$$ds^2 = 2dr\,dv' - \left(1 - \frac{2\mu' v'}{r} + \frac{\mu'^2 v'^2}{a^2 r^2}\right)dv'^2 + r^2 d\Omega^2. \tag{5}$$

In the charged Vaidya solution the local energy density of the null fluid is proportional to $\rho \equiv (4\pi r^2)^{-1}(\dot{M} - Q\dot{Q}/r)$, where a dot denotes $\partial/\partial v'$ [8,5]. At each v', there is a critical r value, $r_c = Q\dot{Q}/\dot{M}$, where the local energy density vanishes. In our case,

$$r_c(v') = M(v')/a^2 = \mu' v'/a^2.$$

This line is located well inside the black hole. The massless particles are presumably falling into the black hole, at $r > r_c$, along ingoing radial null orbits. However, for radial orbits the equation of motion has a branching point at $r = r_c$, and there are two possible continuations of the orbits beyond that point: continuation A, the orbits continue along the same ingoing null direction; continuation B, the orbits switch to the *outgoing* null direction. As was discussed in the previous section, we regard continuation B as the more physical one. (The reasons are explained in detail in Ref. [5].)

Since the null fluid is self-gravitating, the two different continuations of the null fluid's worldlines naturally lead to two different extensions of the geometry beyond $r = r_c$, to

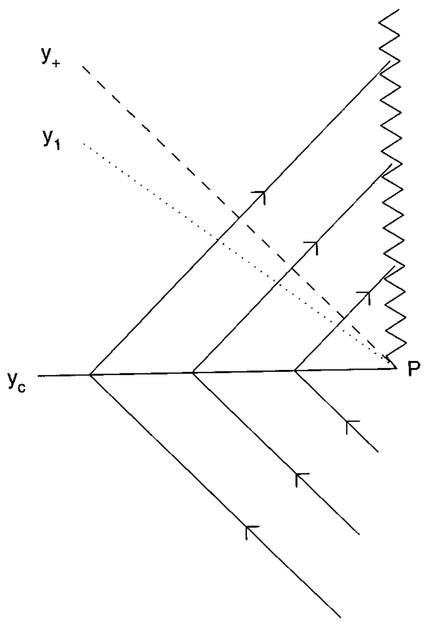

FIG. 2. This figure shows the transition from the ingoing charged Vaidya solution to the outgoing charged Vaidya solution, and the causal structure of the latter. The null fluid's orbits in both regions are denoted by arrows. The matching hypersurface Σ is the horizontal line denoted y_c. The point P corresponds to $r=0$, $v'=0$, and $u=0$ (and also $w=0$ and $W=0$). The vertical singular line denotes the curvature singularity at $r=0$, $u\geqslant 0$. The dotted line is the spacelike hypersurface $y=y_1$, the "inner apparent horizon." The broken line is the null hypersurface $y=y_+$, the Cauchy horizon. The part of the diagram corresponding to the future of y_c can also be viewed as built of two patches of $u-w$ coordinates in the two side of y_+. These two patches are matched at $w=0$ $(y=y_+)$.

which we shall refer as extensions A and B, respectively. The causal structure corresponding to extension A was investigate by Kaminaga [2]. The corresponding Penrose diagram is shown in Fig. 1. In this diagram, a null $r=0$ singularity forms at $v'=0$. This null singularity then intersects the timelike $r=0$ singularity and completely seals the "tunnel."

In what follows we investigate the causal structure corresponding to extension B. We denote the hypersurface $r=r_c$ by Σ. From the line element (5) it follows that Σ is spacelike as long as $\mu<(a^2-1)a^2/2$. We shall assume that this inequality indeed holds (it is consistent with our assumptions about μ and a), and hence Σ is spacelike. Since at this hypersurface the null fluid switches from ingoing to outgoing null orbits, to the future of Σ there will be a piece of *outgoing* charged Vaidya solution (see Fig. 2). The line element has the same form as that of the ingoing charged Vaidya solution, except that now v' is replaced by u (u is an outgoing null coordinate):

$$ds^2=2drdu-F(u,r)du^2+r^2d\Omega^2, \tag{6}$$

where

$$F(u,r)=1-\frac{2M(u)}{r}+\frac{Q^2(u)}{r^2}. \tag{7}$$

The matching of the ingoing charged Vaidya solution to the piece of outgoing charged Vaidya solution through Σ was analyzed in Ref. [5]. Assuming a C^1 matching, one finds that r, M, and Q are all continuous at Σ. That is,

$$M(v')=M(u(v')), \quad Q(v')=Q(u(v')), \tag{8}$$

and the matching is uniquely determined by the function $u(v')$ that relates u to v' at Σ. This function is obtained by comparing the proper distance along $r=r_c$, as measured in both sides of Σ. One finds that[2]

$$\begin{aligned}\frac{du(v')}{dv'}&=\frac{1}{F(v',r)}\left(\frac{2dr_c}{dv'}-F(v',r)\right)\\&=\frac{1}{1-a^2}\left(a^2-1-\frac{2\mu}{a^2}\right)<0.\end{aligned} \tag{9}$$

Therefore, we can take $u(v')$ to be

$$u=\left[\frac{2\mu-a^2(a^2-1)}{a^2(a^2-1)}\right]v'. \tag{10}$$

This specific choice of the constant for the integration of Eq. (9) is convenient, because it assigns the value $u=0$ to the special point $(v'=0,r=0)$ (the point P in Fig. 2). Note that in the region of interest u is positive, because both v' and the term in brackets are negative. From Eq. (8) we now obtain

$$M(u,r)=\overline{\mu}u, \tag{11}$$

where

$$\overline{\mu}=\frac{a^2(a^2-1)}{a^2(a^2-1)-2\mu}\mu>0. \tag{12}$$

Note that in the case $\mu\ll 1$, which we assume, we also find

$$\overline{\mu}\cong\mu\ll 1. \tag{13}$$

III. THE CAUSAL STRUCTURE

Equations (6), (7), (11), and (12) completely satisfy the metric at the future of Σ (i.e., $r<r_c$). In what follows we shall analyze the causal structure of this region. To that end, we shall use methods similar to those developed in Refs. [6,7].

For outgoing ($u=$const) radial null geodesics in the geometry (6), the geodesic equation reads

$$\frac{d^2r}{d\lambda^2}=0,$$

where λ is the affine parameter (which we choose so as to increase in the future direction). Therefore, r is a linear func-

[2]The demand for a continuous matching at Σ yields a quadratic equation, one of whose roots corresponds to Eq. (9). The other root leads to extension A, and will not concern us here.

tion of λ. In particular, r must decrease monotonously (in the future direction) along the lines $u=\text{const}$.[3]

We introduce a new variable $y\equiv u/r$. Note that $y>0$, because both u and r are positive. From the above discussion it is obvious that y increases monotonously along the lines $u=\text{const}$. Therefore, the piece of spacetime which concerns us here (the one covered by the outgoing charged Vaidya solution) is the region $y>y_c$, where

$$y_c \equiv u/r_c = a^2/\bar{\mu} > 0.$$

The function F can be written as

$$F(y) = \frac{\bar{\mu}^2 y^2}{a^2} - 2\bar{\mu} y + 1, \tag{14}$$

and the roots of $F=0$ are

$$y_1 = \frac{a^2}{\bar{\mu}} + \frac{a\sqrt{a^2-1}}{\bar{\mu}}, \quad y_2 = \frac{a^2}{\bar{\mu}} - \frac{a\sqrt{a^2-1}}{\bar{\mu}}.$$

Since $y_2<y_c<y_1$, the line $y=y_1$ exists in our spacetime, but not the line $y=y_2$. The hypersurface $y=y_1$ is spacelike. This follows from the line element (6), as $F=0$ and $du/dr=y_1>0$. This hypersurface is the ''inner apparent horizon.''

The orbits of ingoing ($v=\text{const}$) radial null geodesics in the geometry (6) are described by the equation

$$\frac{dr}{du} = \tfrac{1}{2} F(u/r). \tag{15}$$

Of special importance are solutions of Eq. (15) of the form $u=yr$, with constant y. For this special kind of solution, Eq. (15) reduces to the algebraic equation

$$F(y) - 2/y = 0. \tag{16}$$

This yields the following cubic equation for y:

$$\frac{\bar{\mu}^2 y^3}{a^2} - 2\bar{\mu} y^2 + y - 2 = 0. \tag{17}$$

It is easy to see that, for sufficiently small $\bar{\mu}$, this cubic equation has three different real roots; and from Eq. (13) we see that this is indeed the case. To the leading order in the small parameter $\bar{\mu} \ll 1$, the three roots are

$$y_0 \cong 2, \quad y_\pm \cong \frac{a^2}{\bar{\mu}} \pm \frac{a\sqrt{a^2-1}}{\bar{\mu}}. \tag{18}$$

Since $0<\bar{\mu}\ll 1$, we have $y_c = a^2/\bar{\mu} \gg 1$. It is then obvious that y_0 and y_- are smaller than y_c, and therefore, the lines $y=y_0$ and $y=y_-$ do not exist in the relevant piece of spacetime ($y>y_c$). On the other hand, the hypersurface $y=y_+$ does exist in our spacetime, because $y_+>y_c$. This line is a null hypersurface.

As was mentioned above, along the lines $u=\text{const}>0$, r is decreasing (and, correspondingly, y is increasing) monotonously. Therefore, all these lines will inevitably terminate, in the future, at a curvature singularity at $r=0$, after finite affine time. This singularity, which corresponds to $y\to+\infty$, is located to the future of the null hypersurface $y=y_+$. In order to uncover the causal structure, we shall construct an *ingoing* null coordinate w, and transform the line element (6) to double-null coordinates (u,w):

$$ds^2 = g_{uw}\,du\,dw + r^2 d\Omega^2. \tag{19}$$

We shall further demand that the function $w(u,r)$ will be of the form

$$w(u,r) = u\,h(u/r), \tag{20}$$

where h is a function of $y=u/r$ only. To construct w, we start from the requirement that w will be null, i.e., $g^{ww}=0$:

$$0 = g^{ww} = g^{rr} w_{,r}^2 + 2g^{ur} w_{,u} w_{,r},$$

which implies

$$F(y) w_{,r} + 2 w_{,u} = 0 \tag{21}$$

(note that in the original coordinates u and r, the inverse metric functions are $g^{rr}=F$, $g^{ur}=1$, $g^{uu}=0$). Substituting now Eq. (20) in Eq. (21), one obtains the following differential equation for h:

$$\frac{dh}{dy} = K(y) h, \tag{22}$$

where

$$K(y) \equiv \frac{2}{y^2}[F(y) - 2/y]^{-1}. \tag{23}$$

For any solution $h(y)$ of this equation, $w(u,r)$ defined by Eq. (20) will satisfy the requirement of being null (i.e., $g^{ww}=0$). Since Eq. (22) is linear and homogeneous, we can immediately write down its general solution:

$$h(y) = h_0 \exp[G(y)], \tag{24}$$

where

$$G(y) \equiv \int_{y_i}^{y} K(y')\,dy', \tag{25}$$

h_0 and y_i being arbitrary constants.[4]

It is not difficult to calculate the integral (25) and to get an explicit expression for $h(y)$, see the Appendix. For our purposes, however, it is sufficient to look at the behavior of

[3]That this function must be decreasing (rather than increasing), follows from the fact that r decreases along geodesics $v=\text{const}$ that approach Σ from the past. [Recall also that (i) the matching at Σ is C^1, and (ii) in the neighborhood of the spacelike hypersurface Σ, ∇r is timelike.]

[4]Although Eq. (22) only has a one-parameter family of solutions, we prefer here to keep both parameters, h_0 and y_i, for later convenience.

$K(y)$, and to deduce the features of h from it. From Eq. (23) it is obvious that the only possible singularities of K are the three roots of Eq. (16), i.e., y_+, y_-, and $y=y_0$. However, in the region of interest ($y>y_c$) we have only one of these roots, namely y_+. The divergence of K at $y=y_+$ forces us to treat the two regions $y>y_+$ and $y<y_+$ separately. In particular, we shall have to analyze the behavior of h at $y=y_+$ (and to check that it does not diverge there).

We shall now analyze the causal structure of the region $y>y_+$, using the coordinates (u,w). To that end, we shall choose $y_i>y_+$ and $h_0=1$. The regularity of $K(y)$ at $y>y_+$ implies the regularity of G and h at $y_+<y<\infty$. Also, since K has no roots in $y_+<y<\infty$, it is positive definite throughout this range. [The sign of K in that range can be easily obtained from its asymptotic behavior at $y\to+\infty$, see Eq. (29) below.] Now, a straightforward calculation yields

$$g^{uw}=g^{ur}w_{,r}=-y^2dh/dy=-y^2Kh. \tag{26}$$

Thus, g^{uw} is smooth and nonvanishing at $y_+<y<\infty$, and consequently, the line element (19) is well defined and regular throughout that range.

We shall also need the asymptotic behavior of h at both limits, $y\to y_+$ and $y\to+\infty$. To that end, it is sufficient to consider the behavior of K at both limits.

(i) The limit $y\to y_+$. It is straightforward to show that the asymptotic behavior of K at this limit is

$$K=K_0(y-y_+)^{-1}+O[(y-y_+)^0] \quad (y\to y_+), \tag{27}$$

where K_0 is a positive constant,

$$K_0=\frac{2a^2}{\overline{\mu}^2y_+(y_+-y_0)(y_+-y_-)}.$$

(In the limit $\mu\ll1$ one finds

$$K_0\cong[a\sqrt{a^2-1}(a+\sqrt{a^2-1})^2]^{-1}\mu,$$

so $0<K_0\ll1$.) Correspondingly, $G\to-\infty$, and h vanishes:

$$h\propto(y-y_+)^{K_0}. \tag{28}$$

Thus, the line $y=y_+$ corresponds to $w=0$ (with $w>0$ at $y>y_+$).

(ii) The limit $y\to\infty$. Here the asymptotic behavior is

$$K=\frac{2a^2}{\overline{\mu}^2}y^{-4}+O(y^{-5}) \quad (y\to+\infty), \tag{29}$$

so G is finite, and h approaches a finite limit $h_\infty>0$. We find that in (u,w) coordinates the singularity at $r=0$ $(y=+\infty)$ is located at the line $w=h_\infty u$. This line is *timelike*, because $h_\infty>0$ (and $g_{uw}<0$).

In a similar way, we can use the coordinates (u,w) to analyze the region $y_c<y<y_+$. Now we take the constant y_i to lie inside the range $[y_c,y_+]$, and $h_0=-1$. Again, the important point is that K has no singularities and no roots inside the relevant range (recall that both y_0 and y_- are smaller than y_c). Therefore the line element (19) is regular throughout this range. Also, one finds that h vanishes at y_+

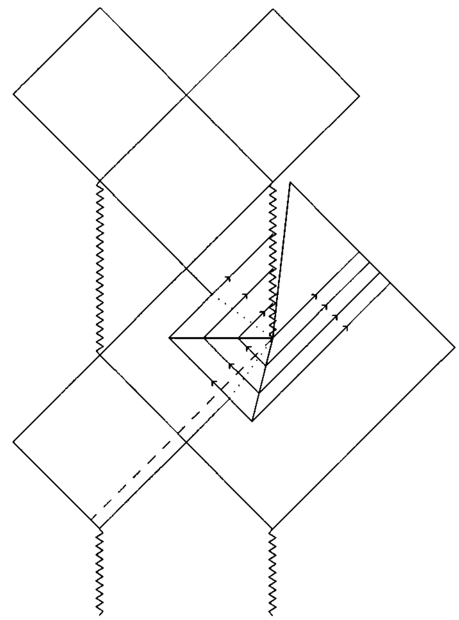

FIG. 3. The global structure of the spacetime corresponding to extension B. As in Fig. 1: The broken line is the event horizon. The dotted lines are the outer and inner apparent horizons. The outgoing lines with arrows outside the black hole represent the Hawking radiation emitted to the external universe. The ingoing lines with arrows that change to outgoing lines at $y=y_c$, represent the orbits of the null fluid.

and is negative at $y<y_+$. This means that the null line $y=y_+$ corresponds to $w=0$ (with $w<0$ at $y<y_+$).

The two patches, $y>y_+$ and $y<y_+$, match together at the hypersurface $w=0$. At this hypersurface, however, the metric (19) becomes singular, as

$$g_{uw}\propto|y-y_+|^{1-K_0}$$

[cf. Eqs. (26)–(28)], so $\det(g)=0$. This is obviously a coordinate singularity, because in the original coordinates (u,r) the metric is manifestly regular at $y=y_+$. This singularity may be removed by transforming w into a new null coordinate $W(w)$, defined by

$$W=c_\pm|w|^\alpha,$$

where $\alpha=1/K_0>0$, and $c_\pm$ stands for two constants: $c_+>0$ for $y>y_+$ and $c_-<0$ for $y<y_+$. The line element in the new double-null coordinates (u,W) can be shown to be regular in the entire range covered by the outgoing charged Vaidya solution, i.e., $y_c<y<\infty$. Figure 2 displays the causal structure of this range.[5]

The overall causal structure of the spacetime corresponding to extension B is displayed in Fig. 3. The differences

[5]For simplicity, in Figs. 2 and 3 the $r=0$ singularity and the other lines $y=$const are displayed as straight lines. These lines are indeed straight as long as the coordinates (u,w) are used. After we transform to the regular double-null coordinates (u,W), however, these lines are no longer straight.

between the two extensions is remarkable: In extension A, there is a null $r=0$ singularity which intersects the ''left-hand'' timelike $r=0$ Reissner-Nordström singularity, and blocks the tunnel (see Fig. 1). In extension B, the singularity is timelike, and the ''Reissner-Nordstrom tunnel'' remains open and traversable (Fig. 3).

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APPENDIX

In this appendix we shall give an explicit expression for $h(y)$ and analyze its asymptotic behavior at the limits $y\to y_+$ and $y\to+\infty$.

The function $K(y)$ is just $2/y$ divided by the left side of Eq. (17). Therefore, we can rewrite $K(y)$ as

$$K(y)=\frac{2}{F(y)y^2-2y}=\frac{2a^2}{\overline{\mu}^2 y(y-y_+)(y-y_0)(y-y_-)}=\frac{A}{y-y_+}+\frac{B}{y-y_0}+\frac{C}{y-y_-}+\frac{D}{y}, \qquad \text{(A1)}$$

where

$$A=2a^2\overline{\mu}^{-2}y_+^{-1}(y_+-y_0)^{-1}(y_+-y_-)^{-1},$$

$$B=2a^2\overline{\mu}^{-2}y_0^{-1}(y_0-y_+)^{-1}(y_0-y_-)^{-1},$$

$$C=2a^2\overline{\mu}^{-2}y_-^{-1}(y_--y_0)^{-1}(y_--y_+)^{-1},$$

$$D=-2a^2\overline{\mu}^{-2}(y_+y_0y_-)^{-1}.$$

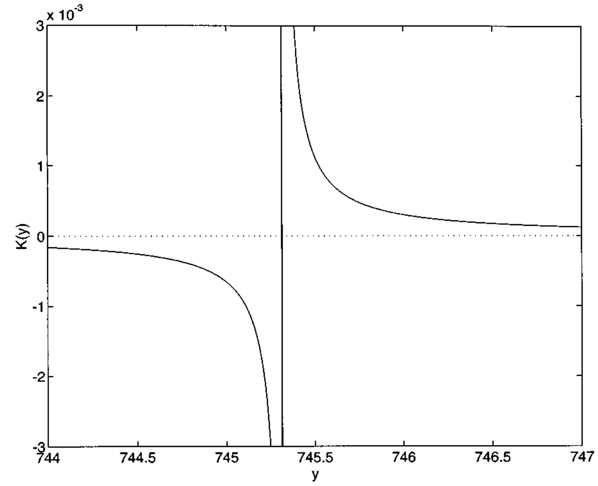

FIG. 4. The function $K(y)$ near $y=y_+$. The values of $K(y)$ were calculated for the parameters $a=2$ and $\mu=0.01$, which yield $\overline{\mu}=0.010\,016\,6$.

These constants satisfy the equations

$$D(-y_+y_0y_-)=2a^2\overline{\mu}^{-2},$$

$$A+B+C+D=0,$$

$$(y_++y_0+y_-)D+A(y_0+y_-)+B(y_++y_-)+C(y_0+y_+)=0,$$

$$(y_+y_0+y_0y_-+y_+y_-)D+y_0y_-A+y_+y_-B+y_+y_0C=0.$$

The function $K(y)$ is drawn in Fig. 4.

$K(y)$ diverges at $y=y+$. Therefore, we must treat the regions $y>y_+$ and $y<y_+$ separately. Then the value of h at $y\geqslant y_+$ is given by Eqs. (24) and (25):

$$h(y)=h_0\exp\{\ln[y^D(y-y_+)^A(y-y_0)^B(y-y_-)^C]|_{y_i}^{y}\}=h_0\exp\left\{\ln\left[y^{A+B+C+D}\left(1-\frac{y_+}{y}\right)^A\left(1-\frac{y_0}{y}\right)^B\left(1-\frac{y_-}{y}\right)^C\right]_{y_i}^{y}\right\}. \qquad \text{(A2)}$$

Using the equality $A+B+C+D=0$ mentioned above, we get a finite value for $h_\infty\equiv h(y=\infty)$:

$$h_\infty=h_0y_i^{-D}(y_i-y_+)^{-A}(y_i-y_0)^{-B}(y_i-y_-)^{-C}. \qquad \text{(A3)}$$

Special choosing of h_0 will give $h(\infty)=1$ which will correspond to Fig. 2. It is very clear from Eq. (A2) that the value of h at $y=y_+$ is zero no matter which h_0 we choose, because $A>0$.

We move our attention now to the region $y_+\geqslant y\geqslant y_c$. By integrating $K(y)$ from $y=y_i$ $(y_+>y_i>y_c)$ to $y=y_+$ we find that $G(y_+)$ is $-\infty$ (no matter which integration constant we choose, see Fig. 4), therefore h is zero at $y=y_+$, the same result that we got from the other region. We found that h is continued at $y=y_+$ and has a finite value h_∞ at the singularity, which is given by Eq. (A3).

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