

One-loop quantum corrections to the thermodynamics of charged black holes

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Quantum corrections are studied for a charged black hole in a two-dimensional (2D) model obtained by spherisymmetric reduction of the 4D Einstein-Maxwell theory. The classical (tree-level) thermodynamics is reformulated in the framework of the off-shell approach, considering systems at arbitrary temperature. This implies a conical singularity at the horizon and modifies the gravitational action by terms defined on the horizon. A consistent variational procedure for the action functional is formulated. It is shown that the free energy reaches an extremum on the regular manifold with $T = T_H$. The one-loop contribution to the action in the Liouville-Polyakov form is reexamined. All the boundary terms are taken into account and the dependence on the state of the quantum field is established. The modification of the Liouville-Polyakov term for a 2D space with a conical defect is derived. The back reaction of the Hawking radiation on the geometry is studied and the quantum-corrected black hole metric is calculated perturbatively. Within the off-shell approach the one-loop thermodynamical quantities, energy, and entropy, are found. They are shown to contain a part due to hot gas surrounding the black hole and a part due to the hole itself. It is noted that the contribution of the hot gas can be eliminated by appropriate choice of the (generally, nonflat) reference geometry. The deviation of the ‘‘entropy - horizon area’’ relation for the quantum-corrected black hole from the classical law is discovered and possible physical consequences are discussed.

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I. INTRODUCTION

That black holes possess some properties of a thermodynamical system characterized by appropriately defined energy, temperature, and entropy was first considered as just an analogy [1] between black hole physics and the laws of thermodynamics. However, the remarkable discovery by Hawking of radiation from a black hole which looks thermal at large distances [2] strongly supported this analogy and forced physicists to think of a black hole as a real thermodynamical object like a heated blackbody. One of the remarkable predictions of the analogy is that one can associate entropy with a hole which in the Einstein theory of gravity is proportional to the area of the horizon. Moreover, in processes involving a hole its entropy plays a role on an equal footing with the entropy of conventional matter. In particular, only their sum is the quantity which is nondecreasing [1,3]. However, it is a mysterious and intriguing puzzle just what states of the hole are counted by the Bekenstein-Hawking entropy. As a possible answer one can relate it to states of quantum fields which are hidden by the horizon and, consequently, remain invisible to an outside observer. The present status of the problem and numerous attempts towards

its resolution have been recently reviewed in [4,5].

The role of quantum effects in black hole physics is two-fold. Semiclassically, a hole can be considered as surrounded by the quantum Hawking radiation which becomes thermal (heat bath) far away from the hole. Since this radiation possesses a nontrivial stress-energy tensor its back reaction leads to deformation of the classical black hole geometry. On the other hand, the quantum corrections lead to modifications of the gravitational effective action. This results in changes to the formulas for calculating the energy and entropy of the hole. As an example of such a modification it was recently observed, in two [6,7] and in four [8] dimensions, that the classical Bekenstein-Hawking expression might be corrected by terms logarithmically dependent on the mass of a black hole. The calculations apply the conformal anomaly argument and take a fixed classical black hole background. However, the quantum deformation of the geometry affects the black hole parameters, like the radius of horizon, introducing some corrections. These also turn out to be of the order $\sim \ln M$ and cannot be neglected. Hence, the back reaction effects necessarily must be included when considering the quantum thermodynamics of the black hole [9,10].

Two-dimensional physics gives us an arena (see Refs.

[11–13]) where the above-noted problems can find a precise solution. The two-dimensional (2D) nonlocal Liouville-Polyakov action [14] incorporates both the Hawking radiation [15] and its back reaction on the geometry (see, e.g., Ref. [16]). Therefore, its inclusion in the gravitational action on an equal footing with the classical counterpart gives the complete semiclassical description for the black hole. It is known but not always stressed that the Liouville-Polyakov action contains some ambiguity which is eliminated by specification of the state of the quantum field. In the case of a black hole in equilibrium with thermal radiation this specification must include the heat bath at large distances from the black hole. As a result, the effective action becomes dependent on the thermal state of the quantum field. In principle, this state can be characterized by a temperature different from the Hawking one. It is a remarkable and long-standing fact that such a state can effectively be described as a quantum field on a singular instanton (i.e., on the Euclidean black hole instanton with conical singularity on the horizon). This probably explains why the Euclidean conical singularity method [17,18,8,7] gives a sensible formulation of black hole thermodynamics. In this method one takes the Gibbons-Hawking [20] Euclidean approach and closes Euclidean time with arbitrary period β , related to the temperature T of the system as $\beta=1/T$. Evaluating the free energy F of the system for arbitrary β , differentiating $F(\beta)$ with respect to β and finally putting β equal to the Hawking value $\beta=\beta_H$, one obtains the thermodynamical quantities (energy and entropy) of the black hole [7,8,10].

Recently, there was great interest in the study of divergent quantum corrections to the entropy [19]. It was realized that these divergences can be absorbed in the renormalization of gravitational couplings in the tree-level gravitational action. In two dimensions this results in a constant (not dependent on the space-time geometry) addition to the entropy originated from the 2D Einstein term in the quantum effective action (see [7]). This term is topological invariant in two dimensions, does not affect the field equations, and it is irrelevant to the analysis of the back reaction problem. Being mainly interested in the UV finite quantum corrections to black hole thermodynamics we will assume throughout the paper that the necessary renormalization was already done and we deal with the renormalized quantum effective action.

In this paper we use the two-dimensional model to study the one-loop quantum effects in the thermodynamics of a charged black hole. We start in Sec. II with the 4D Einstein-Maxwell theory with boundary terms included appropriately [20]. Then, considering only spherically symmetric metrics, this model reduces to an effectively two-dimensional one of the dilaton type. The classical solution describes the well-known Reissner-Nordström charged black hole. The thermodynamics of the classical black hole is reformulated in Sec. III in the framework of the conical singularity method. We especially notice the role of both the terms defined on the external boundary and on the conical singularity in the well-defined variational procedure. The choice of state of the quantum field and the corresponding form of the Liouville-Polyakov action are discussed in Sec. IV. In particular, we take care of the boundary terms and derive the modified Liouville-Polyakov action for a space with a conical defect. The deformation of the geometry of a charged black hole due

to Hawking radiation is calculated perturbatively in Sec. V. The energy and entropy of the quantum-corrected black hole are calculated in Sec. VI. The deviations from the classical Bekenstein-Hawking form are obtained and the possible role of these corrections is discussed.

II. SPHERICALLY SYMMETRIC REDUCTION OF 4D EINSTEIN-MAXWELL THEORY

Let us consider 4D Einstein gravity coupled with a Maxwell field described by the action (we use the Euclidean signature)

$$W_{\text{cl}} = -\frac{1}{16\pi G} \int_{M^4} R^{(4)} \sqrt{g} d^4x + \frac{1}{16\pi G} \int_{M^4} F^2_{\mu\nu} \sqrt{g} d^4x - \frac{1}{8\pi G} \int_{\partial M^4} K^{(4)} \sqrt{h} d^3x, \quad (2.1)$$

where $R^{(4)}$ is the 4D scalar curvature. We have added in Eq. (2.1) the boundary term according to [20]. $K^{(4)}$ is the trace of the extrinsic curvature of the boundary ∂M^4 . If n^μ is the outward unit vector normal to ∂M^4 , then we have

$$K^{(4)} = \nabla_\mu n^\mu. \quad (2.2)$$

The action (2.1) is known to be divergent when the boundary ∂M goes to infinity. The same presumably happens for the one-loop effective action and requires some subtraction procedure. Generally, one proceeds by comparing the divergent quantity with that defined for a specially chosen background. If $g_{\mu\nu}^0$ is the background metric, then we define the subtracted expression as [21]

$$W_{\text{sub}} = W[g_\mu] - W[g_{\mu\nu}^0], \quad (2.3)$$

where W includes both the classical (2.1) and one-loop gravitational action. Presumably, in the quantum case we would have to subtract the contribution of the nonflat reference metric of the asymptotic geometry (see Ref. [10] for such an example). Therefore, we shall consider an arbitrary reference (background) metric hereafter.

Our first goal is to make the reduction of the general action (2.1) to the special case of spherically symmetric space times. Spherically symmetric metrics are of the form

$$ds^2 = \gamma_{\alpha\beta}(z) dz^\alpha dz^\beta + r^2(z) (d\theta^2 + \sin^2\theta d\varphi^2). \quad (2.4)$$

Here $\alpha, \beta, \dots = 0, 1$, $\gamma_{\alpha\beta}(z)$ is the 2D metric on the effective two-dimensional space M^2 covered by coordinates $z^\alpha = (\tau, x)$, and $r^2(z)$ is the scalar field on M^2 . We have, for the scalar curvature of the metric (2.4),

$$R^{(4)} = R^{(2)} + \frac{2}{r^2} (\nabla r)^2 - \frac{2}{r^2} \square r^2 + \frac{2}{r^2}, \quad (2.5)$$

where all the geometrical objects $R^{(2)}, \nabla, \square$ are defined with respect to 2D metric $\gamma_{\alpha\beta}(z)$.

For the spherical reduction of the action it is sufficient to consider boundaries ∂M^4 of the spherically-symmetric space M^4 with metric (2.4) that are a direct product $\partial M^4 = \partial M^2 \times S^2$, where ∂M^2 is a boundary of 2D space M^2 ; S^2 is a 2D sphere. A normal vector n^μ to this boundary

has nonzero components only in the direction tangent to the space M^2 , $n^\mu = (n^\alpha, 0, 0)$. Hence, we obtain, for the trace of the extrinsic curvature of the boundary (2.2),

$$K^{(4)} = k + 2n^\alpha \frac{\partial_\alpha r}{r},$$

$$k \equiv \nabla_\alpha n^\alpha \equiv \frac{1}{\sqrt{\gamma}} \partial_\alpha (\sqrt{\gamma} n^\alpha) = \partial_\alpha n^\alpha + \frac{1}{2\gamma} \partial_\alpha \gamma n^\alpha, \quad (2.6)$$

where $\gamma = \det \gamma_{\alpha\beta}$. If the metric is static and spherisymmetric it can be written in the Schwarzschild form:

$$ds^2 = g(x) d\tau^2 + g^{-1}(x) dx^2 + r^2(x) (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.7)$$

Then we have $n^\alpha = (0, g^{1/2})$ and, hence,

$$K^{(4)} = k + \frac{2}{r} r' g^{1/2}, \quad k = (g^{1/2})'.$$

In accordance with our assumption about spherical symmetry the Maxwell field A_μ is tangent to the space M^2 , i.e., the only nonzero component of the gauge curvature is $F_{\tau r} \neq 0$.

Taking into account that the integration over angles (θ, φ) in Eq. (2.1) induces the measure

$$\int \sqrt{g} d\theta d\varphi = 4\pi r^2 \sqrt{\gamma},$$

we finally get that the action (2.1) for the spherically symmetric metric (2.4) reduces to the effective two-dimensional theory

$$W_{\text{cl}} = -\frac{1}{4G} \int_{M^2} [r^2 R + 2(\nabla r)^2 + 2] \sqrt{\gamma} d^2 z + \frac{1}{4G} \int_{M^2} r^2 F_{\alpha\beta}^2 \sqrt{\gamma} d^2 z - \frac{1}{2G} \int_{\partial M^2} r^2 k. \quad (2.8)$$

In two dimensions $F_{\alpha\beta}$ has only one component

$$F_{\alpha\beta} = e_{\alpha\beta} F, \quad (2.9)$$

where $e_{\alpha\beta}$ is the antisymmetric Levi-Civita tensor. It follows from the equations of motion for the Maxwell field,

$$\nabla_\alpha (r^2 F^{\alpha\beta}) = 0,$$

that

$$F = \frac{Q}{r^2}, \quad Q = \text{const}, \quad (2.10)$$

where Q is the electric charge.

Inserting Eqs. (2.9) and (2.10) into the action (2.8) we find that the whole theory reduces to some type of 2D dilaton gravity:

$$W_{\text{cl}} = -\frac{1}{4G} \int_{M^2} [r^2 R + 2(\nabla r)^2 + 2U(r)] \sqrt{\gamma} d^2 z - \frac{1}{2G} \int_{\partial M^2} r^2 k, \quad (2.11)$$

with the field $r^2(z)$ playing the role of the dilaton field. The dilaton potential reads

$$U(r) = 1 - \frac{Q^2}{r^2}. \quad (2.12)$$

Wick's rotation to the Euclidean metric is typically accompanied by the corresponding complexification of the charge $Q \rightarrow iQ$ assuming that after all calculations we make the continuation back to the real Q [22]. Having this in mind we use the expressions (2.11) and (2.12) where Q is already real. Variation of the action (2.11) with respect to the dilaton r^2 gives the dilaton equation of motion

$$rR - 2\Box r + U'_r = 0, \quad (2.13)$$

while the variation with respect to the metric $\gamma_{\alpha\beta}$ gives

$$G_{\alpha\beta} \equiv -2r \nabla_\alpha \nabla_\beta r + \gamma_{\alpha\beta} [\Box r^2 - (\nabla r)^2 - U] = 0. \quad (2.14)$$

Equation (2.14) implies that the vector $\xi_\alpha = e_\alpha^\beta \partial_\beta r$ is a Killing vector. In the region where $(\nabla r)^2 \neq 0$ the Killing time t ($\xi^\alpha \partial_\alpha = \partial_t$) and r can be used as coordinates on M^2 . The equation $G_\tau^r - G_r^\tau = 0$ implies that the metric is of the form

$$ds^2 = g(r) d\tau^2 + \frac{1}{g(r)} dr^2. \quad (2.15)$$

The trace of Eq. (2.14) is

$$\Box r^2 = 2U(r). \quad (2.16)$$

This relation gives

$$g(r) = g_{\text{cl}}(r) = \frac{1}{r} \int^r U(r') dr' = 1 - \frac{2MG}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2}, \quad (2.17)$$

where M is an integration constant to be identified with the Arnowitt-Deser-Misner (ADM) mass, and $r_\pm = MG \pm \sqrt{(MG)^2 - Q^2}$ are the radii of the outer and inner horizons.

III. TREE-LEVEL BLACK HOLE THERMODYNAMICS

The Euclidean action (2.11) is the starting point for the formulation of the classical thermodynamics of the black hole. The standard procedure for describing the thermodynamical properties of a field system is to go to the Euclidean space by a Wick's rotation $t = i\tau$ and to close the τ direction with period $2\pi\beta = T^{-1}$, where T is the temperature of the system. The system is assumed to be contained in a box of size L . In principle, the field configuration does not necessarily satisfy any field equations. The latter arise as a requirement of extremality of the free energy functional under ap-

appropriately defined boundary conditions.

Analogously the thermodynamics of black holes can be formulated off-shell. We discuss now this formulation in more detail. Consider the Euclidean static metric of the general type

$$ds^2 = g(x)d\tau^2 + \frac{e^{-2\lambda(x)}}{g(x)}dx^2, \quad (3.1)$$

written in the fixed coordinate system (τ, x) where the coordinates range between the limits $0 \leq \tau \leq 2\pi\bar{\beta}$; $x_+ \leq x \leq L$. In what follows we assume that an external boundary is located at $x=L$, while $x=x_+$ is the location of the horizon of the black hole.

The temperature T of the system is fixed at the boundary and can be invariantly defined as $T^{-1} = \int d\tau g_{00}^{1/2}(x=L)$. The system is also characterized by the value of the dilaton field r_B at the boundary, $r_B = r(x=L)$. The fact that the system includes a nonextremal black hole means that at some point $x=x_+$ (horizon) the function $g(x)$ has a simple zero, $g(x_+) = 0$. In this case the Euler characteristic of the space described by Eq. (3.1) is fixed to be $\chi = 1$. Thus the system is specified by (i) fixing temperature T and value of the ‘‘radius’’ r_B on the external boundary, and by (ii) fixing black-hole topology. The statistical ensemble consists of all the functions (g, λ, r) satisfying these conditions. For an arbitrary metric from this class the quantity $\beta_H \equiv (2e^{-\lambda/g'})_{x=x_+}$ is a functional of the metric and it is not fixed by the above conditions. In the general case such a metric describes the Euclidean space with conical singularity at the point $x=x_+$ (horizon) with angle deficit $\delta = (1-\alpha)2\pi$, where $\alpha = \bar{\beta}/\beta_H$. This implies that the scalar curvature has a δ -like contribution coming from the tip of the cone (see details in Ref. [23]):

$$R^{(2)} = 2 \left(\frac{1-\alpha}{\alpha} \right) \delta(x-x_+) + \bar{R}^{(2)}, \quad \alpha = \frac{\bar{\beta}}{\beta_H}, \quad (3.2)$$

where $\bar{R}^{(2)}$ is the regular part of the curvature. The conical singularity vanishes when $\alpha = 1$. Note that only combination $\alpha = \bar{\beta}/\beta_H$ has an invariant meaning while β_H and $\bar{\beta}$ are coordinate dependent.

In many respects, the approach which we use here is similar to the approach developed by York and collaborators [24]. The essential difference, however, is that in [24] only regular metrics are considered. In our approach the statistical ensemble specified by conditions (i) and (ii) includes both the regular metrics and metrics with conical singularities. For a metric of general type (with an arbitrary α) the classical action (2.11) due to Eq. (3.2) takes the form

$$W_{cl} = -\frac{1}{4G} \int_M [r^2 \bar{R} + 2(\nabla r)^2 + 2U(r)] \sqrt{\gamma} d^2z - \frac{1}{2G} \int_{\partial M} r^2 k^{(2)} - \frac{\pi r_+^2}{G} (1-\alpha). \quad (3.3)$$

For the static metric (3.1), action, (3.3) is

$$W_{cl} = -\frac{(2\pi\bar{\beta})}{4G} \int_{x_+}^L [(r^2)' e^\lambda g' + 2g e^\lambda (r'_x)^2 + 2U e^{-\lambda}] - \frac{\pi r_+^2}{G}. \quad (3.4)$$

One can define the free energy F , entropy S , and energy E associated with W_{cl} as

$$F = (2\pi\beta)^{-1} W_{cl}, \quad S = (\beta \partial_\beta - 1) W_{cl}, \quad E = \frac{1}{2\pi} \partial_\beta W_{cl}, \quad (3.5)$$

where $2\pi\beta = T^{-1}$ and $\beta = \bar{\beta} g_B^{1/2}$. Applying these formulas to Eq. (3.4) we obtain that the energy E is given by the expression

$$E = -\frac{1}{4G g_B^{1/2}} \int_{x_+}^L [(r^2)' e^\lambda g' + 2g e^\lambda (r'_x)^2 + 2U e^{-\lambda}], \quad (3.6)$$

and the entropy

$$S_{BH} = \frac{\pi r_+^2}{G} \quad (3.7)$$

takes the standard Bekenstein-Hawking form. In the calculations made up to this point we did not assume that $\alpha = 1$, in other words the calculations were done off-shell. Now, we fix the temperature $T = (2\pi\beta)^{-1}$ and consider the extremum of the free energy $F = E - TS$ or equivalently the extremum of the action W_{cl} . Remarkably, such an equilibrium configuration automatically satisfies the second law of black hole thermodynamics:

$$\delta E = T \delta S \quad (3.8)$$

for small variations around the equilibrium state.

It should be noted that only T and r_B at $x=L$ and condition $g(x_+) = 0$ at the horizon are assumed to be fixed. The functions $g(x)$, $g'(x)$, $r(x)$ and the values on the horizon of $r_+ = r(x_+)$, $g'(x_+)$ (or β_H) are variable. The total variation of the action W_{cl} is $\delta W_{cl} = \delta_r W_{cl} + \delta_g W_{cl} + \delta_\lambda W_{cl}$. For partial variations we have

$$\delta_r W_{cl} = -\frac{2\pi r(x_+)}{G} (1-\alpha) \delta r(x_+) - \frac{(2\pi\bar{\beta})}{4G} \times \int_{x_+}^L \delta r [-2r(e^\lambda g')' - 4(g e^\lambda r')' + 2U_r' e^{-\lambda}] dx, \quad (3.9)$$

$$\delta_g W_{cl} = -\frac{(2\pi\bar{\beta})}{4G} \int_{x_+}^L \delta g [-e^\lambda (r^2)'] + 2e^\lambda r_x'^2] dx, \quad (3.10)$$

$$\delta_\lambda W_{cl} = -\frac{(2\pi\bar{\beta})}{4G} \int_{x_+}^L \delta \lambda [e^\lambda (r^2)' g' + 2e^\lambda g (r'_x)^2 - 2U e^{-\lambda}] dx. \quad (3.11)$$

We see that variation of W_{cl} contains terms due to variations of the functions (r, g, λ) inside the region $x_+ \leq x \leq L$ that leads to the equations of motion

$$\begin{aligned} -2r(e^\lambda g')' - 4(g e^\lambda r')' + 2U_r' e^{-\lambda} &= 0, \\ -(e^\lambda (r^2)')' + 2e^\lambda r_x'^2 &= 0, \\ e^\lambda (r^2)' g' + 2e^\lambda g (r_x')^2 - 2U e^{-\lambda} &= 0, \end{aligned} \quad (3.12)$$

which of course coincide with Eqs. (2.13) and (2.14) written for the metric (3.1). Variations $\delta g'(x_+)$ and $\delta g'(L)$ on the boundaries (x_+, L) are canceled in Eqs. (3.9)–(3.11). This happens because of the presence of the ‘‘surface’’ terms in Eq. (3.3) located on the external boundary and on the singular point (the cone tip).

In some sense, the tip Σ of the cone can be considered as some kind of boundary additional to ∂M of the space M . It is the presence of the additional term located on Σ in the gravitational action (3.3) that makes the variational procedure on the conical space well defined. The term connected with the tip of the cone compensates variations of the normal derivatives of the metric at Σ in the same manner as the standard Gibbons-Hawking terms does at the external boundary ∂M . The variation of the action contains also a term proportional to the variation of the ‘‘radius’’ r_+ of the horizon, δr_+ . The requirement $\delta_r W_{\text{cl}} = 0$ gives the condition $\alpha = 1$. This is the expected result. It means that the equilibrium state is reached on a regular manifold without conical singularity (Gibbon-Hawking instanton).

Equations (3.12) imply that we may choose $r = x$. The metric function $g(r)$ takes the form (2.17)

$$g(r) = \frac{1}{r} \int_{r_+}^r U(\rho) d\rho. \quad (3.13)$$

In particular, we have

$$g(L) = \frac{1}{L} \int_{r_+}^L U(r) dr, \quad g_r'(L) = L^{-1} U(L) - L^{-1} g(L). \quad (3.14)$$

On the other hand, on the horizon we have

$$\frac{2}{\beta_H} \equiv g_r'(r_+) = \frac{U(r_+)}{r_+}. \quad (3.15)$$

The energy functional E , Eq. (3.6), takes the form

$$\begin{aligned} E &= \frac{1}{2Gg_B^{1/2}} \int_{x_+}^L G_0^0 dx + E_{\text{surf}}, \\ E_{\text{surf}} &= -\frac{1}{2G} (e^\lambda (r^2)' g^{1/2})_{x=L}, \end{aligned} \quad (3.16)$$

and modulo the constraint $G_0^0 = 0$ it reduces to the surface terms only. Equivalently, we obtain a coordinate invariant expression for the energy (3.16):

$$E = -\frac{1}{2\pi\beta} \frac{1}{G} \int_{\partial M} r n^\alpha \partial_\alpha r. \quad (3.17)$$

The quantity (3.17) is divergent if ∂M goes to infinity. The subtraction procedure described in Sec. II leads to the result

$$\begin{aligned} E &= E[g] - E[g_0] \\ &= \frac{1}{G} \left(\frac{1}{2\pi\beta_0} \int_{\partial M} r n_0^\alpha \partial_\alpha r - \frac{1}{2\pi\beta} \int_{\partial M} r n^\alpha \partial_\alpha r \right) \partial_\alpha r \\ &= \frac{1}{G} [r(g_0^{1/2} - g^{1/2})]_{r=L}. \end{aligned} \quad (3.18)$$

Here we have chosen $r_0 = r$ for the reference metric. Note that the natural condition to be imposed on the background is that in the limit $L \rightarrow \infty$ the background temperature $T = (2\pi\beta_0)^{-1}$ coincides with the black hole temperature measured at infinity. This is satisfied if $g_0 = \lim_{L \rightarrow \infty} g(L)$. For an asymptotically flat metric,

$$g(L) = 1 - \frac{2MG}{L} + \mathcal{O}\left(\frac{1}{L}\right),$$

we have $g_0 = 1$. Hence for the energy

$$E = \frac{L}{G} [1 - g^{1/2}(L)] \quad (3.19)$$

we find in the limit $L \rightarrow \infty$ that

$$E = M. \quad (3.20)$$

It should be noted that formulating the variational procedure for the charged metric we typically need to augment quantities fixed at the boundary by a quantity characterizing the Maxwell sector of the model: charge Q or potential A_0 [24]. The variation with respect to A_μ would give us the Maxwell equations. Instead of this we first solved the Maxwell sector exactly and all the information about it was collected in the ‘‘dilaton’’ potential $U(r)$, then we formulated the variational problem only for the gravitational sector. These two ways obviously lead to the same results.

The above consideration is valid for an arbitrary potential $U(r)$ provided its form is fixed. For the variations that change the form of the potential $U(r)$ we obtain, from (3.19),

$$\delta E = \delta M - \frac{1}{2G} \int_{r_+}^L \delta U(r) dr. \quad (3.21)$$

For the special choice of the potential $U(r)$ defined by Eq. (2.12) we reproduce the known form of the second law for a charged black hole

$$\delta M = T \delta S + \frac{Q}{G r_+} \delta Q. \quad (3.22)$$

However, the specific form of the potential $U(r)$ is not essential for the above consideration. It can be shown [25] that the quantum corrections change the form of the potential $U(r)$ and result in the deformation of the black hole metric (3.13). Though our methods can deal with such a possibility as well, we do not consider this here.

Special consideration is needed for an extremal black hole. In this case we have

$$U(r_+) = 0, \quad g'(r_+) = 0. \quad (3.23)$$

The geometry of an extremal black hole instanton is very different from the nonextremal one. In the metric

$$ds^2 = g(r)d\tau^2 + g^{-1}(r)dr^2 \quad (3.24)$$

τ can be closed with arbitrary period $2\pi\beta$ not forming any singularity. The horizon lies now at an infinite distance from any other point of the instanton manifold. Near the horizon the extremal instanton resembles a constant curvature space with metric

$$ds^2 = \frac{r_+^2}{z^2}(d\tau^2 + dz^2), \quad (3.25)$$

where $z \rightarrow -\infty$ if $r \rightarrow r_+$. The extremal black hole instanton can be considered as conformally related to a flat cylindrical space.

These features of the extremal geometry are crucial for the formulation of the thermodynamics of the extremal hole [26]. Since there is no conical singularity on the horizon we do not have the additional term in the action and it reads

$$W = 2\pi\beta E, \quad (3.26)$$

where the energy E takes the form (3.19). We obtain from Eq. (3.26) for the free energy of the system $F = E$, and hence the entropy of the extremal hole is formally zero:

$$S_{\text{ext}} = 0. \quad (3.27)$$

Moreover, since the free energy does not depend on the temperature β^{-1} , the requirement of extremality of the free energy under β fixed does not give a relation between parameters of the hole geometry (r_+) and β as we found for the nonextremal case. This can be interpreted as implying that the extremal black hole can be in equilibrium at arbitrary temperature [26]. However, the physical meaning of this formal result is not clear. In particular, quantum effects may change this conclusion. We are going to consider this in a separate publication.

IV. LIOUVILLE-POLYAKOV ACTION AND CHOICE OF THE THERMAL STATE OF THE QUANTUM FIELD

In order to include one-loop quantum effects in the analysis, consider a two-dimensional quantum conformal massless scalar field. This produces the following contribution to the partition function:

$$Z = e^{-\Gamma}, \quad \Gamma = \frac{1}{2} \ln \det \square, \quad (4.1)$$

where $\square = \nabla_\mu \nabla^\mu$ is the two-dimensional Laplacian. The calculation of the effective action Γ is usually made by integrating the conformal anomaly. The result is well known [14]:

$$\Gamma_{\text{PL}}[g] = \frac{1}{96\pi} \int R \square^{-1} R. \quad (4.2)$$

However, if we wish to work with Eq. (4.2) we are confronted with at least two problems. First, the action (4.2) does not transform properly under a constant (global) conformal transformation, $g_{\mu\nu} \rightarrow \Lambda g_{\mu\nu}$. (This was noted by Dowker [27].) Second, when applying Eq. (4.2) to a flat space (where $R=0$), we get that the corresponding mean value of the stress-energy tensor $\langle T^{\mu\nu} \rangle$ obtained by the variation of Eq. (4.2) vanishes. This is certainly valid for the vacuum state, but not for other possible states. In particular, it is not clear how Eq. (4.2) can reproduce the effective action for a thermal radiation. So, writing the effective action in the form (4.2) one loses the information on the concrete choice of the state of the quantum field. We demonstrate that the information about the state of a quantum field is directly connected with the boundary terms which are to be added to Eq. (4.2). Therefore, we begin our consideration of one-loop quantum effects with a more careful treatment of the Liouville-Polyakov action, taking into account all the boundary terms.

It should be emphasized that the integration of the conformal anomaly which is used to derive Eq. (4.2) does not give the absolute value of the effective action $\Gamma[g]$, but rather the difference between the effective actions for two conformally related ($g_{\mu\nu} = e^{2\sigma} \hat{g}_{\mu\nu}$) manifolds [28]:

$$\begin{aligned} \Gamma[g] = \Gamma[\hat{g}] - \frac{1}{24\pi} & \left(\int_M (\hat{\nabla} \sigma)^2 + \int_M \hat{R} \sigma + 2 \int_{\partial M} d\hat{s} \hat{k} \sigma \right) \\ & - \frac{1}{8\pi} \int_{\partial M} d\hat{s} \hat{n}^\mu \partial_\mu \sigma. \end{aligned} \quad (4.3)$$

Here \hat{n}^μ is the outward vector normal to the boundary ∂M , and $\hat{k} = \nabla_\mu \hat{n}^\mu$ is the trace of the second fundamental form of the boundary.

One can write $\Gamma[g]$ in terms of quantities defined only with respect to metric $g_{\mu\nu}$ if we introduce an additional field ψ defined as a solution of the equation

$$\square \psi = R. \quad (4.4)$$

For conformally related metrics $g_{\mu\nu} = e^{2\sigma} \hat{g}_{\mu\nu}$ the respective quantities are related as

$$\begin{aligned} R &= e^{-2\sigma} (\hat{R} - 2\square \sigma), \quad \psi = \hat{\psi} - 2\sigma, \\ k &= e^{-\sigma} (\hat{k} + \hat{n}^\mu \partial_\mu \sigma), \quad n^\mu = e^{-\sigma} \hat{n}^\mu. \end{aligned} \quad (4.5)$$

Using these relations, one can show that the effective action $\Gamma[g]$ of Eq. (4.1), conformally transforming according to Eq. (4.3), takes the form

$$\Gamma[g] = \frac{1}{48\pi} \int_M [\frac{1}{2} (\nabla \psi)^2 + \psi R] + \frac{1}{24\pi} \int_{\partial M} k \psi ds + \Gamma_0, \quad (4.6)$$

where all the quantities are defined with respect to $g_{\mu\nu}$ and the ‘‘integration constant’’ Γ_0 is a conformally invariant functional.

Let us now consider the conformal massless field φ in a thermal state with temperature T in a space time with horizon. The relevant static Euclidean metric reads

$$ds^2 = g(x)d\tau^2 + \frac{1}{g(x)}dx^2, \tag{4.7}$$

or

$$ds^2 = g(\rho)d\tau^2 + d\rho^2, \tag{4.8}$$

where τ lies in the range $0 \leq \tau \leq 2\pi\bar{\beta}$ and $0 \leq \rho \leq L_\rho$. Assume that $g(x)$ has a zero of first order at the point $x = r_+$. This is the Killing horizon. Near the horizon we have $g(\rho) = \rho^2/\beta_H^2$, where $\beta_H = 2/g'_x(r_+)$. For $\bar{\beta} = \beta_H$, Eq. (4.8) describes a regular black hole instanton. If $\bar{\beta} \neq \beta_H$ the metric has a conical singularity at $\rho = 0$ with angle deficit $\delta = 2\pi(1 - \alpha)$, $\alpha = \bar{\beta}/\beta_H$. The metric (4.8) can be written in the conformal form

$$ds^2 = e^{2\sigma}ds_0^2, \quad ds_0^2 = (dz^2 + \alpha^2 z^2 d\tilde{\tau}^2),$$

$$e^{2\sigma} = \beta_H^2 \frac{g}{z^2}, \quad z = z_0 \exp\left[\frac{1}{\beta_H} \int_{L_\rho}^\rho \frac{d\rho}{\sqrt{g}}\right], \tag{4.9}$$

where $\alpha = \bar{\beta}/\beta_H$, $\tau = \bar{\beta}\tilde{\tau}$, and $0 \leq \tilde{\tau} \leq 2\pi$, $0 \leq z \leq z_0$. Note that near the horizon $z \approx \rho$ and hence the conformal factor is regular on the horizon.

For $\bar{\beta} = \beta_H$ Eq. (4.9) conformally relates the metric of the black hole instanton with the metric on the flat disk \mathcal{D} of radius z_0 . For conformally related metrics, $g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu}$, the stress-energy tensors are related as follows:

$$T_{\mu\nu}[g] = T_{\mu\nu}[\hat{g}] + \frac{1}{48\pi}\{-4\hat{\nabla}_\mu\hat{\nabla}_\nu\sigma + 4\partial_\mu\sigma\partial_\nu\sigma + g_{\mu\nu}[4\hat{\square}\sigma - 2(\hat{\nabla}\sigma)^2]\}. \tag{4.10}$$

Thus, for Eq. (4.9) we have

$$T_{\tau\tau} = T_{\tau\tau}^{(0)} + \frac{1}{48\pi}\left(\frac{2}{\beta_H^2} + 2g''_\rho - \frac{3}{2}\frac{(g'_\rho)^2}{g}\right), \tag{4.11}$$

where $T_{\tau\tau}^{(0)}$ is energy density of the quantum field on the flat disk \mathcal{D} . At infinity $\rho = \infty$, $g = 1$, so we have

$$T_{\tau\tau} = T_{\tau\tau}^{(0)} + \frac{1}{24\pi\beta_H^2}. \tag{4.12}$$

Assume that the quantum field on \mathcal{D} is in the state for which $T_{\tau\tau}^{(0)} = 0$. We call this state ‘‘vacuum on the disk.’’ Physically this state is just the usual Minkowski (or Hartle-Hawking) vacuum state in the Rindler space.

For this choice we find that the quantum field on the black hole instanton is in the state of the Hartle-Hawking vacuum with Hawking temperature $T_H = 1/2\pi\beta_H$ since Eq. (4.12) coincides with the energy density of a thermal bath with temperature T_H . Hence, starting with the ‘‘vacuum on the disk’’ state on the flat disk and making the regular conformal transformation (4.9), we obtain the quantum field in the state with Hawking temperature on the regular black hole instanton. If we start with the state at finite temperature $T_0 = (2\pi\beta_0)^{-1}$ on the disk \mathcal{D} we obtain the state with the temperature

$T = (2\pi\beta)^{-1} = (2\pi)^{-1}[\beta_0^{-2} + \beta_H^{-2}]^{-1/2}$ on the black hole instanton, which differs from the Hartle-Hawking state.

After these general remarks consider now a singular black hole instanton M^α with $0 \leq \tau \leq 2\pi\bar{\beta}$ ($\bar{\beta} \neq \beta_H$). Then Eq. (4.9) conformally relates it to the flat cone C_α ($\alpha = \bar{\beta}/\beta_H$), and z_0 is the proper length of the cone’s generator. The conformal factor σ is an everywhere regular function, and we find that the stress-energy tensors $T_{\mu\nu}$ on the two spaces are related by the same expression (4.10) and (4.11), where now $T_{\tau\tau}^{(0)}$ is the energy density on the flat cone C_α ([29], see also [30]):

$$T_\tau^{\tau(0)} = \frac{1}{24\pi} \frac{1}{z^2} \left(\frac{1 - \alpha^2}{\alpha^2}\right), \quad \alpha = \frac{\bar{\beta}}{\beta_H}. \tag{4.13}$$

At infinity, the energy density

$$T_{\tau\tau} \rightarrow \frac{1}{24\pi\bar{\beta}^2}$$

takes the thermal form with temperature $T_\infty = 2\pi\bar{\beta}^{-1}$. Hence we may conclude that the thermal state of the quantum field with $T \neq T_H$ in the gravitational field of a black hole can be effectively described as a quantum field on a singular instanton (i.e., on the instanton with a conical singularity on the horizon).

One can calculate $T_{\mu\nu}$ directly in terms of the metric on the black hole instanton M^α with a conical singularity ($\bar{\beta} \neq \beta_H$) (see [7]). Equation (4.4) for the metric (4.8) has the solution

$$\psi = -\ln g + b \int \frac{dx}{g} + C = -\ln g + b \int \frac{\rho d\rho}{\sqrt{g}} + C. \tag{4.14}$$

In order to fix constant b in Eq. (4.14) consider the renormalized stress-energy tensor which is expressed via ψ as [31]

$$T_{\mu\nu} = \frac{1}{48\pi}\{2\nabla_\mu\nabla_\nu\psi - \partial_\mu\psi\partial_\nu\psi + g_{\mu\nu}[-2R + \frac{1}{2}(\nabla\psi)^2]\}. \tag{4.15}$$

The conformal transformation of Eq. (4.15) is given by Eq. (4.10). Inserting ψ [Eq. (4.14)] into Eq. (4.15) we obtain

$$T_{\tau\tau} = \frac{1}{48\pi}\left(2g''_\rho - \frac{3}{2}\frac{(g'_\rho)^2}{g} + \frac{b^2}{2}\right). \tag{4.16}$$

In order to have at infinity thermal behavior with $T = (2\pi\bar{\beta})^{-1}$ we must fix the constant $b = 2/\bar{\beta}$ in Eq. (4.14).

This identification automatically gives us that in the limit $\rho \rightarrow 0$ the function ψ [Eq. (4.14)],

$$\psi \rightarrow \psi_c = -2\left(1 - \frac{\beta_H}{\bar{\beta}}\right)\ln\rho, \tag{4.17}$$

coincides with the solution of the cone equation:

$$\square_c\psi_c = R_c, \quad R_c = 2\left(\frac{1 - \alpha}{\alpha}\right)\delta(\rho), \tag{4.18}$$

where \square_c is the Laplacian on the flat cone C_α . Thus, the stress-energy tensor $T_{\mu\nu}$ for the state with temperature $T=(2\pi\bar{\beta})^{-1}$ at infinity coincides with the $T_{\mu\nu}$ of a quantum field on the black hole instanton (4.9) with conical singularity on the horizon ($\bar{\beta}\neq\beta_H$).

In order to fix the constant C in Eq. (4.14), which in fact can depend on the characteristics of the system, consider the conformal transformation determined by $\sigma(x)$ [Eq. (4.9)]:

$$2\sigma(x)=\text{lg}(x)+\frac{2}{\beta_H}\int_x^L\frac{dx}{g(x)}+2\text{ln}\frac{\beta_H}{z_0}, \quad (4.19)$$

which relates [see Eq. (4.9)] our singular black hole instanton with a flat cone C_α with radius z_0 . Then we have that the functions $\psi(x)$ on these spaces are related as follows:

$$\psi_{M^\alpha}(x)=\psi_{C_\alpha}(z)-2\sigma(x), \quad (4.20)$$

where $z(x)$ is given by Eq. (4.9). On the other hand, for each function ψ_{M^α} and ψ_{C_α} we have the representation (4.14):

$$\begin{aligned} \psi_{M^\alpha}(x) &= -\text{lg}(x) - \frac{2}{\beta} \int_x^L \frac{dx}{g(x)} + C, \\ \psi_{C_\alpha}(z) &= -2 \left(1 - \frac{1}{\alpha} \right) \text{ln} \frac{z}{z_0} + C(\alpha, z_0). \end{aligned} \quad (4.21)$$

Here $C(\alpha, z_0)$ is a function of only α and z_0 .

Plugging Eqs. (4.19) and (4.21) into Eq. (4.20), we find for the constant $C = -2\text{ln}(\beta_H/z_0) + C(\alpha, z_0)$. Really, there is no dependence of C on z_0 since under rescaling $z_0 \rightarrow e^\gamma z_0$ we have $C(\alpha, z_0) \rightarrow C(\alpha, z_0) - 2\text{ln}\gamma$. Thus, finally we have

$$\psi_{M^\alpha}(x) = -\text{lg}(x) - \frac{2}{\beta} \int_x^L \frac{dx}{g(x)} - 2\text{ln}\frac{\beta_H}{z_0} + C(\alpha, z_0). \quad (4.22)$$

In order to write down the Polyakov-Liouville action for this case it should be noted that in the presence of the conical singularity the conformal transformation of the effective action (4.1) must be modified. If two conical spaces M^α and \hat{M}^α with the angle deficit $\delta=2\pi(1-\alpha)$ and a tip Σ are related by a regular conformal transformation $g_{\mu\nu}=e^{2\sigma}\hat{g}_{\mu\nu}$, then the corresponding effective actions are related as follows [32]:

$$\begin{aligned} \Gamma[g] &= \Gamma[\hat{g}] - \frac{1}{24\pi} \left(\int_{\hat{M}^\alpha} (\hat{\nabla}\sigma)^2 + \int_{\hat{M}^\alpha} \hat{R}\sigma + 2 \int_{\partial\hat{M}^\alpha} d\hat{s}\hat{k}\sigma \right) \\ &\quad - \frac{1}{8\pi} \int_{\partial\hat{M}^\alpha} d\hat{s}\hat{n}^\mu \partial_\mu \sigma - \frac{1}{12} \frac{(1-\alpha^2)}{\alpha} \sigma_h, \end{aligned} \quad (4.23)$$

where σ_h is the value at the tip Σ of the cone.

Taking into account the transformation law (4.5) of ψ , the effective action on M^α , transforming according to Eq. (4.23), can be written in the form

$$\begin{aligned} \Gamma[M^\alpha] &= \frac{1}{48\pi} \int_{M^\alpha} \left[\frac{1}{2} (\nabla\psi)^2 + \psi\bar{R} \right] + \frac{1}{24} \frac{(1-\alpha^2)}{\alpha} \psi_h \\ &\quad + \frac{1}{24\pi} \int_{\partial M^\alpha} k\psi ds + \Gamma_0. \end{aligned} \quad (4.24)$$

Here \bar{R} is the regular part of the scalar curvature, and $\psi(x)$ in Eq. (4.24) is the solution of the equation $\square\psi=R\equiv 2[(1-\alpha)/\alpha]\delta_\Sigma+\bar{R}$. For a static metric (4.7) ψ takes the form (4.22).

We denote by $\psi_h=\psi(\Sigma)$ the value of ψ on the horizon (tip of the cone) and by Γ_0 a conformally invariant functional.

It is worthwhile to note that the expression (4.24) can be rewritten in two equivalent forms. The first one,

$$\begin{aligned} \Gamma[M^\alpha] &= \frac{1}{48\pi} \int_{M^\alpha} \left[\frac{1}{2} (\nabla\psi)^2 + \psi R \right] + \frac{(1-\alpha)^2}{24\alpha} \psi_h \\ &\quad + \frac{1}{24\pi} \int_{\partial M^\alpha} k\psi ds + \Gamma_0, \end{aligned} \quad (4.25)$$

involves quantities defined on the full conical space M^α : $R\equiv 2(\alpha^{-1}-1)\delta_\Sigma+\bar{R}$, $\square\psi=R$.

Another way to present the effective action on the conical space M^α is to write it by using quantities defined only on the regular part $M^\alpha\setminus\Sigma$:

$$\begin{aligned} \Gamma[M^\alpha] &= \frac{1}{48\pi} \int_{M^\alpha\setminus\Sigma} \left[\frac{1}{2} (\nabla\bar{\psi})^2 + \bar{\psi}\bar{R} \right] + \frac{1}{12} (1-\alpha) \bar{\psi}_h \\ &\quad + \frac{1}{24\pi} \int_{\partial M^\alpha} \bar{k}\bar{\psi} ds + O((1-\alpha)^2), \end{aligned} \quad (4.26)$$

where $\bar{\psi}=\psi_{\alpha=1}$, $\square\bar{\psi}=\bar{R}$. The effective action in this form was written in [10].

V. QUANTUM-CORRECTED BLACK HOLE GEOMETRY

In the semiclassical approximation (when the metric is not quantized) the one-loop quantum effects are taken into account by adding to the classical gravitational action the quantum counterpart obtained by integrating out the matter fields:

$$W = W_{\text{cl}} + \Gamma. \quad (5.1)$$

Following our spherically symmetric considerations we take the classical part W_{cl} to have the form (2.11) (with correct subtraction of the contribution due to the reference metric, as has been explained in Sec. III) while the one-loop contribution Γ is the Polyakov-Liouville action (4.24). Of course, in a self-consistent treatment the quantum effective action Γ must be obtained by the same spherically symmetric reduction of the 4D matter fields as has been done for the gravitational part W_{cl} . However, the effective action becomes a rather complicated quantity which makes the analysis difficult. Therefore, we consider here the simplest case when the effective 2D matter is conformal and Γ is the nonlocal Polyakov-Liouville functional.

We begin our consideration of one-loop quantum effects by studying the corrections to the classical geometry of the

black hole induced by quantum corrections to the action (5.1). Variation of Eq. (5.1) with respect to the metric gives the equations

$$G_{\alpha\beta} = -T_{\alpha\beta}, \quad (5.2)$$

$$T_{\alpha\beta} = \frac{G}{24\pi} \{2\nabla_\alpha \nabla_\beta \psi - \partial_\alpha \psi \partial_\beta \psi - \gamma_{\alpha\beta} [2R - \frac{1}{2}(\nabla\psi)^2]\}, \quad (5.3)$$

where $G_{\alpha\beta}$ is given by Eq. (2.14). The variation with respect to the dilaton field $r^2(x)$ gives the same equation as in the classical case:

$$2R - 2\Box r + U'_r = 0. \quad (5.4)$$

An important consequence of Eqs. (5.2) and (5.4) is that the space-time singularity now is placed at finite radius (value of the dilaton) $r^2 = r_{cr}^2 \equiv G/12\pi$. This typically happens in two-dimensional models of gravity, as has been previously observed in the string context [33] and for the theory under consideration in [34,25]. For this value of the dilaton the kinetic term in Eq. (5.1) becomes degenerate. On the other hand, taking the trace of Eq. (5.2) we have

$$\Box r^2 - 2U(r) = \frac{G}{12\pi} R. \quad (5.5)$$

Combining this relation with Eq. (5.4) we obtain, for the curvature,

$$R = \frac{2U - rU' - 2(\nabla r)^2}{r^2 - r_{cr}^2}, \quad (5.6)$$

which implies a singularity at $r = r_{cr}$. We do not investigate here the behavior of the solution of Eqs. (5.2) and (5.4) near this point. Instead, we assume that the outer horizon lies at $r_+ \gg r_{cr}$. Then, in the region $r \gg r_+$ we may solve Eqs. (5.2) and (5.4) perturbatively (with respect to r_{cr}/r_+) considering $T_{\alpha\beta}$ in the right-hand side (r.h.s.) of Eq. (5.2) as a small perturbation. This gives the correction to the black hole geometry to first order in the Planck constant \hbar .

As earlier we consider a static solution. We define functions f and M as

$$f = (\nabla r)^2, \quad M = \frac{1}{2}r[1 - (\nabla r)^2] + \frac{Q^2}{r}, \quad (5.7)$$

and choose r as one of the coordinates, while the Killing time t as the other coordinate. For this choice of the coordinates we get

$$ds^2 = f(r)e^{2\Phi(r)} dt^2 + \frac{1}{f(r)} dr^2, \quad (5.8)$$

$$f(r) = 1 - \frac{2M(r)}{r} + \frac{Q^2}{r^2}. \quad (5.9)$$

Equation (5.2) takes the form

$$2r\nabla_\alpha \nabla_\beta r = \gamma_{\alpha\beta} \frac{2M}{r} - \gamma_{\alpha\beta} T + T_{\alpha\beta}, \quad T = \gamma_{\alpha\beta} T^{\alpha\beta}. \quad (5.10)$$

Differentiating Eq. (5.7) and using Eq. (5.10) we obtain

$$2\partial_\alpha M = \partial_\beta r (\delta_\alpha^\beta T - T_\alpha^\beta). \quad (5.11)$$

This equation is identically satisfied for the value of index $\alpha=0$, while for $\alpha=1$ it gives

$$\partial_r M = \frac{1}{2} T_t^t. \quad (5.12)$$

By taking the trace of Eq. (5.10) we obtain the equation for the function $\Phi(r)$:

$$\partial_r \Phi = \frac{1}{2rf} (T_t^t - T_r^r). \quad (5.13)$$

We consider the r.h.s. of Eqs. (5.12) and (5.13) as a perturbation. Then, solving these equations perturbatively, we must take their right-hand sides on the classical background. At the classical level we have $\Phi(r)=0$ and $M = \text{const}$, and for the static metric (5.8) the stress-energy tensor $T_{\alpha\beta}$ [Eq. (5.2)] reads

$$T_t^t = \kappa \left[+2f'' - \frac{1}{2f} \left(f'^2 - \frac{4}{\beta_H^2} \right) \right],$$

$$T_r^r = \kappa \left[\frac{1}{2f} \left(f'^2 - \frac{4}{\beta_H^2} \right) \right]. \quad (5.14)$$

Here $\kappa = G/24\pi$, $\beta_H = 2/f'(r_+)$. We must put the classical metric (2.17) with $f = g_{ct}(r) = r^{-2}(r-r_+)(r-r_-)$, $r_\pm = MG \pm \sqrt{(MG)^2 - Q^2}$ into Eq. (5.14).

It should be noted that $T_{\alpha\beta}$ given by Eq. (5.14) is divergent at the inner horizon $r = r_-$. This is the well-known divergence [35] which makes the perturbation scheme non-applicable near $r = r_-$. To derive the conditions of applicability of the perturbation scheme consider T_β^α first at the outer horizon $r = r_+$ and then take $r_- \sim r_+$. Then we observe that both T_t^t and T_r^r defined by Eq. (5.14) remain finite in this limit, while the combination $f^{-1}(T_t^t - T_r^r)$ appearing in Eq. (5.13) diverges as $\kappa[(r_+ - r_-)r_+]^{-1}$. The perturbation analysis works if the parameters r_+, r_- are such that this dangerous term is eliminated by the condition $\kappa[(r_+ - r_-)r_+]^{-1} \ll 1$ which implies that $\kappa[r_+]^{-2} \ll 1 - r_-/r_+$. Thus, taking r_+ to be large enough we always can come arbitrarily close to extremality $r_- \sim r_+$. This important circumstance allows us to apply our consideration to charged black holes with $Q \sim M$, which guarantees stability of the thermodynamical ensemble for an arbitrary large ‘radius’ r_B of the external boundary.

Equations (5.12) and (5.13) are easily integrated. Denote

$$m(r) = 2\kappa^{-1}[M - M(r)]. \quad (5.15)$$

Then the integration of Eq. (5.12) gives

$$\begin{aligned}
m(r) &= -\frac{1}{\kappa} \int_r^L T'_i(r) dr = C(r) + A \ln \frac{(r-r_-)}{l} + B \ln \frac{r}{l}, \\
C(r) &= -\frac{2}{\beta_H^2} r - \frac{(r_+ - r_-)^2}{2r_+ r_-} - \frac{2(r_+ + r_-)}{r^2} + \frac{10r_+ r_-}{3r^3}, \\
A &= -\frac{(r_+ - r_-)^2 (r_+ + r_-) (r_+^2 + r_-^2)}{2r_+^4 r_-^2}, \\
B &= \frac{(r_+ - r_-)^2 (r_+ + r_-)}{2r_+^2 r_-^2}. \tag{5.16}
\end{aligned}$$

As earlier, r_{\pm} denotes the ‘‘radius’’ of the classical inner and outer horizons. The following useful identity between constants A and B is worth noting: $A + B = -4MG\beta_H^{-2}$. In Eq. (5.16) we have introduced a distance l in order to have dimensionless quantities under the logarithms. The final results for the energy and entropy calculated in Sec. VI do not depend on this parameter. It seems natural to assume l to be of order of the Planckian length $l \sim r_{cr}$. However, this point is not essential for our further considerations.

Similarly the integration of Eq. (5.13),

$$\Phi(r) = \frac{1}{2} \int_r^L \frac{1}{rf} (T_r^r - T_t^t) dr, \tag{5.17}$$

for $f = g_{ct}(r)$ with the imposed condition $\Phi(L) = 0$ gives

$$\begin{aligned}
\Phi(r) &= \frac{1}{2} \kappa [F(L) - F(r)], \\
F(r) &= -\frac{(r_+^4 - r_-^4)}{r_+^4 r_- (r - r_-)} + \frac{4}{r^2} + \frac{4(r_+ + r_-)}{r_+ r_-} \\
&\quad + D \ln[(r - r_-)/l] + E \ln(r/l), \\
D &= \frac{1}{r_+^4 r_-^2} (3r_+^4 + 2r_+^3 r_- + 2r_+^2 r_-^2 + 2r_+ r_-^3 - r_-^4), \\
E &= \frac{1}{r_+^2 r_-^2} (-3r_+^2 - 2r_+ r_- - 3r_-^2). \tag{5.18}
\end{aligned}$$

Consider now the special case of the *uncharged black hole* ($Q=0$). The classical metric function is $g_{cl}(r) = 1 - r_+/r$, $r_+ = 2MG$, $\beta_H = 2r_+$. For the quantum-corrected metric we get

$$f(r) = 1 - \frac{2MG}{r} + \frac{\kappa m(r)}{r}, \tag{5.19}$$

$$m(r) = -\frac{7r_+}{4r^2} + \frac{1}{2r} - \frac{2r}{\beta_H^2} - \frac{1}{2r_+} \ln \frac{r}{l} \tag{5.20}$$

and

$$\begin{aligned}
\Phi(r) &= \frac{1}{2} \kappa [F(L) - F(r)], \\
F(r) &= \frac{3}{2r^2} + \frac{2}{r_+ r} - \frac{1}{r_+^2} \ln \frac{r}{l}. \tag{5.21}
\end{aligned}$$

For a large size L of the box we have

$$\exp(2\Phi(r)) = \left(\frac{r}{L}\right)^{\kappa/r_+^2} \exp\left[-\kappa\left(\frac{3}{2r^2} + \frac{2}{r_+ r}\right)\right]. \tag{5.22}$$

One of the important characteristics of a black hole is the radius of its horizon. In our model its role is played by the value \bar{r}_+ of the dilaton field on the horizon. For the quantum-corrected solution (5.15) it is shifted with respect to the classical value r_+ . To see this, take the condition $f(\bar{r}_+) = 0$, which is solved as $\bar{r}_+ = M(\bar{r}_+)G + \sqrt{(M(\bar{r}_+)G)^2 - Q^2}$.

Expanding this with respect to κ we finally have $\bar{r}_+ = r_+ - \kappa\beta_H m(r_+)/(2r_+)$, where the quantities r_+, β_H are classical quantities calculated for mass M and charge Q . From this it immediately follows that

$$\bar{r}_+^2 = r_+^2 - \kappa\beta_H m(r_+). \tag{5.23}$$

This identity can be interpreted as the deformation of the ‘‘horizon area’’ because of the quantum corrections.

VI. QUANTUM CORRECTIONS TO BLACK-HOLE THERMODYNAMICS

Our approach to the one-loop thermodynamics described by the action W [Eq. (5.1)] is essentially the same as in the tree-level approximation considered in Sec. III. We fix r_B , the temperature $T = (2\pi\beta)^{-1}$ on the boundary $x=L$ of the system and the black hole topology of the space-time geometry, and define the off-shell entropy and energy by the relations

$$S = (\beta\partial_\beta - 1)W, \quad E = \frac{1}{2\pi} \partial_\beta W. \tag{6.1}$$

Then, taking the Euclidean static metric in the form (3.1) with arbitrary functions $g(x), \lambda(x)$ satisfying the above conditions [$g(x)$ has simple zero at $x=x_+$], we find the equilibrium state of the system described by the extremum of the functional $W[g(x), r(x), \lambda(x)]$:

$$\delta W \equiv \delta_r W + \delta_g W + \delta_\lambda W = 0. \tag{6.2}$$

For our choice of the action for the quantum field the one-loop part Γ does not depend on the dilaton field $r(x)$. Therefore, a variation of W with respect to $r(x)$ is exactly the same as for the classical action W_{cl} , $\delta_r W = \delta_r W_{cl}$ [see Eq. (3.9)], where now $r(x_+) = \bar{r}_+$ is the quantum value of the dilaton field on the horizon. This means that the extremum configuration satisfies the condition

$$\frac{2}{g'(x_+)} \equiv \beta_H = \bar{\beta}, \tag{6.3}$$

i.e., the extremum as in the classical case is attained on the regular manifold without conical singularity¹. The extremum

¹Note that in principle the one-loop effective action Γ can be a functional of both $g(x)$ and $r(x)$ leading to a more complicated equation than Eq. (6.3). In consequence, the extremum configuration can be singular ($\beta \neq \beta_H$). We do not consider this possibility here.

of functional W describes the quantum-corrected black hole configuration, the perturbative form of which we found in Sec. V [Eqs. (5.8), (5.9), and (5.15)].

In variation with respect to metric, $\delta_g W$, the terms depending on $\delta g'(x_+)$ and $\delta g'(L)$ are absent in the same manner as in the classical case [see Eq. (3.10)]. Thus, for the one-loop effective action W we also have a well-defined variational procedure when the contribution of variations of the normal derivatives of metric at the external boundary ($x=L$) and at the tip of the cone ($x=x_+$) are compensated by the corresponding boundary terms.

Calculating the off-shell quantities (6.1) it is convenient to write metric in the Schwarzschild-like form:

$$ds^2 = g(x) d\tau^2 + g^{-1}(x) dx^2, \quad (6.4)$$

where $0 \leq \tau \leq 2\pi\bar{\beta}$. This can always be done using the residual gauge freedom in Eq. (3.1) allowing us to choose the coordinate system where $\lambda(x)=0$. This change of coordinates $x \rightarrow \int x e^{-\lambda(x)} dx$ must be accompanied by the corresponding change of the integration limits (x_+, L). On-shell they become dependent on r_B and β_H . However, for calculation of coordinate-invariant off-shell quantities (like effective action), the using of Eq. (6.4) instead of the general form (3.1) is just a convenient choice of the coordinate system. The corresponding Polyakov-Liouville action Γ reads

$$\begin{aligned} \Gamma[g] = & \frac{1}{24} \int_{x_+}^L \left(\frac{2}{\bar{\beta}g} - \frac{\bar{\beta}}{2} \frac{g'^2}{g} \right) dx + \frac{1}{12} \left(\alpha + \frac{(1-\alpha^2)}{2\alpha} \right) \psi(x_+) \\ & - \frac{\bar{\beta}}{8} g'(L) + \Gamma_0, \end{aligned} \quad (6.5)$$

where $\alpha = \beta/\beta_H$; $\beta_H = 2[g'(x_+)]^{-1}$, and $\psi(x)$ is defined by Eq. (4.22). It should be noted that Eq. (6.5) is divergent at the lower limit. Taking the regularization $x^+ \rightarrow x^+ + \epsilon$ we have, for the divergent part of Eq. (6.5),

$$\Gamma_{\text{div}} = \ln \epsilon \frac{(1-\alpha)^2}{24\alpha^2}. \quad (6.6)$$

This is the physical divergence due to the infinity of the renormalized $T_{\mu\nu}$ [Eqs. (4.15) and (4.16)] at the tip of the cone (for $\bar{\beta} \neq \beta_H$). Note that Γ_{div} is proportional to $(1-\alpha)^2$. Hence the divergence does not affect physical quantities calculated at the Hawking temperature ($\bar{\beta} = \beta_H$). In principle, one can regularize this divergence by subtracting in Eq. (6.5) the Polyakov action calculated for the Rindler space with metric function $g_R(x) = (2/\beta_H)(x-x_+)$. But we do not do this here.

Equation (6.3) allows us to calculate the energy E for the equilibrium state (for $\bar{\beta} = \beta_H$)

$$E = E_{\text{cl}} + E_q,$$

where the classical part E_{cl} takes the form (4.5) while the quantum part reads

$$\begin{aligned} E_q = & \frac{1}{2\pi g^{1/2}(L)} \partial_{\bar{\beta}} \Gamma|_{\bar{\beta}=\beta_H} = \frac{1}{96\pi} \int_{x_+}^L \frac{1}{g} \left(\frac{4}{\beta_H^2} - g'^2(x) \right) dx \\ & - \frac{1}{16\pi g^{1/2}(L)} g'(L). \end{aligned} \quad (6.7)$$

For the quantum-corrected metric obtained in the previous section $g'(L)$ vanishes in the limit $L \rightarrow \infty$. Therefore, we will neglect such a term below.

Analogously, we have for the entropy in the equilibrium state

$$S = \frac{\pi \bar{r}_+^2}{G} + S_q, \quad (6.8)$$

where

$$\begin{aligned} S_q = & (\beta \partial_{\beta} - 1) \Gamma|_{\beta=\beta_H} = -\frac{1}{12} \psi(x_+) \\ = & \frac{1}{12} \int_{x_+}^L \frac{dx}{g(x)} \left(\frac{2}{\beta_H} - g'(x) \right) + \frac{1}{6} \ln \frac{\beta_H g^{1/2}(L)}{z_0} + c(z_0). \end{aligned} \quad (6.9)$$

In Eqs. (6.8) and (6.9), \bar{r}_+ and β_H are quantum position of the horizon and quantum inverse Hawking temperature, respectively, and $g(x)$ is the metric of the quantum black hole. Note that both E_q and S_q are free of divergences at the lower limit. For a metric written in the conformally flat form $g_{\mu\nu} = e^{2\sigma} \delta_{\mu\nu}$, we have $\psi(x) = -2\sigma(x)$ and the entropy (6.9) coincides with that previously obtained in [36,6,23].

For the energy functional we have

$$E = \frac{1}{2Gg^{1/2}(L)} \int_{x_+}^L (G_0^0 + T_0^0) dx + \frac{1}{12\pi\beta_H g^{1/2}(L)} + E_{\text{surf}}, \quad (6.10)$$

where the surface term E_{surf} is the same as in Eq. (3.16).

Remembering that the temperature $T = [2\pi\beta_H g^{1/2}(L)]^{-1}$ is fixed on the external boundary we obtain that when the equations of motion (5.2) hold E reduces to

$$E = E_{\text{surf}} + \frac{T}{6}, \quad (6.11)$$

or, in invariant form,

$$E = \frac{T}{G} \int_{\partial M} r n^\alpha \partial_\alpha r + \frac{T}{6} = -\frac{1}{G} (rg^{1/2})_{r=L} + \frac{T}{6}. \quad (6.12)$$

Note that both the terms in Eq. (6.12) are defined on the external boundary $r=L$.

Subtracting now the energy of the background g_0 we obtain

$$E[g] - E[g_0] = \frac{L}{G} [r(g_0^{1/2} - g^{1/2})]_{r=L} + \frac{1}{6} (T - T_0), \quad (6.13)$$

where $T_0 = [2\pi\beta_H^0 g_0^{1/2}(L)]^{-1}$ is the temperature of the background metric. The temperature T which enters Eqs. (6.11) and (6.12) is measured at the external boundary. Nevertheless the terms which contain it originated from the horizon when one integrates by parts in passage from Eq. (6.7) to Eq. (6.10). Thus, $T/6$ in Eqs. (6.11) and (6.12) is a consequence of the black hole topology. In the non-black-hole case (hot space) this term is absent. Taking $T_0 = T$, the second contribution in Eq. (6.13) due to differences of temperatures vanishes and we get the classical expression (4.9) for the energy. But now g and g_0 are the corresponding quantum corrected metrics.

The above expressions for the energy and entropy were given for the static metric in the form (6.4). The quantum-corrected metric found in Sec. V takes this form by means of the coordinate transformation $r \rightarrow x(r)$, $\partial_r x = e^{\Phi(r)}$, and identification $g(x) = f e^{2\Phi}$. Since $\Phi(L) = 0$, near the boundary $r = L$ we have $x \approx r$ and $g(L) = f(L)$.

A. Mass of the quantum-corrected black hole

The quantum-corrected solution (5.15) and (5.16) found in the previous section behaves for large size L of the box as

$$g(L) \approx 1 - \frac{2MG}{L} - \frac{2\kappa}{\beta_H^2} - \frac{4MG\kappa}{L\beta_H^2} \ln \frac{L}{l} + O\left(\frac{1}{L}\right). \quad (6.14)$$

We see that in the limit $L \rightarrow \infty$ the metric function on the boundary of the box $g(L)$ goes to the constant value $g_0 = 1 - 2\kappa/\beta_H^2$ rather than to 1. Introducing the Planck temperature $T_{\text{Pl}} = (2\pi r_{cr})^{-1}$ this can be rewritten as $g_0 = 1 - (T/T_{\text{Pl}})^2$. We see that the modification of the asymptotic behavior of g and of the background is essentially due to temperature effects. Indeed, if we would take background $g_0 = 1$ as in classics and apply Eq. (6.13) for the metric (6.14) we would obtain for the energy the divergent term $E_{th} = (\pi/6)T^2 L$ which is the energy of the hot gas surrounding the black hole.

We can interpret this as follows. The system under consideration represents a rather complicated interaction of two objects: black hole and hot gas. Far from the horizon the effect of the gas is more important, while near the horizon the hole dominates. Therefore, extensive characteristics (such as energy or entropy) of the system presumably contain different contributions due to these two subsystems. The contribution of the hot gas can be identified and eliminated through its dependence on the size of the system L . On the other hand, the contribution of the hole itself does not depend on L .

It is a remarkable fact that the contribution of the hot gas can be subtracted and the contribution of the hole itself extracted by an appropriate choice of the reference metric² in

²This has been demonstrated for the string-inspired 2D model in [10].

the expression (6.13). Indeed, let us choose $g_0 = 1 - 2\kappa\beta_H^{-2}$ for the reference metric. Then we get for the energy³

$$E = M + \frac{\kappa M}{\beta_H^2} \eta, \quad (6.15)$$

where $\eta = 1 + 2\ln(L/l)$. As we can see the part E_{th} disappeared in Eq. (6.15), however the logarithmically divergent term is still present. We think that this divergence is due to the infrared behavior typical of massless fields. One might expect that it is absent when massive matter is considered. Therefore, we will keep the size of the box L regularizing this infrared behavior to be finite though large enough with respect to the characteristic size of the hole, $L \gg r_+$. We can rewrite Eq. (6.15) in the form

$$E = M \left[1 + \frac{1}{2} \left(\frac{T}{T_{\text{Pl}}} \right)^2 \eta \right]. \quad (6.16)$$

Comparing expressions (6.15) and (6.16) we can conclude that the [first-order (in G)] quantum corrections are identical to the temperature corrections to the mass of the hole.

B. Entropy of the quantum-corrected black hole

Substituting the classical metric function $g_{cl}(r) = (r - r_+)(r - r_-)r^{-2}$ into the expression for S_q we find that

$$S_q = \frac{\pi}{3} T_H (L - r_+) - \frac{1}{12} \left(\frac{r_-}{r_+} \right)^2 \ln \left(\frac{L - r_-}{r_+ - r_-} \right) + \frac{1}{12} \ln \left(\frac{L - r_+}{r_+ - r_-} \right) + \frac{1}{6} \ln \frac{r_+}{z_0}, \quad (6.17)$$

where z_0 is the proper cone generator length appearing in Eq. (4.9). Again, as for the calculation of the energy, we observe that S_q is divergent in the limit $L \rightarrow \infty$. The first, linearly divergent, term on the r.h.s. of Eq. (6.17) coincides with the entropy of the 2D hot gas contained in the box with size $(L - r_+)$ and temperature T_H , $S_{th} = (\pi/3)T_H(L - r_+)$. We may subtract the hot gas contribution S_{th} from the expression of the entropy since we are interested in the entropy of the hole itself.

In Eq. (6.8) the first term is defined with respect to the quantum-corrected radius of the horizon, \bar{r}_+ . Near the outer horizon $r = r_+$ the quantum-corrected metric (5.15) and (5.16) reads as $f(r) = (r - \bar{r}_+)(r - \bar{r}_-)r^{-2}$, where $\bar{r}_\pm = r_\pm \pm \kappa r_\pm^q$, and r_\pm are classical values. Therefore, in S_q (which is really proportional to \hbar) we may take the quantum-corrected values \bar{r}_\pm instead of the classical one. Then, taking the limit $L \rightarrow \infty$, we derive the complete quantum entropy of the hole in terms of the quantum-corrected horizon values \bar{r}_\pm :

³Applying formula (6.13) to calculate the energy E we must take into account two different regimes: the perturbative expansion in κ and the limit $L \rightarrow \infty$. Therefore, our steps are the following: we first expand the expression (6.12) with respect to κ for L fixed and then take the limit $L \rightarrow \infty$.

$$S = \frac{\pi \bar{r}_+^2}{G} + \frac{1}{12} \left[1 - \left(\frac{\bar{r}_-}{\bar{r}_+} \right)^2 \right] \ln \frac{L}{(\bar{r}_+ - \bar{r}_-)} + \frac{1}{6} \ln \frac{\bar{r}_+}{z_0}. \quad (6.18)$$

This illustrates the modification of the classical “entropy - horizon area” relation at the quantum level.⁴

A few regimes are of special interest. The first one is the extremal limit, $\bar{r}_+ \rightarrow \bar{r}_-$. Note that the correction to the mass (6.15) and (6.16) vanishes then. On the other hand, S_q [Eq. (6.17)] has the well-defined limit

$$S_q^{\text{ext}} = \frac{1}{6} \ln \frac{\bar{r}_+}{z_0} = \frac{1}{12} \ln \left(\frac{A_+}{\pi z_0^2} \right) \quad (6.19)$$

giving the logarithmic correction to the entropy. In the other regime we take $\bar{r}_- = 0$ (uncharged hole) and get for the entropy

$$S = \frac{A_+}{4G} + \frac{1}{24} \ln \frac{A_+}{\pi z_0^2}, \quad (6.20)$$

where $A_+ = \pi \bar{r}_+^2$ is the area of the horizon and we omitted a term $\sim \ln L/z_0$. This result is similar to that obtained in [8] for the four-dimensional Schwarzschild black hole.

It is not quite clear in which phenomena involving black holes the logarithmic corrections to the entropy (6.17)–(6.20) might be important. We may speculate that they play some role in the final stage of black hole evaporation. However, this problem needs further investigation.

⁴One can expect that the geometry drastically changes near the inner horizon r_- due to quantum corrections as was previously indicated in [37]. As a result, the inner horizon area probably becomes a nonanalytical function of the quantum perturbation parameter κ . Therefore, \bar{r}_- in the expression (6.18) is not a real inner horizon radius but as it is seen from the form of the metric in the region $r \geq r_+$.

VII. CONCLUSION

In concluding, several remarks are in order. The entropy of a black hole in classical theory is determined by data on the horizon surface Σ . In four-dimensional Einstein gravity it is just the area of Σ . In an R^2 theory of gravity the entropy is given by an integral over Σ of the curvature tensors projected onto the subspace normal to Σ [38,23]. When the quantum matter contribution is taken into account we find, at least in the 2D case, that the correction to the entropy is also given by some data on the horizon, namely by $\psi|_{\Sigma} = \psi(x_+)$ [see Eqs. (6.8) and (6.9)] [36,23]. In fact, the quantum correction contains the contribution of the hot gas surrounding the hole and a correction to the entropy of the hole itself. The value of $\psi(x_+)$ involves both of them [see Eqs. (6.17) and (6.18)]. It may be unexpected that information on the hot gas is encoded in data at the horizon located far from the region where the gas really contributes. But this becomes less surprising if we recall that the function $\psi(x)$ is in fact a nonlocal object [$\psi = \square^{-1}R$, see Eq. (4.4)] and its value at one point can, in principle, contain information on the whole space. It is not clear whether this is a general rule, applicable to the four-dimensional case as well. In the two-dimensional model, which is a reduction of the 4D theory, we obtain the one-loop entropy of the hole (6.18) which is a rather complicated function of the quantum-corrected geometry. Probably, there must be an equivalent derivation of the formula (6.18) in terms of the 4D geometric (presumably nonlocal) invariants integrated over Σ .

Also it is of interest to make the derivation of entropy presented in this paper directly in four dimensions. This is a much more complicated problem. However, some scaling arguments such as that given in [8] might be helpful in this project.

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