Black hole entropy: Off shell versus on shell

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(Received 22 December 1995)

Different methods of calculation of quantum corrections to the thermodynamical characteristics of a black hole are compared. The general relation between on-shell and off-shell approaches is discussed. We consider the simplified 2D model with the dilaton gravity and show in its framework that the observable thermodynamical black hole entropy can be presented in the form $S^{\mathrm{TD}}=\pi\overline{r}_+^{\,2}+S^{\mathrm{SM}}-S^{\mathrm{SM}}_{\mathrm{Rindler}}$. Here, $\overline{r_+}$ is the radius of the horizon shifted because of the quantum back reaction effect, S^{SM} is the statistical-mechanical entropy, determined as $S^{\mathrm{SM}}=-\,\mathrm{Tr}(\hat{\rho}^H\ln\hat{\rho}^H)$ for the density matrix $\hat{\rho}^H$ of a black hole, and $S^{\mathrm{SM}}_{\mathrm{Rindler}}$ is the analogous entropy in the Rindler space. [S0556-2821(96)00516-4]

PACS number(s): 04.70.Dy, 04.62.+v

I. INTRODUCTION

According to the thermodynamical analogy in black hole physics, the entropy of a black hole in the Einstein theory of gravity is

$$S^{\mathrm{BH}}=A^H/(4l_{\mathrm{P}}^2), \tag{1.1}$$

where A^H is the area of a black hole horizon and $l_{\mathrm{P}}=(\hbar G/c^3)^{1/2}$ is the Planck length [1–4]. In black hole physics the Bekenstein-Hawking entropy S^{BH} plays basically the same role as in the usual thermodynamics. It can be determined by the response of the free energy of a system containing a black hole to the change of the temperature of the system.

In the Euclidean approach [5–9] the free energy F is directly related to the Euclidean action calculated for the regular Euclidean solution of the vacuum Einstein equations (the Gibbons-Hawking instanton). According to the first law of thermodynamics, the thermodynamical entropy of a black hole S^{TD} is defined by the relation

$$dF=-S^{\mathrm{TD}}dT, \tag{1.2}$$

where T is the temperature of the system containing a black hole. The free energy F, in addition to the classical (tree-level) contribution, includes quantum (one-loop) corrections. For this reason the thermodynamical entropy, in addition to the classical (tree-level) part S^{BH}, acquires also a quantum correction S_1^{TD}:

$$S^{\mathrm{TD}}=S^{\mathrm{BH}}+S_1^{\mathrm{TD}}. \tag{1.3}$$

To find S^{TD} one must compare two equilibrium configurations. That is why all the calculations which are required to determine S^{TD} can be made by using the regular Gibbons-Hawking instanton as the background metric. One usually refers to this type of calculations as to the *on shell* method.

The fundamental problem of black hole thermodynamics is its statistical-mechanical foundation. The problem consists of the following three parts: (1) a definition of internal degrees of freedom of a black hole; (2) the calculation of the statistical-mechanical entropy S^{SM} of a black hole defined as $S^{\mathrm{SM}}=-\,\mathrm{Tr}(\hat{\rho}^H\ln\hat{\rho}^H)$ by counting the dynamical degrees of freedom described by the black hole density matrix $\hat{\rho}^H$; and (3) establishing the relation between the statistical-mechanical S^{SM} and the thermodynamical S^{TD} entropies.

In order to escape possible misleadings, let us note that we use the name ''statistical-mechanical entropy'' in order to stress that the quantity S^{SM} is calculated according to the standard statistical-mechanical rules. As for the density matrix $\hat{\rho}$, its form and properties depend on the concrete model. In the present paper we restrict ourselves by considering the class of the models in which the internal degrees of freedom of a black hole are identified with its quantum excitations. This idea has different realizations (see, e.g., Refs. [10,11] and references therein) and it has been widely discussed recently. The common feature of these models is that the corresponding density matrix $\hat{\rho}$ which enters the consideration is thermal. There is enormous number of papers, where the

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statistical-mechanical entropy has been calculated for different black hole models. The main purpose of our paper is to establish the relation between the results of these calculations and the observable thermodynamical black hole entropy S^{TD}.

It should be stressed that the problem of relations between S^{TD} and S^{SM} is very nontrivial for black holes. The quantities S^{TD} and S^{SM} are equal for the usual thermodynamical systems. Black holes possess a property which singles them out of the other thermodynamical systems. Namely, in a state of thermal equilibrium a mass m of a black hole is a universal function of a temperature T. But the mass uniquely determines the geometry of a black hole, and hence the internal parameters of the Hamiltonian describing its quantum excitations. This property has two important consequences: (i) S^{TD} and S^{SM} do not coincide for a black hole [12], and (ii) calculation of S^{SM} and its comparison with S^{TD} require *off-shell* methods. The latter means that one needs to consider the temperature T and the mass of a black hole m as independent parameters. The problem which arises is that when $T \neq T^{\mathrm{BH}} \equiv (8\pi m)^{-1}$, there is no regular complete vacuum Euclidean solutions. For this reason it is necessary either to consider the background metric which is not a solution of the vacuum gravitational equations, or to exclude some region of spacetime near the horizon and to make a solution incomplete. In both cases the calculation of the free energy meets problems. Moreover, the result may depend on the chosen concrete off-shell procedure [13].

In this paper we obtain the relation between different definitions of the black hole entropy. We also discuss and compare different off-shell methods (brick wall, conical singularity, blunt cone, and volume cutoff), and their relations to the on shell approach. We illustrate these relations for a simplified two-dimensional model, where all the calculations can be performed exactly. It is explicitly demonstrated that the thermodynamical entropy S^{TD} of a black hole differs from the statistical-mechanical entropy S^{SM}. Although, the statistical-mechanical interpretation of the tree-level Bekenstein-Hawking entropy remains the problem for the models we are dealing with in the present paper, we can find the relations between one-loop corrected S^{TD} and S^{SM}. One of the main results is the observation that in the considered two-dimensional (2D) model the one-loop contribution S_1^{TD} of a quantum field to the thermodynamical entropy can be presented in the form

$$S_1^{\mathrm{TD}} = S^{\mathrm{SM}} - S^{\mathrm{SM}}_{\mathrm{Rindler}} + \Delta S. \tag{1.4}$$

Here, $S^{\mathrm{SM}}_{\mathrm{Rindler}}$ is the statistical-mechanical entropy calculated in the Rindler space, and ΔS is an additional, finite correction caused by the shift of the black hole horizon because of quantum effects. The entropy calculated using the brick-wall and volume cutoff methods is directly related with S^{SM}. This quantity is divergent [in two-dimensional (2D) case] as $\ln\epsilon$, where ϵ is the proper distance to the horizon. On the other hand, the entropy calculated using the conical singularity and blunt-cone methods coincides with the difference $S^{\mathrm{SM}} - S^{\mathrm{SM}}_{\mathrm{Rindler}}$. It is finite because logarithmical divergence in S^{SM} is exactly canceled by the divergence of the Rindler entropy $S^{\mathrm{SM}}_{\mathrm{Rindler}}$.

It is well known that one-loop effective action which defines the free energy contains local ultraviolet divergences. In order to work with well-defined finite quantities it is necessary to renormalize it. Usually, one assumes that the bare classical action contains the same local structures that arise in the one-loop calculations. In the procedure of the renormalization one excludes the local one-loop divergences by a simple redefinition of coupling constants of the classical action. In our approach we assume that this renormalization procedure has been done from the very beginning. We use renormalized observable quantities as parameters of on shell solutions. In this case the renormalized one-loop effective action is finite (at least on shell). Quantum effects which change this solution can be considered as small perturbations for black holes with mass much larger than the Planckian mass. This also allows us to restrict ourselves by considering only those off-shell solutions which are close to the renormalized on-shell one [17]. As a result of our analysis we find out that all thermodynamical characteristics of a black hole expressed in terms of observable parameters are finite and their definition does not require the knowledge of physics at Planckian scales.

The paper is organized as follows. In Sec. II we remind the main features of the Euclidean approach and give the general definition of the thermodynamical entropy which is used throughout this paper. The description of a two-dimensional model is given in Sec. III. This section also contains the derivation of the on shell free energy and the thermodynamical entropy for this model. The general scheme of the off-shell methods is discussed in Sec. IV. The off-shell effective action, free energy, and statistical-mechanical entropy are exactly calculated for four, most common off-shell approaches: brick-wall (Sec. V), conical singularity (Sec. VI), blunt-cone (Sec. VII), and volume cutoff (Sec. VIII) methods. Section IX includes the comparison of the off-shell expressions for free energy and entropy, as well as the relation between statistical-mechanical and thermodynamical entropies of a black hole. Section X contains concluding remarks. Important results concerning conformal transformations of the effective action in the presence of conical singularities, derivation of the effective action on a cylinder, and the role of the vacuum polarization effect in the brick-wall model, which are used in the main text, are collected in appendices.

II. EUCLIDEAN APPROACH AND THERMODYNAMICAL ENTROPY

The starting point of the Euclidean approach to the black hole thermodynamics is the partition function $Z(\beta)$ and the effective action $W(\beta)$ which, for a canonical ensemble in the presence of black holes, are defined by the path integral

$$e^{-W(\beta)} = Z(\beta) = \int [D\phi] e^{-I[\phi]}. \tag{2.1}$$

Here, $I[\phi]$ is the Euclidean classical action and all the physical variables ϕ, including the gravitational field $g_{\mu\nu}$, are assumed to be periodic or antiperiodic, depending on their statistics, in the Euclidean time τ with the period β_∞. As usual, the class of metrics involved in Eq. (2.1) is supposed to be asymptotically flat. The parameter β_∞ has the meaning of the inverse temperature measured at the spatial infinity. It is also assumed that the integration measure $[D\phi]$ is defined as the covariant measure.

The standard way to calculate W is to use quasiclassical approximation. Thus, if ϕ_0 is a stationary point of $I[\phi]$,

$$\left.\frac{\delta I}{\delta\phi}\right|_{\phi=\phi_0}=0, \tag{2.2}$$

then one has the decomposition

$$I[\phi_0+\tilde{\phi}]=I[\phi_0]+I_2[\tilde{\phi}]+\cdots, \tag{2.3}$$

where I_2 is a quadratic in fluctuations $\tilde{\phi}$ part of the linearized action and the ellipsis in the right-hand side denotes the terms of the higher order in $\tilde{\phi}$. Using this relation, one gets

$$Z(\beta)=e^{-I[\phi_0]}\int[D\tilde{\phi}]e^{-I_2[\tilde{\phi}]}\equiv e^{-I[\phi_0]}Z_1(\beta). \tag{2.4}$$

The result of the Gaussian integration over $\tilde{\phi}$ in Eq. (2.4) can be expressed in terms of the determinants of the corresponding wave operators D_j for the different spins j:

$$Z_1(\beta)\equiv Z_1[\phi_0(\beta)]=\prod_j\{\det[-\mu^2 D_j(\phi_0)]\}^{\mp 1/2}. \tag{2.5}$$

Operators D_j are determined by the quadratic part $I_2=\frac{1}{2}\int dx\sqrt{g}\tilde{\phi}D_j\tilde{\phi}$ of the action and their explicit form depends on the spin j. For instance, for the conformally invariant massless scalar field in d-dimensional space $D_0=\triangle-(d-2)[4(d-1)]^{-1}R$, where $\triangle=\nabla_\mu\nabla^\mu$ is the Laplace operator and R is the scalar curvature. A constant μ^2 in Eq. (2.5) is an arbitrary renormalization parameter with the dimension of the length. It does not depend on the field configuration ϕ. Equation (2.5) enables one to represent the effective action in the one-loop approximation as the sum

$$W(\beta)=I[\phi_0(\beta)]-\ln Z_1(\beta)\equiv I[\phi_0(\beta)]+W_1[\phi_0(\beta)]. \tag{2.6}$$

The one-loop contribution [20] $W_1[\phi_0]$ to the effective action is ultraviolet divergent and, as usual, the classical action I is assumed to be chosen in such a way that the corresponding local divergences of W_1 can be removed by simple redefinition of the coupling constants in I. From now on we suppose that it has been done and that the classical action is written in terms of renormalized coefficients, ϕ_0 is its extremum, and W_1 is the *renormalized* one-loop action [21]. The ambiguity in the choice of the parameter μ in Eq. (2.5) corresponds to a freedom in the choice of *finite* counterterms which can be added to the action after renormalization.

To apply this general scheme to a black hole we assume that it is nonrotating, uncharged, and that there is no spontaneous symmetry breaking, so that average values of all fields except the gravitational one vanish. Also, it is worth taking the renormalized cosmological constant to be zero to provide an asymptotically flat black hole solution g_0 of the (vacuum) gravitational equations. The solution represents a Gibbons-Hawking instanton which is regular at the Euclidean horizon. For the Einstein theory such an instanton is described by the Schwarzschild metric and depends only on one constant, mass m of a black hole. The condition of regularity of this metric at the horizon implies that $\beta_\infty=\beta_H=8\pi m$.

When considering quantum corrections it is worth keeping in mind that a system for the chosen boundary conditions (periodicity in τ) necessarily consists of a black hole in thermal equilibrium with a surrounding thermal radiation which also contributes into observable thermodynamical quantities. This contribution is infinite for the thermal bath of the infinite size. Moreover, an equilibrium of a black hole with an infinite bath is unstable. For this reason it is important from the very beginning to consider a black hole surrounded by a boundary surface B of a finite size [7–9]. We assume this surface cannot be penetrated by fields. This is provided by the corresponding boundary conditions on it. For simplicity, B is assumed to be spherical (of a radius r_B) and a hole to be located in the center. For the Schwarzschild black hole, thermal stability is guaranteed if $r_B<3m$. Finally, in such a formulation of the problem the parameter β is the inverse temperature measured on B. Further, we suppose that all the necessary requirements of this kind are satisfied and we omit their discussion.

Equation (2.6) contains the renormalized effective action W calculated on a particular classical solution. This renormalized action itself is defined as a functional,

$$W[\phi]=I[\phi]+W_1[\phi], \tag{2.7}$$

for an arbitrary field ϕ with appropriately chosen boundary conditions. The extremum $\bar{\phi}$ of this functional

$$\left.\frac{\delta W}{\delta\phi}\right|_{\phi=\bar{\phi}}=0 \tag{2.8}$$

describes a modified field configuration which differs from a classical solution by quantum corrections: $\bar{\phi}=\phi_0+\hbar\phi_1$. The important observation is that, if one is interested in the one-loop effects, the difference between the values of W on ϕ_0 and $\bar{\phi}$ turns out to be of the second order in the Planck constant $\hbar$:

$$W(\beta)=W[\phi_0(\beta)]=W[\overline{\phi(\beta)}]+O(\hbar^2). \tag{2.9}$$

This follows from Eq. (2.8), provided the quantum-corrected and classical solutions obey the same boundary conditions.

The *thermodynamical entropy* of a black hole $S^{\rm TD}$ is defined by the response of the free energy $F(\beta)=\beta^{-1}W(\beta)$ to the change of the inverse temperature β for fixed r_b:

$$S^{\rm TD}(\beta)=\beta^2\frac{dF(\beta)}{d\beta}=\left(\beta\frac{d}{d\beta}-1\right)W(\beta). \tag{2.10}$$

We remind that the renormalized effective action $W(\beta)$ is calculated on shell, that is for $\beta_\infty=8\pi m$. The thermodynamical entropy $S^{\rm TD}$ can be written as

$$S^{\rm TD}=S_0^{\rm TD}+S_1^{\rm TD}. \tag{2.11}$$

It can be shown [7,9] that

$$S_0^{\rm TD} = \left(\beta \frac{d}{d\beta} - 1\right) I[\phi_0(\beta)] \quad (2.12)$$

coincides with the Bekenstein-Hawking entropy $S^{\rm BH}$ given by Eq. (1.1), while

$$S_1^{\rm TD}(\beta) = \left(\beta \frac{d}{d\beta} - 1\right) W_1[\phi_0(\beta)] \quad (2.13)$$

describes the quantum correction to it. This correction contains also the entropy of the thermal radiation outside the black hole as its part. By its construction the thermodynamical entropy $S^{\rm TD}$ is well defined and finite. All the calculations required to obtain this quantity can be performed *on shell*, that is, on a regular complete vacuum Euclidean solution of the gravitational equations. The parameters of this solution are expressed only in terms of the renormalized coupling constants.

III. DESCRIPTION OF THE MODEL: ON-SHELL RESULTS

In four dimensions the calculation of $S_1^{\rm TD}$ is a quite complicated problem. To discuss the properties of $S_1^{\rm TD}$ and its relation to $S^{\rm SM}$, it is instructive to consider a simplified two-dimensional model where the calculations can be done explicitly. Certainly, the explicit forms of these quantities in two and in four dimensions are different. Nevertheless, the study of 2D model allows us to make definite conclusions concerning the physically interesting case of a four-dimensional spacetime. To preserve the maximal similarity with the four-dimensional case we consider a 2D dilaton gravity described by the action

$$I = -\frac{1}{4}\int_{M^2} [r^2 R + 2(\nabla r)^2 + 2]\sqrt{\gamma} d^2x - \frac{1}{2}\int_{\partial M^2} r^2(k - k_0) dy + \frac{1}{2}\int \sqrt{\gamma}\varphi_{,\mu}\varphi^{,\mu}. \quad (3.1)$$

The 2D metric γ, dilaton field r, and a scalar field φ are dynamical variables of the problem. We denote by R the curvature of γ, and by k the extrinsic curvature of ∂M^2. This model is similar to the one which has been extensively studied [22] as an example of a renormalizable exactly solvable theory of two-dimensional dilaton gravity coupled to matter. In the absence of the scalar field φ, this action can be obtained from the 4D Euclidean Einstein action

$$I^{(4)} = -\frac{1}{16\pi}\int_{M^4} R^{(4)}\sqrt{g} d^4x - \frac{1}{8\pi}\int_{\partial M^4} (K^{(4)} - K_0^{(4)})\sqrt{h} d^3x, \quad (3.2)$$

by its reduction to the spherically symmetric metrics of the form

$$ds^2 = \gamma_{ab} dx^a dx^b + r^2 d\omega^2. \quad (3.3)$$

Here, γ_{ab} is a 2D metric, r is a scalar function on the two-dimensional manifold, and $d\omega^2$ is the line element on the unit sphere. $K_0^{(4)}$ is the standard subtraction term, and $k_0 = K_0^{(4)}$.

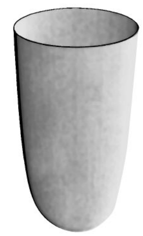

FIG. 1. Embedding diagram for a two-dimensional Gibbons-Hawking instanton. Regularity condition at the Euclidean horizon $r = r_+$ requires $\beta_\infty = \beta_H \equiv 8\pi m$.

Since the 2D action I is related with 4D action $I^{(4)}$ by the reduction procedure, the pair of fields (γ_0, φ_0), where $\varphi_0 = 0$ and γ_0 is a 2D Schwarzschild metric

$$ds^2 = f d\tau^2 + f^{-1} dr^2, \quad f = 1 - r_+/r, \quad (3.4)$$

is evidently the extremum of the functional I. The regularity condition at $r = r_+$ requires τ to be periodic with the period $\beta_H = 4\pi r_+$. The Gibbons-Hawking instanton, i.e., the regular complete Euclidean manifold with the metric (3.4), is shown in Fig. 1.

Consider a region M_B of the Gibbons-Hawking instanton within the external boundary Σ_B at $r = r_B$ (see Fig. 2). If the boundary conditions are fixed on the surface Σ_B, and β is the proper length of the line $r = r_B$, then the classical Euclidean action calculated for the region M_B and expressed in terms of the boundary conditions (β, r_B) is

$$I(\beta, r_B) = I[\gamma_0, \varphi_0] = 3\pi r_+^2 - 4\pi r_+ r_B + \beta r_B, \quad (3.5)$$

where r_+ is defined by the equation

$$\beta = 4\pi r_+ (1 - r_+/r_B)^{1/2}, \quad (3.6)$$

and β is the inverse temperature at $r = r_B$. In the limit $r_B \to \infty$, when $\beta = 4\pi r_+$, the classical action takes the simple form

$$I(\beta) = \frac{1}{16\pi}\beta^2. \quad (3.7)$$

In accordance with the general discussion of Sec. II, the one-loop contribution to the effective action is

$$W_1(\beta) = \tfrac{1}{2} \ln \det(-\mu^2 \Delta). \quad (3.8)$$

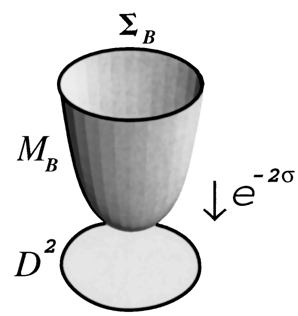

FIG. 2. A region M_B of the Gibbons-Hawking instanton with the external boundary Σ_B at $r = r_B$. This region is conformal to the 2D flat unit disk D^2.

Here, the renormalized determinant is taken for the region M_B of the 2D instanton (3.4). To make discussion more concrete we assume that the field φ obeys the Dirichlet boundary condition at the mirrorlike boundary Σ_B surrounding the black hole. The divergent part which has been removed from the action is

$$W_1^{\rm div}[M_B]=-\frac{1}{8\pi\delta}\int_{M_B}\sqrt{\gamma}d^2x+\frac{\ln\delta}{12}\chi[M_B], \quad (3.9)$$

$$\chi[M_B]=\frac{1}{4\pi}\left(\int_{M_B}R\sqrt{\gamma}d^2x+2\int_{\Sigma_B}k\sqrt{h}dy\right), \quad (3.10)$$

where δ is the parameter of the ultraviolet regularization and $\chi[M_B]$ is the Euler characteristic of the Gibbons-Hawking instanton M_B, which is the same as for a disk $\chi[M_B]=1$. To remove the volume divergence $\sim\int_{M_B}$, one must introduce in the bare classical action a cosmological constant λ, which we put after renormalization to be $-1/2$, see Eq. (3.1). Removing the other divergence in Eq. (3.9) requires introduction of the additional term in Eq. (3.1), but because it is just a topological invariant it can be neglected.

Using the conformal transformation, the one-loop effective action $W_1(\beta)$ can be found explicitly. Note that metric (3.4) can be represented in the form

$$ds^2=\left(1-\frac{r_+}{r}\right)d\tau^2+\left(1-\frac{r_+}{r}\right)^{-1}dr^2=e^{2\sigma}d\tilde{s}^2, \quad (3.11)$$

$$d\tilde{s}^2=\mu^2(x^2d\tilde{\tau}^2+dx^2). \quad (3.12)$$

Here,

$$\tilde{\tau}=\frac{\tau}{2r_+}, \quad 0\leqslant\tilde{\tau}\leqslant 2\pi,$$

$$x=\left(\frac{r-r_+}{r_B-r_+}\right)^{1/2}e^{(r-r_B)/2r_+}, \quad 0\leqslant x\leqslant 1, \quad (3.13)$$

and the conformal factor σ is defined as

$$\sigma(r)=\frac{1}{2}\left[\ln\left(\frac{r_B-r_+}{r}\right)+\frac{r_B-r}{r_+}+2\ln\left(\frac{2r_+}{\mu}\right)\right]. \quad (3.14)$$

In order to preserve the dimensionality, we introduce the parameter μ with the dimension of length into the flat space metric (3.12). The above conformal transformation

$$\gamma_{\mu\nu}\rightarrow\tilde{\gamma}_{\mu\nu}=e^{-2\sigma}\gamma_{\mu\nu} \quad (3.15)$$

is a map of the region M_B onto the flat 2D disk D^2 of the unit radius (measured in units of μ), see Fig. 2. It will be shown that the physical results do not depend on the particular choice of μ [23].

For a conformal field the transformation law of W_1 under this map can be obtained by an integration of a conformal anomaly. The corresponding formulas are collected in Appendix A. Denote by C the renormalized one-loop effective action for the unit disk D^2, Eq. (3.12), then using the relation (A9), we get

$$W_1(\beta,r_B)=\tilde{W}_1[\beta,y(\beta,r_B)], \quad (3.16)$$

where $y=r_+/r_B$ and

$$\tilde{W}_1(\beta,y)=\frac{1}{48}\left[-\frac{2}{y}+2\ln y+17-2y-13y^2\right]-\frac{1}{6}\ln\frac{\beta}{2\pi\mu}+C. \quad (3.17)$$

The relations (3.16) and (3.17) require some explanations. First of all, the one-loop effective action $W_1(\beta,r_B)$, in addition to the inverse temperature β at the boundary, also depends on its ''radius'' r_B. For given β and r_B, the gravitational radius r_+ is defined by the relation (3.6). To simplify the expressions we use the dimensionless variable $y=r_+/r_B$ instead of r_B. The relation (3.6) implies that this dimensional variable y is the function of β and r_B defined by the implicit relation

$$y(1-y)^{1/2}=\frac{\beta}{4\pi r_B}. \quad (3.18)$$

The one-loop contributions to the free energy F_1 and to the thermodynamical entropy $S_1^{\rm TD}$ are defined by the formulas

$$F_1(\beta,r_B)=\beta^{-1}W_1(\beta,r_B),$$

$$S_1^{\rm TD}=\beta\left.\frac{\partial W_1(\beta,r_B)}{\partial\beta}\right|_{r_B}-W_1(\beta,r_B). \quad (3.19)$$

The derivative of W_1 can be expressed in terms of the partial derivatives of $\tilde{W}_1$:

$$\left.\frac{\partial W_1(\beta,r_B)}{\partial\beta}\right|_{r_B}=\left.\frac{\partial\tilde{W}_1(\beta,y)}{\partial\beta}\right|_{y}+\left.\frac{\partial\tilde{W}_1(\beta,y)}{\partial y}\right|_{\beta}\left.\frac{\partial y}{\partial\beta}\right|_{r_B}, \quad (3.20)$$

where

$$\left.\frac{\partial y}{\partial\beta}\right|_{r_B}=\frac{2y(1-y)}{\beta(2-3y)}. \quad (3.21)$$

The latter equality results from Eq. (3.18). Using the relations (3.19)–(3.21), we finally obtain

$$S_1^{\rm TD}(y,\beta)=\frac{1}{48(2-3y)}\left[\frac{8}{y}-13y-28y^2+13y^3\right]-\frac{1}{24}\ln y+\frac{1}{6}\ln\frac{\beta}{2\pi\mu}-\frac{17}{48}-C. \quad (3.22)$$

This quantity is finite. The dimensionless constant C does not depend on the parameters of the system and reflects the ambiguity in the definition of the entropy. For further consideration this ambiguity is not important, so that this and other similar constants can be omitted. For a large value of the radius r_B of the boundary ($r_B\gg r_+$ or $y\ll 1$), the leading term in $S_1^{\rm TD}$ is $(\pi/3)\,r_B\beta^{-1}$. This leading term coincides with the entropy of the one-dimensional thermal gas of massless scalar quanta. It should be noted that we always consider the case when $r_B<3/2r_+$, so that the limit discussed above has only formal meaning. The quantity $S_1^{\rm TD}$ is infinite when $r_B=\frac{3}{2}r_+$. This singularity also results in the infinite heat capacity at $y=3/2$. One can expect the same behavior of these quantities in four-dimensional case.

IV. OFF-SHELL METHODS

In the above consideration we used the relation (3.6) which can be rewritten as $\beta_\infty=\beta_H$, where $\beta_\infty=\beta(1-r_+/r_B)^{-1/2}$ denotes the inverse temperature on the boundary Σ_B as seen from infinity, and $(1-r_+/r_B)^{1/2}$ is the redshift factor. β_H is the inverse Hawking temperature (also measured at infinity). The relation $\beta_\infty=\beta_H$ has evident meaning of the equilibrium condition between the thermal radiation and the black hole and it is this relation which is assumed when we are speaking about the on-shell quantities.

In the next sections we consider different off-shell approaches in which the condition $\beta_\infty=\beta_H$ is violated for the background geometries. The one-loop contribution to the effective action in these cases is the function of the three variables β,r_B,r_+: $W_1^\bullet(\beta,r_B,r_+,\dots)$. We use the superscript • to indicate that this quantity depends on the chosen off-shell procedure. The ellipsis in the argument of $W_1^\bullet$ indicates that it may also depend on some additional parameters, which are different for different off-shell procedures. These parameters are not important now and will be specified later.

In the general case, the off-shell entropy is defined by the response of the off-shell free energy $F^\bullet=\beta^{-1}W^\bullet$ on the change of the temperature, under the condition that the other parameters which specify the system (r_B) as well as the black hole (r_+) are fixed. According to this definition, the one-loop off-shell entropy is

$$S_1^\bullet=\beta\left.\frac{\partial W_1^\bullet}{\partial\beta}\right|_{r_B,r_+,\dots}-W_1^\bullet. \tag{4.1}$$

It is assumed that the on shell limit in Eq. (4.1) is taken at the end of the computation. This means that r_+ which enters $S_1^\bullet$ is put equal to its on shell value, determined by solving the corresponding gravitational equations.

It occurs that the explicit formulas for $W_1^\bullet$ and $S_1^\bullet$ are greatly simplified if, instead of r_B and r_+, the following dimensionless variables are used

$$y=y(r_B,r_+)=\frac{r_+}{r_B},$$

$$\alpha=\alpha(\beta,r_B,r_+)=\frac{\beta_\infty}{\beta_H}=\frac{\beta}{4\pi r_+\sqrt{1-\frac{r_+}{r_B}}}. \tag{4.2}$$

The variable α is the *off-shell parameter* so that the condition that a system is on shell reads $\alpha=1$. The parameter y is the ratio of the values of the dilaton field on the external boundary Σ_B and on the horizon. We shall use the notation

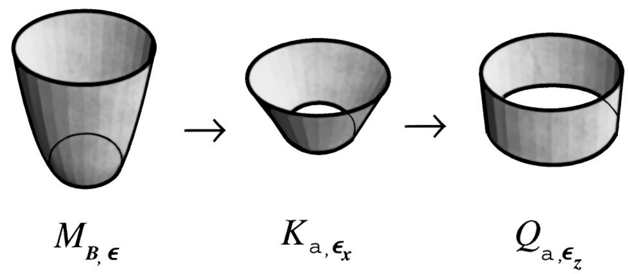

FIG. 3. Conformal maps of the region $M_{B,\epsilon}$ of the Gibbons-Hawking instanton onto the part K_{α,ϵ_x} of the cone C_α, and of the region K_{α,ϵ_x} onto the cylinder Q_{α,ϵ_z}. ϵ is the proper distance of the inner boundary Σ_ϵ of $M_{B,\epsilon}$ to the horizon. The parameter ϵ_x is the distance from Σ_B to the vertex of the cone along the cone generator, and ϵ_z is the length of the cylinder generator (both measured in the units of μ). The circumference length of the cylinder, as well as the circumference length of of the external boundary Σ_B of the cone, measured in units μ is $2\pi\alpha$.

$$W_1(\beta,r_B,r_+,\dots)=\widetilde{W}_1(\beta,\alpha(\beta,r_B,r_+),y(r_+,r_B),\dots). \tag{4.3}$$

For fixed values of r_+ and r_B, the quantity $y=r_+/r_B$ is also fixed, while Eq. (4.2) implies that α is proportional to β. Thus, one has

$$S_1^\bullet=\beta\left.\frac{\partial\widetilde{W}_1^\bullet(\beta,\alpha,y,\dots)}{\partial\beta}\right|_{\alpha,y,\dots}+\alpha\left.\frac{\partial\widetilde{W}_1^\bullet(\beta,\alpha,y,\dots)}{\partial\alpha}\right|_{\beta,y,\dots}-W_1^\bullet. \tag{4.4}$$

As earlier, it is assumed that after the calculations, one must put $\alpha=1$ in the right-hand side of this relation. Then, the corresponding on shell value of $S_1^\bullet$ depends only on the boundary conditions β and r_B. After these general remarks consider concrete off-shell methods.

V. BRICK-WALL MODEL

A. Effective action

As the first example of the off-shell procedure we consider the so-called *brick-wall model*, proposed by 't Hooft [24] and discussed later in many subsequent papers [25,18,26–28,30,19]. The basic idea of this method is to introduce at some small proper distance ϵ from the black hole horizon an additional mirrorlike boundary Σ_ϵ. Denote by $M_{B,\epsilon}$ the region located between Σ_B and Σ_ϵ (see Fig. 3). To be more specific, assume, following 't Hooft, that the field φ obeys the Dirichlet condition on both boundaries Σ_B and Σ_ϵ. The starting point of the brick-wall model is the partition function $Z_1^{\rm BW}(\beta)$ of massless scalar field in the region $M_{B,\epsilon}$ near the Schwarzschild black hole of the mass m:

$$\ln Z_1^{\rm BW}(\beta)=-\tfrac{1}{2}\ln\det(-\mu^2\triangle). \tag{5.1}$$

Here, β is the inverse temperature measured at Σ_B, ''ln det'' is understood as renormalized quantity, and $\triangle$ is the Laplace operator for the scalar field in the region $M_{B,\epsilon}$ with the Dirichlet boundary conditions. Because of the presence of the inner boundary Σ_ϵ, the region near the black hole horizon where the thermal gas cannot penetrate is completely excluded. For this reason the system is nonsingular for any relation between the parameters β and m, and the brick-wall model can be used for an off-shell extension. To distinguish the quantities calculated in this off-shell procedure we use the abbreviation BW as the superscript. The corresponding partition function Z_1^{BW} and action W_1^{BW} depend, in addition to β and r_B, on ϵ and the value r_+ of the dilaton field on the horizon. Our purpose, now, is to find $W_1^{\mathrm{BW}}(\beta,r_B,r_+,\epsilon)$.

Obviously, this problem can be reduced to the calculation of the effective action for some ''standard'' 2D flat region. We choose a cylinder as such a region (see Fig. 3).

It is convenient to make the conformal transformation into two steps.

First, use the map Eq. (3.15) with σ given by Eq. (3.14). Under this transformation, the metric takes the form

$$d\tilde{s}^2=\mu^2(x^2d\tilde{\tau}^2+dx^2),\quad 0\leqslant\tilde{\tau}\leqslant 2\pi\alpha,\quad \epsilon_x\leqslant x\leqslant 1. \tag{5.2}$$

The embedding diagram for this space is shown in Fig. 3. It is a part K_{α,ϵ_x} of the cone C_α between the surfaces Σ_B located at $x=1$ and Σ_ϵ at ϵ_x. The value of $x=\epsilon_x$ is related with the proper distance ϵ as

$$\epsilon_x=\epsilon\frac{2\pi\alpha}{\beta}\sqrt{y}\exp\left(\frac{y-1}{2y}\right), \tag{5.3}$$

where the parameters y and α are defined in Eq. (4.2).

Second, map K_{α,ϵ_x} onto a cylinder Q_{α,ϵ_z} with the metric $\mu^2(d\tilde{\tau}^2+dz^2)$:

$$d\tilde{s}^2=\mu^2(x^2d\tilde{\tau}^2+dx^2)=x^2[\mu^2(d\tilde{\tau}^2+dz^2)],\quad z=-\ln x. \tag{5.4}$$

The cylinder has the circumference length $2\pi\alpha$ and the length of its generator is $\epsilon_z=-\ln\epsilon_x$ (in the μ units) (see Fig. 3).

Thus, the effective action $W_1^{\mathrm{BW}}(\beta,r_B,r_+,\epsilon)$ can be obtained by conformal transformation, provided one knows the action $W_1[Q_{\alpha,\epsilon_z}]$ for the ''standard'' cylinder Q_{α,ϵ_z}. It can be shown (see Appendix B) that

$$W_1[Q_{\alpha,\epsilon_z}]=-\ln \mathrm{Tr}\, e^{-2\pi\alpha\mu\hat{H}}, \tag{5.5}$$

where $\hat{H}$ is the Hamiltonian for the scalar massless field on the interval $(0,\mu\epsilon_z)$ with the Dirichlet boundary conditions at the ends. Using this fact we get, for $\epsilon_z\gg 1$ (see Appendix B),

$$W_1[Q_{\alpha,\epsilon_z}]=-\frac{1}{12\alpha}\epsilon_z-\frac{1}{2}\ln\frac{\pi\alpha}{\epsilon_z}+o\left(\frac{1}{\epsilon_z}\right). \tag{5.6}$$

The scale parameter μ disappears from this expression because of the scale invariance of the action on the cylinder. The effective action $W_1[K_{\alpha,\epsilon_x}]$ for the region K_{α,ϵ_x} obtained from $W_1[Q_{\alpha,\epsilon_z}]$ by conformal transformation has the form

$$W_1[K_{\alpha,\epsilon_x}]=W_1[Q_{\alpha,\epsilon_z}]-\frac{\alpha}{12}\epsilon_z \tag{5.7}$$

while the transformation (3.15) gives

$$W_1[M_{B,\epsilon}]=W_1[K_{\alpha,\epsilon_x}]+\alpha f(y), \tag{5.8}$$

$$f(y)=-\frac{1}{48}\left(-\frac{2}{y}+2\ln y+2y+13y^2-13\right). \tag{5.9}$$

The final result is obtained by using the formulas (5.6)–(5.8). The effective action $W_1^{\mathrm{BW}}(\beta,r_B,r_+,\epsilon)$, written as the function of $(\beta,\alpha,y,\epsilon)$, is

$$W_1^{\mathrm{BW}}(\beta,r_B,r_+,\epsilon)=\tilde{W}_1^{\mathrm{BW}}(\beta,\alpha(\beta,r_B,r_+),y(r_B,r_+),\epsilon), \tag{5.10}$$

$$\begin{aligned}\tilde{W}_1^{\mathrm{BW}}(\beta,\alpha,y,\epsilon)=&\frac{1}{12}\left(\alpha+\frac{1}{\alpha}\right)\ln\frac{2\pi\alpha\epsilon}{\beta}-\frac{1}{2}\ln\frac{\pi\alpha}{\ln(\beta/2\pi\alpha\epsilon)}\\&+\frac{\alpha}{48}(15-2y-13y^2)\\&+\frac{1}{24\alpha}\left(1-\frac{1}{y}+\ln y\right)+o(\ln^{-1}(\beta/\epsilon)).\end{aligned} \tag{5.11}$$

For $\alpha=1$, i.e., on shell this action can be represented as the sum

$$\tilde{W}_1^{\mathrm{BW}}(\beta,\alpha=1,y,\epsilon)=\tilde{W}_1(\beta,y)+\frac{1}{6}\ln\epsilon-\frac{1}{2}\ln\frac{\pi}{\ln(\beta/2\pi\epsilon)}+o(\ln^{-1}(\beta/\epsilon)) \tag{5.12}$$

of the thermodynamical action $\tilde{W}_1(\beta,y)$ for the region M_B given by Eq. (3.17) and an additional term which arises because of the presence of the wall. The latter diverges logarithmically in the limit $\epsilon\to 0$ [29].

B. Entropy

The entropy S_1^{BW} for the brick-wall model is defined by Eq. (4.1) using W_1^{BW}. Written in terms of $(\beta,\alpha,y,\epsilon)$, it reads

$$\begin{aligned}S_1^{\mathrm{BW}}(\beta,\alpha,y,\epsilon)=&\frac{1}{12\alpha}\left(2\ln\frac{\beta}{2\pi\alpha\epsilon}-\ln y+\frac{1}{y}-1\right)\\&+\frac{1}{2}\ln\frac{\pi\alpha}{\ln(\beta/2\pi\alpha\epsilon)}+o(\ln^{-1}(\beta/\epsilon)).\end{aligned} \tag{5.13}$$

The on shell value of S_1^{BW} is obtained if one puts $\alpha=1$ in this expression.

Note that the renormalization parameter μ does not enter Eqs. (5.11) and (5.13), so that neither the brick-wall action W_1^{BW} nor the entropy S_1^{BW} depends on it. It happens because under a constant conformal transformation, the effective action acquires an addition proportional to the Euler characteristic of the manifold. But the topology of $M_{B,\epsilon}$ is the topology of a cylinder and its Euler number is zero. Thus, the effective action is invariant under the constant rescaling, and it does not depend on μ. On the other hand, the Euler characteristic of the complete regular instanton is the same as that of the disk D^2, and it does not vanish. As a result, the integral of the anomaly also does not vanish, and μ appears in the thermodynamical action and entropy as a parameter of the dimensional transmutation.

We show now that the brick-wall entropy (5.13) coincides with the statistical-mechanical entropy and can be represented in the form

$$S_1^{\rm BW}(\beta,\alpha,y,\epsilon)=-\,{\rm Tr}[\hat\rho_\epsilon^H(\beta)\ln\hat\rho_\epsilon^H(\beta)]. \qquad (5.14)$$

Here, $\hat\rho_\epsilon^H(\beta)$ is the thermal density matrix for the massless gas in the region $M_{B,\epsilon}$ near the black hole, β being the inverse temperature measured at Σ_B. In the 't Hooft's brick-wall model this thermal gas is identified with internal degrees of freedom of the black hole.

To prove Eq. (5.14) we obtain at first expression (5.13) for $S_1^{\rm BW}$ in a slightly different way. Equations (5.7) and (5.8) show that

$$W_1^{\rm BW}(\beta,r_B,r_+,\epsilon)=\alpha f(y)-\frac{\alpha\epsilon_z}{12}+W_1[Q_{\alpha,\epsilon_z}]. \qquad (5.15)$$

To get $S_1^{\rm BW}$, we keep the variables r_B, r_+, and ϵ fixed. Under these conditions, y does not depend on β, while α is proportional to β. As a result, the first two terms in Eq. (5.15) do not contribute into $S_1^{\rm BW}$, so that

$$S_1^{\rm BW}=\left(\alpha\frac{\partial}{\partial\alpha}-1\right)W_1[Q_{\alpha,\epsilon_z}]=\frac{1}{6\alpha}\epsilon_z+\frac{1}{2}\ln\frac{\pi\alpha}{\epsilon_z}+O(\epsilon_z^{-1}). \qquad (5.16)$$

It can be easily verified that this expression coincides with Eq. (5.13). Note that $W_1[Q_{\alpha,\epsilon_z}]$ is given by Eq. (5.5). The quantity $(1-\beta(\partial/\partial\beta))\ln{\rm Tr}\,e^{-\beta\hat H_L}$ can be identically rewritten as $-\,{\rm Tr}[\hat\rho_L(\beta)\ln\hat\rho_L(\beta)]$, where $\hat H_L$ is the Hamiltonian on the interval of the length L, and $\hat\rho_L(\beta)=\rho_0 e^{-\beta\hat H_L}$. Using this relation, we can present Eq. (5.16) in the form

$$S_1^{\rm BW}=-\,{\rm Tr}[\hat\rho_{\mu\epsilon_z}(2\pi\mu\alpha)\ln\hat\rho_{\mu\epsilon_z}(2\pi\mu\alpha)]. \qquad (5.17)$$

This relation explicitly demonstrates that $S_1^{\rm BW}$ is the entropy of the one-dimensional thermal gas on the interval $\mu\epsilon_z$ and with the temperature $(2\pi\mu\alpha)^{-1}$. [The parameter μ is absent in Eq. (5.16) for the reason explained above.]

This result can be used to prove the formula (5.14) because the density matrix $\hat\rho_{\mu\epsilon_z}(2\pi\mu\alpha)$ coincides with the black hole density matrix $\hat\rho_\epsilon^H(\beta)$. Indeed, we used conformal transformations which preserve the symmetry (a Killing vector) and do not affect the boundary conditions. Under these conditions the Hamiltonian of the conformal massless field is invariant, so that the density matrix is also invariant. Note, however, that scales we used to define the temperature and distance may change. In order to define energy, temperature, etc., we must fix the normalization of the Killing vector. For the problem in question we chose the condition $(\xi^2)_B=1$ at the external boundary Σ_B. If the conformal factor σ does not vanish on the boundary $(\sigma_B\neq0)$, one must rescale $\xi^\mu\to\tilde\xi^\mu=\exp(-\sigma_B)\xi^\mu$ to get $\tilde\xi^2=1$ at the boundary after the conformal transformation. We have

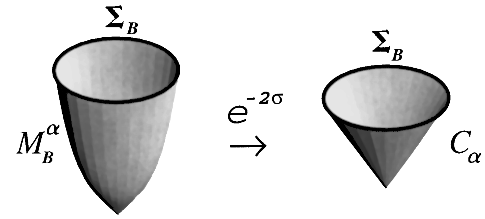

FIG. 4. Conformal map of a singular instanton M_β^α onto the standard cone C_α.

$$e^{-\beta\hat H_L}=e^{-\tilde\beta\tilde H_{\tilde L}}, \qquad (5.18)$$

where $\tilde\beta=\exp(-\sigma_B)\beta$, $\hat H=\exp(\sigma_B)\hat H$, and $\tilde L$ is the proper length of the interval in the conformally related metric $\tilde\gamma_{\mu\nu}=e^{-2\sigma}\gamma_{\mu\nu}$.

In particular for the conformal maps Eqs. (3.11) and (3.14) which we used as the first step, Eq. (5.18) implies

$$\hat\rho_\epsilon^H(\beta)=\hat\rho_{\mu\epsilon_x}^R(2\pi\mu\alpha). \qquad (5.19)$$

Here, $\hat\rho^H$ is the original black-hole density matrix, and $\hat\rho^R$ is a thermal density matrix in a Rindler space with the metric

$$d\tilde s^2=\mu^2[x^2d\tilde\tau^2+dx^2]=\left(\frac{X}{\mu}\right)^2dT^2+dX^2. \qquad (5.20)$$

The inverse temperature $2\pi\mu\alpha$ in the Rindler space is measured at the point of the boundary $X=\mu$, where the $g_{TT}=1$. The parameter $\mu\epsilon_x$ is the proper distance from the inner boundary to the horizon, measured in the Rindler metric. Note, that the proper distance is not conformal invariant. Finally, by mapping Rindler space onto the flat one [the corresponding transformation of the effective action from K_{α,ϵ_x} to Q_{α,ϵ_z} is given by Eq. (5.4)], one receives the identity

$$\hat\rho_{\mu\epsilon_x}^R(2\pi\mu\alpha)=\hat\rho_{\mu\epsilon_z}(2\pi\mu\alpha) \qquad (5.21)$$

between the Rindler density matrix and that on the interval. The statistical-mechanical formula (5.14) for $S_1^{\rm BW}$ follows from the identities (5.17), (5.19), and (5.21).

VI. CONICAL SINGULARITY METHOD

Instead of excluding the ϵ-domain near the horizon, one can work directly on the complete black hole geometry. However, if β_∞ differs from the Hawking value β_H, the spacetime is not anymore regular because of the presence of the conical singularity with the angle deficit $2\pi(1-\alpha)$ at the horizon $r=r_+$ (fixed point of the Killing vector). Such a space has the δ-like curvature located on the cone vertex. For this reason it is not a solution of the vacuum Einstein equations. We call such a space a *singular instanton* and denote it M_B^α see Fig. 4.

It is possible to develop the one-loop calculations working directly on the manifolds with this kind of singularities. We refer to the corresponding approach as the *conical singular-*

ity method [18,31–33,35–40]. The difference between it and quantum theory on the regular spaces is in the structure of the ultraviolet divergences [41,33,42]. Conical singularities result in appearing, in the effective action, of additional divergent terms concentrated on the horizon surface and their renormalization requires new counterterms. The important property, however, is that these counterterms turn out to be of the order $(\beta_\infty-\beta_H)^2\sim(1-\alpha)^2$ and hence, when taken on shell, they contribute neither to the entropy, nor to the free energy of the black hole [32,34,19,37].

In two dimensions, as follows from Eqs. (A2) and (A3), the divergent part of the action on the singular instanton M_B^α can be represented as

$$W_1^{\rm div}[M_B^\alpha]=-\frac{1}{8\pi\delta}\int_{M_B^\alpha}\sqrt{\gamma}d^2x+\frac{\ln\delta}{12}\left(\chi[M_B^\alpha]+\frac{1}{2\alpha}(1-\alpha)^2\right), \tag{6.1}$$

$$\chi[M_B^\alpha]=\frac{1}{4\pi}\left(\int_{M_B^\alpha}Rd^2x+2\int_{\Sigma_B}kdy+4\pi(1-\alpha)\right), \tag{6.2}$$

where, as in Eq. (3.9), δ is the ultraviolet cutoff parameter, and R is the regular curvature. The quantity $\chi[M_B^\alpha]$ is the Euler characteristic of M_B^α and it is identical to that of the Gibbons-Hawking instanton [43]: $\chi[M_B^\alpha]=\chi[M_B]=1$. Thus, up to the terms $(1-\alpha)^2$, the divergences on a regular instanton and those on a singular one coincide [compare Eqs. (3.9) and (6.1)] and the difference between them, being taken on shell, does not affect the entropy. As earlier, we assume that the renormalization has already been done and further we use only renormalized quantities.

Let us calculate the off-shell effective action $W_1^{\rm CS}$ and entropy $S_1^{\rm CS}$ by the conical singularity method.

As earlier, β is the inverse temperature on Σ_B and $\alpha=\beta_\infty/\beta_H$ is the off-shell parameter. We again use the conformal transformation (3.11), but now it maps a singular instanton onto the standard cone C_α with the unit (in the units of μ) length of the generator

$$d\tilde{s}^{\,2}=\mu^2(x^2d\tilde{\tau}^{\,2}+dx^2),\quad 0\leqslant x\leqslant 1,\quad 0\leqslant\tau\leqslant 2\pi\alpha. \tag{6.3}$$

Equations (3.11), (3.14), and (A9) enable one to relate the effective action $W_1^{\rm CS}$ to the action on C_α. Written as earlier, in terms of variables (β,α,y), this action takes the form

$$W_1^{\rm CS}(\beta,r_B,r_+)=\widetilde{W}_1^{\rm CS}(\beta,\alpha(\beta,r_B,r_+),y(r_B,r_+)), \tag{6.4}$$

$$\widetilde{W}_1^{\rm CS}(\beta,\alpha,y)=-\frac{\alpha}{48}\left(2y+13y^2-15+4\ln\frac{\beta}{2\pi\mu\alpha}\right)-\frac{1}{24\alpha}\left(\frac{1}{y}-1-\ln y+2\ln\frac{\beta}{2\pi\mu\alpha}\right)+C(\alpha). \tag{6.5}$$

Here, $C(\alpha)$ is the effective action for the unit cone which for $\alpha=1$, coincides with the effective action on the unit disk

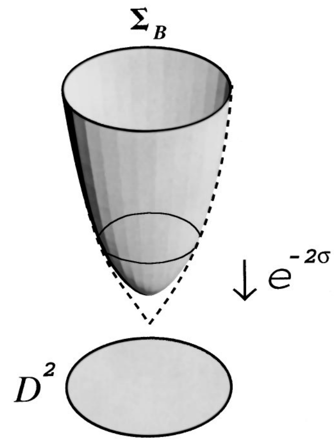

FIG. 5. Blunt instanton and its conformal transformation onto a unit disk D^2.

D^2 denoted earlier as C: $C(\alpha=1)=C$. The function $C(\alpha)$ does not depend on μ and results in a numerical addition to the entropy. Its form is not important for our consideration.

For the on shell limit $\alpha=1$, the cone singularity disappears, so that one has

$$\widetilde{W}_1^{\rm CS}(\beta,\alpha=1,y)=\widetilde{W}_1(y,\beta), \tag{6.6}$$

where $\widetilde{W}_1(y,\beta)$ is the on shell effective action given by Eq. (3.17).

The entropy $S_1^{\rm CS}$ is defined from $\widetilde{W}_1^{\rm CS}(\beta,\alpha,y)$ by Eq. (4.4). The calculation gives

$$S_1^{\rm CS}(\beta,\alpha,y)=\frac{1}{12\alpha}\left(\frac{1}{y}-1-\ln y+2\ln\frac{\beta}{2\pi\mu\alpha}\right)+C^{\rm CS}(\alpha), \tag{6.7}$$

where

$$C^{\rm CS}(\alpha)=\left(\alpha\frac{\partial}{\partial\alpha}-1\right)C(\alpha) \tag{6.8}$$

is an irrelevant constant at $\alpha=1$. Note that in the conical singularity approach both the renormalized action $W_1^{\rm CS}$ and the entropy $S_1^{\rm CS}$ are finite quantities.

VII. BLUNT-CONE METHOD

Consider, as earlier, the singular instanton M_B^α shown in Fig. 4 and a set of regular manifolds that modify its geometry in the narrow vicinity of the sharp cone vertex (see Fig. 5). The Riemann curvature for such geometries is regular everywhere and it differs from the Riemann curvature on a singular instanton only near the horizon. We call this geometry the ''blunt instanton'' and refer to this off-shell extension [32,43] as to the *blunt-cone* method. In this approach we can avoid the problems connected with the formulation of the quantization and renormalization procedures on manifolds with infinite curvatures. The regularization of the cone singularity is supposed to be removed at the very end of calculations.

For simplicity of calculations we choose a special form of the off-shell extension characterized by only two parameters:

an off-shell parameter $\alpha=\beta_\infty/\beta_H$ and a new parameter η which describes the width of the rounded tip of the blunt instanton. We choose the metric on a blunt instanton in the form

$$ds^2=\left(\frac{\beta}{2\pi}\right)^2(\rho^2 d\tau^2+b^2 d\rho^2),\quad 0\leqslant\tau\leqslant 2\pi,\quad 0\leqslant\rho\leqslant 1; \tag{7.1}$$

$$b=\frac{1}{(1-\rho^2+y\rho^2)^2}\,\frac{\rho^2+\alpha\eta^2}{\alpha\rho^2+\alpha\eta^2}.$$

The boundary Σ_B of the region under consideration is located at $\rho=1$, and its length is β. The parameter of the black hole mass enters, as earlier, through the dimensionless quantity $y=r_+/r_B$. The parameters that uniquely fix a *blunt instanton* are β, r_B, r_+, and η. For $\alpha=1$, the metric is identical to the metric of the Gibbons-Hawking instanton.

To calculate the renormalized one-loop effective action on the blunt instanton we map the latter onto a unit disk D^2. Consider, at first, an arbitrary static Euclidean 2D manifold with the line element ds^2 that is conformally related to the unit disk with the element $d\tilde{s}^2$:

$$ds^2=\left(\frac{\beta}{2\pi}\right)^2[a^2 d\tau^2+b^2 d\rho^2]=\exp(2\sigma)\mu^2[x^2 d\tilde{\tau}^2+dx^2], \tag{7.2}$$

where $0\leqslant\tau\leqslant 2\pi$, $0\leqslant\tilde{\tau}\leqslant 2\pi$, $0\leqslant\rho\leqslant 1$, and $0\leqslant x\leqslant 1$. Then, the metric coefficients a,b, and the conformal factor σ,

$$\sigma(\rho)=\ln\frac{a(\rho)}{a(1)}+\int_\rho^1 d\rho\,\frac{b}{a}+\ln\left(\frac{\beta}{2\pi\mu}\right), \tag{7.3}$$

depend only on ρ. The normalization of σ is fixed by the requirements $\sigma(1)=\ln(\beta/2\pi\mu)$ and $\tilde{\tau}=\tau$.

Integration of the conformal anomaly (see Appendix A), when applied to the metric (7.2), gives the one-loop effective action

$$W_1^{BC}=-\frac{1}{6}\ln\left(\frac{\beta}{2\pi\mu}\right)-\frac{1}{12}\int_0^1 d\rho\,\frac{(a'-b)^2}{ab}-\left(\frac{a'}{4b}\right)_{\rho=1}+\frac{1}{4}+C. \tag{7.4}$$

Here, $a'=da/d\rho$ and the constant C is, as earlier, the effective action for the unit disk D^2. To derive this formula, the regularity condition $(a'/b)|_{\rho=0}=1$ of the metric at the horizon has been used. For the metric (7.1) of the blunt instanton, one has

$$a=\rho,\quad b=\frac{1}{(1-\rho^2+y\rho^2)^2}\,\frac{\rho^2+\alpha\eta^2}{\alpha\rho^2+\alpha\eta^2}, \tag{7.5}$$

$$\sigma=\ln\rho+\frac{1}{2}\int_{\rho^2}^1 dz\,\frac{z+\alpha\eta^2}{z(\alpha z+\alpha\eta^2)(1-z+yz)^2}+\ln\left(\frac{\beta}{2\pi\mu}\right),$$

and the blunt-cone effective action W_1^{BC} reads

$$W_1^{BC}(\beta,r_B,r_+,\eta)=\widetilde{W}_1^{BC}(\beta,\alpha(\beta,r_B,r_+),y(r_B,r_+),\eta), \tag{7.6}$$

$$\begin{aligned}\widetilde{W}_1^{BC}(\beta,\alpha,y,\eta)=&-\frac{1}{6}\ln\left[\frac{\beta}{2\pi\mu}\right]-\frac{(\alpha-1)}{24\alpha}\,\frac{1}{(1+\eta^2-y\eta^2)^2}\ln\left|\frac{\eta^2}{1+\eta^2}\right|+\frac{\alpha-1}{24}(1+\alpha\eta^2-y\alpha\eta^2)^2\ln\left|\frac{\alpha\eta^2}{1+\alpha\eta^2}\right|\\&+\frac{1}{24}\ln|y|\left\{1-\frac{\alpha-1}{\alpha}\,\frac{1}{(1+\eta^2-y\eta^2)^2}\right\}+\frac{1}{24}(1-y)\left\{2\alpha-\frac{1+\alpha\eta^2-y\alpha\eta^2}{\alpha y(1+\eta^2-y\eta^2)}\right\}\\&-\frac{1}{48}\alpha(1-y)^2\{1-2(\alpha-1)\eta^2\}-\frac{1}{4}\,\frac{\alpha+\alpha\eta^2}{1+\alpha\eta^2}y^2+\frac{1}{4}+C.\end{aligned} \tag{7.7}$$

The parameter η in the blunt-cone method plays the role similar to the cutoff parameter ϵ in the brick-wall method. When the regularization parameter η tends to zero $\eta\to 0$, the action becomes

$$\begin{aligned}\widetilde{W}_1^{BC}(\beta,\alpha,y,\eta)=&-\frac{1}{6}\ln\frac{\beta}{2\pi\mu}+\frac{1}{48}\left[-\frac{2}{\alpha y}+\frac{2}{\alpha}\ln y-2\alpha y-13\alpha y^2+2(\alpha-1)\ln\alpha+\frac{2}{\alpha}+3\alpha+12\right]\\&+C+\frac{1}{24\alpha}(\alpha-1)^2\ln\eta^2+O(\eta^2).\end{aligned} \tag{7.8}$$

The metric (7.1) on shell ($\alpha=1$) becomes the metric of the Gibbons-Hawking instanton and the corresponding on shell effective action reads

$$\widetilde{W}_1^{BC}(\beta,\alpha=1,y,\eta)=-\frac{1}{6}\ln\frac{\beta}{2\pi\mu}+\frac{1}{48}\left[-\frac{2}{y}+2\ln y-2y-13y^2+17\right]+C. \tag{7.9}$$

It is identical to the on shell action $\widetilde{W}_1(\beta,y)$ given by expression (3.17). The corresponding blunt-cone entropy remains finite in the limit $\eta=0$ and reads

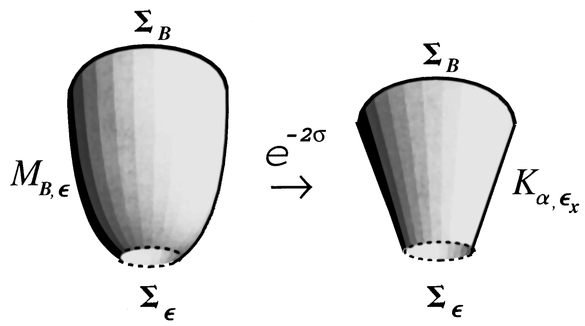

FIG. 6. Volume cutoff.

$$S_1^{BC}(\beta,1,y,0)=\frac{1}{12y}-\frac{1}{12}\ln y+\frac{1}{6}\ln\frac{\beta}{2\pi\mu}-\frac{1}{2}-C. \tag{7.10}$$

This result coincides (up to an unimportant constant) with the entropy S_1^{CS} found by the conical singularity method.

VIII. METHOD OF THE VOLUME CUTOFF

Finally we discuss here one more method of the off-shell definition of the black hole effective action W_1. Note that W_1 can be represented as the volume integral over the background space of some Lagrange density $\mathcal{L}_1(x)$:

$$W_1=\int \sqrt{g}dx\mathcal{L}(x). \tag{8.1}$$

The corresponding density $\mathcal{L}_1(x)$ can be written in terms of the diagonal elements of the heat kernel operator in the coordinate representation

$$\mathcal{L}_1(x)=-\frac{1}{2}\int_0^\infty\frac{ds}{s}\langle x|e^{s\triangle}|x\rangle, \tag{8.2}$$

so that, for the action itself, one has the standard formula

$$W_1=\frac{1}{2}\ln\,\det(-\mu^2\triangle)=-\frac{1}{2}\int_0^\infty\frac{ds}{s}\mathrm{Tr}e^{s\mu^2\triangle}. \tag{8.3}$$

Consider now a singular instanton, and calculate $\mathcal{L}_1(x)$ for its regular points $r>r_+$. Denote by Σ_ϵ a surface located at a small proper distance ϵ from the horizon, and restrict the integration in Eq. (8.3) by the region $M_{B,\epsilon}$ located outside Σ_ϵ, see Fig. 6. As a result, the action W_1 depends on a new parameter ϵ. We call this off-shell procedure as the *volume cutoff* method and denote the corresponding quantities with the superscript VC.

The volume (or spatial) cutoff method arises naturally in the dynamical-interior approach to the black hole entropy, proposed in Ref. [44]. In this approach the internal degrees of freedom of a black hole are identified with the states of fields propagating in its interior in the close vicinity to the horizon. Because of the quantum fluctuations of the horizon, the separation of the modes into external (propagating outside the horizon) and internal (propagating inside the horizon) becomes impossible for modes located closer to the horizon than the amplitude of its quantum fluctuations. For this reason the summation of the modes which contribute to the statistical-mechanical entropy of a black hole in the approach [44] is restricted only to the modes, which are located outside the fluctuation region of the horizon. This is equivalent to the spatial cutoff in the volume integral for the effective action described above. The volume cutoff procedure has been also used in many other papers [45–52]. In works [46,48–52] the black hole metric has been mapped onto an optical (ultrastatic) metric. The horizon then maps to infinity and the proper volume of the optical space becomes infinite. In order to deal with this divergence it is natural to restrict the volume integration by a finite region. This approach enables one to get a number of interesting results for the entropy corrections even for the massive fields in spaces with the dimension larger than two [50,51] and for conformal fields with nonzero spins [52].

Up to a certain extent the volume cutoff method resembles the brick-wall approach. However, they are certainly different because the volume cutoff method does not require any special boundary conditions on Σ_ϵ. It is also nonsensitive to the behavior of the quantum field in the region lying closer than Σ_ϵ to the horizon.

The calculation of the Lagrangian $\mathcal{L}_1$ on the off-shell black hole solution can again be carried out with the help of the conformal transformation to the conical space. Using Eq. (A9), one can write the relation

$$\mathcal{L}_1=e^{-2\sigma}\mathcal{L}_1(C_\alpha)-\frac{1}{24\pi}\times[R\sigma-(\nabla\sigma)^2+(2k\sigma+3\sigma_{,\mu}n^\mu)\delta(r,r_B)] \tag{8.4}$$

between $\mathcal{L}_1$ and the Lagrangian $\mathcal{L}_1(C_\alpha)$ on a unit cone C_α valid in the region outside the horizon. Here, $\delta(r,r_B)$ is the invariant delta function which is included to reproduce the surface terms on the external boundary in the action. The factor σ is given by Eq. (3.14). Note that the terms in Eq. (A9), which are determined by the value of the conformal parameter σ on the cone apex do not contribute to W_1^{VC} in Eq. (8.4).

To find $\mathcal{L}_1(C_\alpha)$, one can use the Sommerfeld representation for the heat kernel $K_\alpha(x,x')=<x|e^{s\triangle}|x'>$ of the Laplace operator on the conical space (6.3)

$$K_\alpha(x,x',\tilde{\tau}-\tilde{\tau}')=K(x,x',\tilde{\tau}-\tilde{\tau}')+\frac{i}{4\pi\alpha}\times\int_\Gamma\cot\left(\frac{w}{2\alpha}\right)K(x,x',\tilde{\tau}-\tilde{\tau}'+w)dw \tag{8.5}$$

relating it to the heat kernel $K(x,x',\tilde{\tau}-\tilde{\tau}')$ on a unit disk D^2. Here, the integration contour Γ lies in the complex plane and consists of two curves, going from $\mp\pi-(\tilde{\tau}-\tilde{\tau}')\pm i\infty$ to $\mp\pi-(\tilde{\tau}-\tilde{\tau}')\pm i\infty$ and intersecting the real axis between the poles of the integrand $-2\pi\alpha$, 0, and $2\pi\alpha$. A derivation and discussion of this formula can be found in [54–57]. The Lagrange density on a cone can be easily calculated if one substitutes Eqs. (8.5) and (8.3). The result has a simple form

$$\mathcal{L}_1(C_\alpha)=\mathcal{L}_1(D^2)-\frac{1}{24\pi x^2}\left(\frac{1}{\alpha^2}-1\right). \tag{8.6}$$

Here, $\mathcal{L}_1(D^2)$ is the Lagrange density on the unit disk D^2. In what follows we omit $\mathcal{L}_1(D^2)$ which results in an irrelevant constant in W_1^{VC}. The second term, vanishing at $\alpha=1$, arises as the result of integration in Eq. (8.5). In the calculations the integral over s is taken first and then the formula

$$\frac{i}{8\pi\alpha}\int_\Gamma \frac{\cot(w/2\alpha)}{\sin^2 w/2}dw=\frac{1}{6}\left(\frac{1}{\alpha^2}-1\right) \tag{8.7}$$

is used.

Let $W_1^{VC}[C_\alpha]$ be the effective action on a cone C_α obtained by the integration of Eq. (8.6) till the point $x=\epsilon_x$. As earlier, ϵ_x is related with the invariant distance ϵ to the horizon by Eq. (5.3). This functional reads

$$W_1^{VC}[C_\alpha]=\frac{1}{12}\left(\alpha-\frac{1}{\alpha}\right)\ln\epsilon_x^{-1}. \tag{8.8}$$

Then, by using Eqs. (8.6) and (5.3), one can write the complete effective action in the volume cutoff method as

$$W_1^{VC}(\beta,r_B,r_+,\epsilon)=W_1^{VC}[C_\alpha]-\frac{1}{24\pi}\left(\int_{M_{B,\epsilon}}[R\sigma-(\nabla\sigma)^2]+\int_{\Sigma_B}(2K\sigma+3\sigma_{,\mu}n^\mu)\right). \tag{8.9}$$

So eventually we have

$$W_1^{VC}(\beta,r_B,r_+,\epsilon)=\widetilde{W}_1^{VC}(\beta,\alpha(\beta,r_B,r_+),y(r_B,r_+),\epsilon),$$

$$\widetilde{W}_1^{VC}(\beta,\alpha,y,\epsilon)=\frac{1}{12}\left(\alpha-\frac{1}{\alpha}\right)\left(\ln\frac{\mu}{\epsilon}-\ln\frac{2\pi\mu\alpha}{\beta}-\frac{1}{2}\ln y-\frac{1}{2}+\frac{1}{2y}\right)+\frac{\alpha}{48\pi}\left(-\frac{2}{y}+2\ln y-2y-13y^2+17+8\ln\frac{2\pi\mu\alpha}{\beta}\right)+o(\epsilon). \tag{8.10}$$

When taken on shell $(\alpha=1)$, the divergence $\ln\epsilon$ of this functional disappears and $\widetilde{W}_1^{VC}$coincides with the action (3.17) on the regular space

$$\widetilde{W}_1^{VC}(\beta,\alpha=1,y,\epsilon)=\widetilde{W}_1(\beta,y). \tag{8.11}$$

The entropy $S_1^{VC}(\beta,\alpha,y,\epsilon)$, calculated from the action (8.10), reads

$$S_1^{VC}(\beta,\alpha,y,\epsilon)=\frac{1}{12\alpha}\left(2\ln\frac{\mu}{\epsilon}+2\ln\frac{\beta}{2\pi\alpha}-\ln y-1+\frac{1}{y}\right). \tag{8.12}$$

On shell S_1^{VC} differs from the conical singularity entropy S_1^{CS} only by a singular term depending on ϵ

$$S_1^{VC}(\beta,\alpha=1,y,\epsilon)=S_1^{\mathrm{CS}}(\beta,\alpha=1,y)+\frac{1}{6}\ln\frac{\mu}{\epsilon}. \tag{8.13}$$

The entropy S_1^{VC} can be also written as

$$S_1^{VC}(\beta,\alpha,y,\epsilon)=\frac{1}{6\alpha}\ln\epsilon_x^{-1}. \tag{8.14}$$

So, this quantity coincides with the entropy computed from the action $W_1^{VC}(C_\alpha)$. The coincidence takes place because the anomaly, which differentiates $W_1^{VC}(\beta,\alpha,y,\epsilon)$ from $W_1^{VC}(C_\alpha)$, is proportional to β and does not contribute into S_1^{VC}.

Another observation is that S_1^{VC} coincides with the thermal entropy of the quantum gas in the volume of the size $\ln\epsilon_x^{-1}$ The volume cutoff entropy does not contain the term $\ln\ln\epsilon^{-1}$ which is present in the brick-wall entropy S_1^{BW} since the boundary condition on the quantum field at Σ_ϵ is not imposed, and the field can freely fluctuate on this boundary, see Appendix C.

IX. OFF SHELL VERSUS ON SHELL

A. Off-shell and On-shell effective actions

In this section we discuss and compare the results of the off-shell and on shell calculations of the thermodynamical characteristics of a black hole. We begin by discussing the obtained results for the effective action. It is convenient to introduce the notation

$$U(\beta,\alpha,y)=-\frac{1}{6}\ln\left[\frac{\beta}{2\pi\mu}\right]+\frac{1}{48}\left[-\frac{2}{y}+2\ln y+17-2y-13y^2\right]+\frac{\alpha-1}{48\alpha}\left(\frac{2}{y}-2\ln y-2+15\alpha-2\alpha y-13\alpha y^2\right)-\frac{(\alpha-1)^2}{12\alpha}\ln\left[\frac{\beta}{2\pi\mu}\right]+\left(\alpha+\frac{1}{\alpha}\right)\ln\alpha. \tag{9.1}$$

Then, the one-loop contributions to the effective action calculated by different off-shell methods can be presented in the form

$$\widetilde{W}_1^{\mathrm{CS}}(\beta,\alpha,y)=U(\beta,\alpha,y)+C(\alpha), \tag{9.2}$$

$$\widetilde{W}_1^{\mathrm{BW}}(\beta,\alpha,y,\epsilon)=U(\beta,\alpha,y)+\frac{1}{12}\left(\alpha+\frac{1}{\alpha}\right)\ln\left(\frac{\epsilon}{\mu}\right)-\frac{1}{2}\ln\frac{\pi\alpha}{\ln(\beta/2\pi\alpha\epsilon)}, \tag{9.3}$$

$$\widetilde{W}_1^{BC}(\beta,\alpha,y,\eta)=U(\beta,\alpha,y)+\frac{(\alpha-1)^2}{12\alpha}\ln\left[\frac{\eta\beta}{2\pi\alpha\mu}\right]+\frac{\alpha-1}{24}\ln\alpha-\frac{\alpha-5}{4}+C, \tag{9.4}$$

$$\widetilde{W}_1^{VC}(\beta,\alpha,y,\epsilon)=U(\beta,\alpha,y)-\frac{1}{12}\left(\alpha-\frac{1}{\alpha}\right)\ln\frac{\epsilon}{\mu}. \tag{9.5}$$

Here, we again use the notations $y=r_+/r_B$ and $\alpha(\beta,r_B,r_+)=\beta/(4\pi r_+\sqrt{1-r_+/r_B})$. The constants C and $C(\alpha)$ which enter these relations are the effective actions $W_1=\frac{1}{2}\ln\det(-\mu^2\triangle)$ on the unit disk D^2 and on the unit cone C_α, respectively.

In the same notations, the on shell one-loop effective action is

$$\widetilde{W}_1(\beta,y)=U(\beta,\alpha=1,y)+C. \tag{9.6}$$

A simple comparison of Eqs. (9.2), (9.4), and (9.5) with Eq. (9.6) shows that the relations

$$\widetilde{W}_1^{\rm CS}(\beta,\alpha=1,y)=\widetilde{W}_1^{BC}(\beta,\alpha=1,y,\eta)$$
$$=\widetilde{W}_1^{VC}(\beta,\alpha=1,y,\epsilon)=\widetilde{W}_1(\beta,y) \tag{9.7}$$

take place [after neglecting unessential numerical constants in Eqs. (9.4) and (9.5)]. In other words, the on-shell values of the one-loop effective actions calculated by conical singularity, blunt-cone, and volume cutoff methods coincide with the on shell one-loop effective action $\widetilde{W}_1(\beta,y)$. $\widetilde{W}_1^{\rm CS}$ is always finite, while $\widetilde{W}_1^{BC}$ and $\widetilde{W}_1^{VC}$ are finite (i.e., do not contain either $\ln\eta$ or $\ln\epsilon$ divergence) only on shell (for $\alpha=1$). The only divergent on shell quantity is the brick-wall effective action $\widetilde{W}_1^{\rm BW}$.

The relation (9.3) can be interpreted in the following way. Let us remind that the effective action $W_1^{\rm CS}$ has been computed by the conformal map onto the cone C_α, [see Eq. (6.3)]. So $W_1^{\rm CS}$ is defined up to addition of the action $W_1[C_\alpha]=C(\alpha)$. Alternatively, one could use the map onto a cone $C_{\alpha,\epsilon}$ with the size ϵ. The results of two computations can be compared by using the difference between $W_1[C_\alpha]$ and $W_1[C_{\alpha,\epsilon}]$. This difference can be easily found because both cones are related by the trivial rescaling:

$$ds^2(C_\alpha)=\left(\frac{\mu}{\epsilon}\right)^2 ds^2(C_{\alpha,\epsilon}). \tag{9.8}$$

Then, Eq. (A9) gives

$$W_1[C_\alpha]=W_1[C_{\alpha,\epsilon}]+\frac{1}{12}\left(\frac{1}{\alpha}+\alpha\right)\ln\frac{\epsilon}{\mu}. \tag{9.9}$$

It enables one to represent the result (9.3) as

$$W_1^{\rm BW}(\beta,\alpha,y,\epsilon)=W_1^{\rm CS}(\beta,\alpha,y)-W_1[C_{\alpha,\epsilon}]+W_1^{\rm Cas}(\beta,\alpha,\epsilon), \tag{9.10}$$

where

$$W_1^{\rm Cas}(\beta,\alpha,\epsilon)=-\frac{1}{2}\ln\frac{\pi\alpha}{\ln(\beta/2\pi\alpha\epsilon)} \tag{9.11}$$

is the contribution due to the Casimir effect. The detailed discussion of this term and its relation to the brick-wall boundary conditions is given in Appendix C.

B. Why the on shell and off-shell one-loop contributions to the entropy are different

The equality (9.7) of all (except brick-wall) off-shell effective actions and the on shell effective action does not guarantee that the same is true for the corresponding values of entropy. Moreover, as we shall see, all the off-shell calculations give the results for the entropy which differ from the on shell result. Before giving the concrete relations between these quantities, let us discuss why does it happen.

Our starting point in the off-shell calculations is the one-loop action $W_1^\bullet$ which is the function of the parameters β, r_B, and r_+. In the "brick-wall" and volume cutoff approaches, it also depends on the additional parameter ϵ, and on ϵ and η in the blunt-cone method. The dependence on these additional parameters is not important at the moment, so we will not indicate it explicitly. The quantities β and r_B are external parameters fixing the problem and r_+ is determined on shell in terms of them by the condition

$$\alpha(\beta,r_B,r_+)=\frac{\beta}{4\pi r_+\sqrt{1-r_+/r_B}}=1. \tag{9.12}$$

Consider first cone-singularity, blunt cone, and volume cutoff methods for which the effective actions, when taken on shell (9.12), coincide with the thermodynamical action $W_1(\beta,r_B)$ given by Eqs. (3.16) and (3.17)

$$W_1^\bullet(\beta,r_B,r_+)|_{\alpha=1}=W_1(\beta,r_B). \tag{9.13}$$

Here the symbol • replaces CS, BC, and VC notations. The thermodynamical entropy $S_1^{\rm TD}$ is defined by Eq. (3.19)

$$S_1^{\rm TD}=\beta\frac{\partial W_1(\beta,r_B)}{\partial\beta}\bigg|_{r_B}-W_1(\beta,r_B), \tag{9.14}$$

while the off-shell entropy $S_1^\bullet$ is defined by Eq. (4.1)

$$S_1^\bullet=\beta\frac{\partial W_1^\bullet(\beta,r_B,r_+)}{\partial\beta}\bigg|_{r_B,r_+}-W_1^\bullet(\beta,r_B,r_+). \tag{9.15}$$

Note that in the calculation of $S_1^\bullet$ the parameter r_+ is assumed to be fixed. This results in the difference $\Delta S^\bullet$ between two entropies:

$$\Delta S^\bullet=S_1^{\rm TD}-S_1^\bullet$$
$$=\beta\left(\frac{\partial}{\partial\beta}W_1(\beta,r_B)-\frac{\partial}{\partial\beta}W_1^\bullet(\beta,r_B,r_+)\right)\bigg|_{\alpha=1}. \tag{9.16}$$

Together with the Eq. (9.13), it gives

$$\Delta S^\bullet=\beta\left(\frac{\partial r_+}{\partial\beta}\frac{\partial W_1^\bullet}{\partial r_+}\bigg|_{\beta,r_B}\right)_{\alpha=1} \tag{9.17}$$

which, obviously, is nonzero quantity. This shows why in the general case the one-loop contribution to the black hole entropy found by an off-shell procedure differs from the contribution inferred in the thermodynamical computation, based on the on-shell action.

C. Relations between off-shell and on-shell entropies

We obtain now explicit formulas relating different off-shell entropies. As earlier, we assume that after the calculations of the entropy the limit $\alpha=1$ is taken. The calculated entropies are always understood as the function of the parameters β,r_B characterizing the system. For simplicity, we omit these arguments. Note also, that the effective actions contain an arbitrary constants, which we denoted as C and $C(\alpha)$. It is evident that similar constants enter also the expressions for the entropies. We indicated these constants explicitly earlier in the expressions for the entropies. They may be important for the discussion of the questions connected with the third law of black hole thermodynamics. But they are not important for us now. For this reason, in order to simplify the expressions we simply omit them from now on. We also omit the terms which vanish when the additional parameters (such as ϵ and η) take their limiting value ($\epsilon=0$ and $\eta=0$).

It is convenient to begin with the entropy $S_1^{\rm CS}$ calculated by the conical singularity method. It is obtained from the effective action $W_1^{\rm CS}$ given by Eq. (9.2) with $C(\alpha=1)=0$, or what is equivalent from U, given by Eq. (9.1):

$$S_1^{\rm CS}=\frac{1}{12}\left(\frac{1}{y}-1-\ln y+2\ln\frac{\beta}{2\pi\mu}\right). \tag{9.18}$$

Let us denote

$$S_1^T(\epsilon)=\frac{1}{6}\ln\frac{\mu}{\epsilon},\quad S_1^{\rm Cas}(\epsilon)=\frac{1}{2}\ln\frac{\pi}{\ln\frac{\beta}{2\pi\epsilon}}. \tag{9.19}$$

Then, the results of the previous sections can be summarized as

$$S_1^{\rm BW}=S_1^{\rm CS}+S_1^T+S_1^{\rm Cas}, \tag{9.20}$$

$$S_1^{VC}=S_1^{\rm CS}+S_1^T, \tag{9.21}$$

$$S_1^{BC}=S_1^{\rm CS}. \tag{9.22}$$

Thus, the blunt-cone and conical singularity methods give the same finite result for the entropy, while the brick-wall and volume cutoff methods give expressions containing ($\ln\epsilon$) divergence. The difference $S_1^{\rm Cas}$ between $S_1^{\rm BW}$ and S_1^{VC} occurs because the different boundary conditions in these methods are imposed. All the above off-shell expressions for the entropy differ from the one-loop contribution $S_1^{\rm TD}$ to the thermodynamical entropy given by Eq. (3.22). The latter can be presented as

$$S_1^{\rm TD}=S_1^{\rm CS}+\Delta S, \tag{9.23}$$

where

$$\Delta S\equiv\beta\left(\frac{\partial r_+}{\partial\beta}\frac{\partial W_1^{\rm CS}}{\partial r_+}\bigg|_{\beta,r_B}\right)_{\alpha=1}=\frac{1}{48(2-3y)}(-14+26y-28y^2+13y^3)+\frac{1}{24}\ln y. \tag{9.24}$$

The relation (9.20) can be rewritten in a different form which is more convenient for interpretation. Note that according to Eqs. (5.16), (5.17), and (5.21),

$$S_1^{\rm BW}=-\,{\rm Tr}[\hat\rho_\epsilon^H(\beta)\ln\hat\rho_\epsilon^H(\beta)]. \tag{9.25}$$

On the other hand, $S_1^T+S_1^{\rm Cas}$ can be identically rewritten as

$$S_1^T+S_1^{\rm Cas}=S_\epsilon^R(2\pi\mu)=-\,{\rm Tr}[\hat\rho_\epsilon^R(2\pi\mu)\ln\hat\rho_\epsilon^R(2\pi\mu)]. \tag{9.26}$$

That is, this expression coincides with the entropy of a massless thermal radiation in the Rindler space between two mirrors located at the proper distances ϵ and μ from the horizon. The temperature of the radiation measured at the distance μ from the horizon is $1/(2\pi\mu)$. Thus, we have

$$S_1^{\rm CS}=-\{{\rm Tr}[\hat\rho_\epsilon^H(\beta)\ln\hat\rho_\epsilon^H(\beta)]-{\rm Tr}[\hat\rho_\epsilon^R(2\pi\mu)\ln\hat\rho_\epsilon^R(2\pi\mu)]\}. \tag{9.27}$$

It is easy to verify that the same relation is valid also if the inner mirrorlike boundary (at ϵ) is absent provided the quantities in the right-hand side are defined by using the volume cutoff method. For both brick-wall and volume cutoff methods, each of the terms in the right-hand side of Eq. (9.27) is divergent as $\epsilon\to0$, while the difference remains finite in this limit. If we formally define the density matrices $\hat\rho^H(\beta)$ and $\hat\rho^R(2\pi\mu)$ on the black hole and Rindler backgrounds as the limits

$$\hat\rho^H(\beta)=\lim_{\epsilon\to0}\hat\rho_\epsilon^H(\beta),\quad \hat\rho^R(2\pi\mu)=\lim_{\epsilon\to0}\hat\rho_\epsilon^R(2\pi\mu), \tag{9.28}$$

then for both volume cutoff and brick-wall methods, we have

$$S_1^{\rm CS}(\beta,\alpha=1,y)=-\{{\rm Tr}[\hat\rho^H(\beta)\ln\hat\rho^H(\beta)]-{\rm Tr}[\hat\rho^R(2\pi\mu)\ln\hat\rho^R(2\pi\mu)]\}. \tag{9.29}$$

Using Eq. (9.23), we finally get

$$S_1^{\rm TD}=-\{{\rm Tr}[\hat\rho^H(\beta)\ln\hat\rho^H(\beta)]-{\rm Tr}[\hat\rho^R(2\pi\mu)\ln\hat\rho^R(2\pi\mu)]\}+\Delta S. \tag{9.30}$$

This relation indicates that the one-loop correction to the thermodynamical entropy can be obtained from the statistical-mechanical black hole entropy by the following procedure. First, one needs to subtract the Rindler entropy which removes the divergence, and then add a finite correction $\triangle S$. In the next section we show that the second term ΔS coincides with the change of the classical Bekenstein-Hawking entropy due to the quantum deformation of the background geometry.

It is worth mentioning that a similar subtraction procedure naturally arises in the membrane paradigm [60]. Namely, in order to obtain the correct expression for the flux of the entropy onto a black hole, Thorne and Zurek [59,60] proposed to subtract from the entropy, calculated by a statistical-mechanical method, the entropy of a thermal atmosphere of the black hole. The later entropy close to the horizon coincides with $S_{\rm Rindler}^{\rm SM}$. Equation (9.30) can be used to prove this conjecture. However, it should be stressed that Thorne and Zurek did not consider quantum corrections to the entropy discussed in the present paper. Equation (9.30) not only explains how the volume infinities in $S^{\rm SM}$ are separated, but also gives an exact dependence of the quantum corrections to the entropy on physical characteristics.

D. Entropy and back reaction effects

The thermodynamical entropy of a black hole with quantum one-loop corrections is

$$S^{\mathrm{TD}} = S^{\mathrm{BH}}(r_+) + S_1^{\mathrm{TD}}, \tag{9.31}$$

where $S^{\mathrm{BH}}(r_+) = \pi r_+^2$ is the Bekenstein-Hawking entropy. As the result of quantum effects, a ''real'' solution $(\overline{\gamma,r})$, including quantum corrections, is different from the classical Schwarzschild solution (γ,r) [61]. In particular, the value $\overline{r}_+$ of the dilaton field at the horizon of $\overline{\gamma}$ differs from its classical value r_+. We demonstrate now that Eq. (9.31) can be identically rewritten as

$$S^{\mathrm{TD}} = \pi \overline{r}_+^2 + S_1^{\mathrm{CS}}. \tag{9.32}$$

A first step in the proof is to obtain an equation which determines $\overline{r_+}$. For given boundary conditions (β, r_B), the extremum of the Euclidean effective action W defines a regular quantum solution. This solution can be obtained by solving the field equations $\delta W/\delta\overline{\gamma} = \delta W/\delta\overline{r} = 0$ and fixing an arbitrary constant which enters the solution by the regularity condition on the horizon. This determines $\overline{r_+}$ as a function of (β, r_B): $\overline{r_+} = \overline{r_+}(\beta, r_B)$. For any other choice of the constant, the solution has a conelike singularity. We call such a singular solution a *quantum singular instanton*. It obeys local field equations but does not provide a global extremum for W. The quantum singular instanton is specified by (β, r_B) and an arbitrary parameter $\overline{r_+}$. We write the solution as $[\overline{\gamma(r_+)}, \overline{r(r_+)}]$. The effective action $W(\beta, r_B, \overline{r_+})$ calculated on the quantum singular instanton is

$$\begin{aligned} W(\beta, r_B, \overline{r_+}) &\equiv W[\beta, r_B, \overline{\gamma(r_+)}, \overline{r(r_+)}] \\ &= I[\beta, r_B, \overline{\gamma(r_+)}, \overline{r(r_+)}] \\ &\quad + W_1^{\mathrm{CS}}[\beta, r_B, \overline{\gamma(r_+)}, \overline{r(r_+)}]. \end{aligned} \tag{9.33}$$

The condition of the global extremality of W

$$\frac{\partial W(\beta, r_B, r_+)}{\partial r_+} = 0 \tag{9.34}$$

determines the horizon radius $\overline{r_+} = \overline{r_+}(\beta, r_B)$ for the regular quantum instanton.

In the calculations we keep only terms up to the first order in $\hbar$. For this reason we can replace $W_1^{\mathrm{CS}}[\beta, r_B, \overline{\gamma(r_+)}, \overline{r(r_+)}]$ in the right-hand side of Eq. (9.33) by its value calculated on the classical singular instanton $W_1^{\mathrm{CS}}[\beta, r_B, \gamma(\overline{r_+}), r(\overline{r_+})]$. What is much less trivial, we can also replace $[\overline{\gamma(r_+)}, \overline{r(r_+)}]$ in the classical action I in Eq. (9.33) by the solution $[\gamma(\overline{r_+}), r(\overline{r_+})]$ for a classical singular instanton provided the value of the dilaton field $\overline{r_+}$ on the horizon is preserved the same. To show this, consider the general variation of the classical action I given by Eq. (3.1). For fixed r_B and β, we have

$$\begin{aligned} I[\beta, r_B, \overline{\gamma,r}] &= I[\beta, r_B, \gamma, r] \\ &\quad + \int \left[\left. \frac{\delta I}{\delta \gamma_{ab}} \right|_{\gamma_{ab}} (\overline{\gamma_{ab}} - \gamma_{ab}) + \frac{\delta I}{\delta r} \delta r \right] \\ &\quad + r_{,\mu} n^{\mu}|_{r=r_+} \delta r_+ \\ &\quad - 2\pi(1-\alpha) r_+ \delta r_+ + O(\hbar^2). \end{aligned} \tag{9.35}$$

We assume that the value of the dilaton field on the cone singularity is r_+, and denote by $2\pi(1-\alpha)$ the corresponding deficit angle which is defined by (γ, r) at r_+ [58]. The relation (9.35) shows that when the value r_+ for γ and $\overline{\gamma}$ is the same, and (γ, r) is a solution of classical equations $(\delta I/\delta\gamma_{ab} = 0,\ \delta I/\delta r = 0)$, the value of the classical action calculated on $(\overline{\gamma,r})$ differs from the classical value $I[\beta, r_B, \gamma, r]$ only by terms of the order $O(\hbar^2)$. That is why we can replace $I[\beta, r_B, \overline{\gamma(r_+)}, \overline{r(r_+)}]$ in Eq. (9.33) by $I(\beta, r_B, r_+)$, the value of I calculated on the classical singular instanton. The latter can be easily found

$$I(\beta, r_B, r_+) = \beta E(r_B, r_+) - \pi r_+^2,$$

$$E(r_B, r_+) \equiv r_B[1 - (1 - r_+/r_B)^{1/2}], \tag{9.36}$$

where E is a quasilocal energy [7,53].

The Eq. (9.34) which defines the ''position'' $\overline{r_+}$ of the quantum horizon can be written as

$$\frac{\partial W_1^{\mathrm{CS}}(\beta, r_B, r_+)}{\partial r_+} = -2\pi \overline{r_+}(\overline{\alpha} - 1). \tag{9.37}$$

Here, $\alpha = \alpha(\beta, r_B, r_+) = \beta[4\pi r_+ \sqrt{1 - r_+/r_B}]^{-1}$, and $\overline{\alpha}$ is the value of the classical off-shell parameter α calculated for $r_+ = \overline{r_+}$. For the classical regular instanton $\alpha = 1$. It means that up to the second order in $\hbar$ we can write

$$2\pi \overline{r_+}(\overline{\alpha} - 1) = 2\pi r_+ \left(\frac{\partial \alpha}{\partial r_+} \right)_{\alpha=1} \Delta r_+. \tag{9.38}$$

Here, $\Delta r_+ = \overline{r_+} - r_+$ is the change of the ''position'' of the black hole horizon because of the quantum corrections. Using the explicit expression for α, it is easy to show that

$$\left(\frac{\partial \alpha}{\partial r_+} \right)_{\alpha=1} = -\left[\beta \frac{\partial r_+}{\partial \beta} \right]_{\alpha=1}^{-1}. \tag{9.39}$$

The latter relation allows one to write

$$2\pi r_+ \Delta r_+ = \beta \left[\frac{\partial r_+}{\partial \beta} \frac{\partial W_1^{\mathrm{CS}}}{\partial r_+} \right]_{\alpha=1}, \tag{9.40}$$

and hence using Eq. (9.24), one gets

$$\Delta S = 2\pi r_+ \Delta r_+. \tag{9.41}$$

Therefore, up to the terms $O(\hbar^2)$, the quantity ΔS can be represented as the difference $\Delta S = S^{\mathrm{BH}}(\overline{r_+}) - S^{\mathrm{BH}}(r_+)$. On the other hand, taking into account Eq. (9.23), we can write the thermodynamical entropy given by Eq. (9.31) as the sum $S^{\mathrm{TD}} = S^{\mathrm{BH}}(r_+) + \triangle S + S_1^{\mathrm{CS}}$. These equalities prove the desired relation (9.32).

X. SUMMARY AND CONCLUSIONS

Discuss now some lessons we have learned by comparing on shell results with the results of the different off-shell methods in black hole thermodynamics. First of all, direct calculations demonstrate that the thermodynamical entropy of a black hole $S^{\rm TD}$ determined by the response of the free energy to the change of the temperature, and the statistical-mechanical entropy $S^{\rm SM}$, defined as $S^{\rm SM}=-\,{\rm Tr}(\hat{\rho}^H \ln \hat{\rho}^H)$ for density matrix $\hat{\rho}^H$ of black hole internal degrees of freedom, are different. The thermodynamical entropy, in addition to the tree-level Bekenstein-Hawking part $S^{\rm BH}=A/4$, contains also finite quantum one-loop correction $S_1^{\rm TD}$. The latter can be obtained from the on shell effective action. The statistical-mechanical entropy $S^{\rm SM}$ is defined as a one-loop quantity, and it requires an off-shell procedure for its calculation. $S^{\rm SM}$ can be identified with the volume cutoff entropy S_1^{VC}. Then, it contains the divergence ($\ln\epsilon$) where ϵ is a proper-distance cutoff of the volume integration, required to make this quantity finite. This leading logarithmical part of $S^{\rm SM}$ also presents in the brick-wall model, but generally due to the Casimir effect, $S_1^{\rm BW}$ has the additional divergence ($\ln|\ln\epsilon|$).

The physical reason why $S^{\rm TD}$ and $S^{\rm SM}$ are different is connected with a special property of a black hole as a thermodynamical system [12]. Namely, the internal degrees of freedom of a black hole are defined as excitations propagating on the background geometry. This geometry is uniquely determined by the mass parameter which, in the state of thermal equilibrium, is a function of the external temperature. For this reason, to find $S_1^{\rm TD}$ one must change the temperature. This results in the change of Hamiltonian, describing these internal excitations. On the other hand, in the calculations of $S^{\rm SM}$ the black hole mass and the Hamiltonian are to be fixed.

We proved that the thermodynamical entropy of a black hole can be presented in the form

$$S^{\rm TD}=S^{\rm BH}(\overline{r_+})+[S^{\rm SM}-S^{\rm SM}_{\rm Rindler}]. \qquad (10.1)$$

$S^{\rm BH}(\overline{r_+})=\pi\overline{r_+^2}$ is the Bekenstein-Hawking entropy, and $\overline{r_+}$ is the "radius" of the horizon of a "quantum" black hole. The term in the square brackets is the difference between the statistical-mechanical entropies calculated for a black hole $\{S^{\rm SM}=-\,{\rm Tr}[\hat{\rho}^H(\beta)\ln\hat{\rho}^H(\beta)]\}$ and for a Rindler space $\{S^{\rm SM}_{\rm Rindler}=-\,{\rm Tr}[\hat{\rho}^R(2\pi\mu)\ln\hat{\rho}^R(2\pi\mu)]\}$. This *subtraction procedure* automatically removes all the divergences from $S^{\rm SM}$ and results in an invariant regularization-independent quantity.

We proved the relation (10.1) by explicit calculations in 2D case, but it seems to be of the general nature and it (or its generalization) must be valid in the 4D case. The reason is that the on shell renormalized quantity $S^{\rm TD}$ is always finite, so that the subtraction terms in Eq. (10.1) will always be of the form required for the complete cancellation of the volume divergences of $S^{\rm SM}$ [12]. One of the possible ways to derive in four dimensions the relation analogous to Eq. (10.1) is to use an optical metric, where the required subtraction terms can be calculated by using high-temperature expansion. For this reason, the coefficients, which enter the subtraction terms with different order of singularity in ϵ must be connected with the Schwinger-DeWitt coefficients.

A remarkable property of the conical singularity method is that (at least in 2D case) it gives the finite result immediately:

$$S_1^{\rm CS}=S^{\rm SM}-S^{\rm SM}_{\rm Rindler}. \qquad (10.2)$$

The mathematical reason why $S_1^{\rm CS}$ is finite while S_1^{VC} contains volume ($\ln\epsilon$) divergence is connected with the difference of the topologies of the manifolds used to calculate the corresponding effective actions. For S_1^{VC} the standard manifold has the topology of a cylinder (or a ring), while in for $S_1^{\rm CS}$ the topology is of D^2, i.e., the same as the topology of the Gibbons-Hawking instanton. The mathematical operation when one cuts a small disk of the radius ϵ, from the standard unit disk D^2 to transform it into a ring, can be interpreted as the subtraction of an entanglement entropy [63,64,27,31,65] $S^{\rm SM}_{\rm Rindler}=-\,{\rm Tr}(\hat{\rho}^R\ln\hat{\rho}^R)$.

We stress once again that in our approach all the renormalizations are to be done from the very beginning so that only observable finite coupling constants enter the results. We demonstrated that some of the off-shell methods require an additional cutoff parameter which we denoted by ϵ. This cutoff parameter is completely independent from the ultraviolet cutoff δ, see Eqs. (3.9) and (6.1). Moreover, the parameter ϵ enters only some intermediate quantities and never appears in the final observable results. We demonstrated explicitly that quantum corrections to the physically observable quantities can be always obtained by working only with on shell quantities. As a result, for a black hole of a mass, much greater than the Planckian mass, the quantum corrections to observables are small and independent of the physics at Planckian scales. This differentiates on shell quantities from the off-shell ones, such as $S^{\rm SM}$.

There remains one more general question to be clarified. All the observables characterizing a black hole in a thermal equilibrium, or its slow transition from one equilibrium state to another, can be found by using only on shell quantities. Why at all does one need to use off-shell methods in the black hole thermodynamics? We have already seen that one of the reasons is the desire to establish a relation between statistical-mechanical and thermodynamical entropies. In this sense, the off-shell methods can be considered as a useful tool for calculation and interpretation of the on shell quantities. But we believe that, in addition to this trivial reason, there may exist another more deep one. The off-shell approaches may also be relevant for description of nonequilibrium processes in a system including a black hole. In this case quantum and thermal fluctuations of a thermodynamical system can be described by introducing stochastic noise [66], which effectively takes a system off shell. For this reason, one may guess that such processes, for instance, as transition to a thermal equilibrium of a black hole initially exited by high energy explosion near its horizon, may require for their consideration some of the above-mentioned off-shell characteristics.

ACKNOWLEDGMENT

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

APPENDIX A: CONFORMAL TRANSFORMATIONS OF THE EFFECTIVE ACTION IN TWO DIMENSIONS

For completeness we derive in this Appendix the conformal transformations for the effective action

$$W_1[\gamma]=\frac{1}{2}\ln\det[-\triangle]=-\frac{1}{2}\int_0^\infty\frac{ds}{s}\mathrm{Tr}(e^{s\triangle}) \tag{A1}$$

defined on a 2D Euclidean manifold $\mathcal{M}_\alpha$ with the boundary $\partial\mathcal{M}_\alpha$ and a point x_s where $\mathcal{M}_\alpha$ has the conical singularity with the deficit angle $2\pi(1-\alpha)$. We will follow the method developed in [62] and use for this aim the dimensional regularization. Consider the effective action W_1 for the conformally invariant operator $D=\triangle-(d-2)[4(d-1)]^{-1}R$ in a d-dimensional space. The divergent part W_1^{div} of W_1 can be found from the asymptotic heat kernel expansion

$$\mathrm{Tr}(e^{sD})=\frac{1}{(4\pi s)^{d/2}}\sum_{n=0,1/2,\ldots}^{\infty}a_n^{(d)}s^n. \tag{A2}$$

In two-dimensional case for the dimensional regularization

$$W_1^{\mathrm{div}}=\frac{1}{d-2}\,\frac{a_1^{(d)}}{4\pi}, \tag{A3}$$

where for an arbitrary α [41,42]

$$a_1^{(d)}=\left(\frac{1}{6}-\frac{d-2}{4(d-1)}\right)\int_{\mathcal{M}_\alpha}R+\frac{\pi}{3}\left(\frac{1}{\alpha}-\alpha\right)\int_\Sigma+\frac{1}{3}\int_{\partial\mathcal{M}_\alpha}k. \tag{A4}$$

In Eq. (A4) the singular point x_s is replaced by a singular surface Σ of the dimension $d-2$ and the integral of the scalar curvature R is taken over the regular part of $\mathcal{M}_\alpha$. k is the second fundamental form of the spatial boundary $\partial\mathcal{M}_\alpha$ defined in terms of its normal as $k=\nabla^\mu n_\mu$.

The renormalized action is defined as the difference of the nonrenormalized (bare) action W_1^{bare} and its divergent part W_1^{div}

$$W_1=W_1^{\mathrm{bare}}-W_1^{\mathrm{div}}. \tag{A5}$$

Under conformal transformation $\tilde{\gamma}_{\mu\nu}=e^{-2\sigma}\gamma_{\mu\nu}$ of the metric on $\mathcal{M}_\alpha$, the renormalized action changes as [62]

$$W_1(\tilde{\gamma})-W_1(\gamma)=\frac{1}{4\pi}\lim_{d\to2}\frac{1}{2-d}[a_1^{(d)}(\tilde{\gamma})-a_1^{(d)}(\gamma)]. \tag{A6}$$

Further, we will consider only those transformations which do not ''squash'' the conical singularity. Then, by making use of the relations

$$\tilde{R}=e^{2\sigma}\{R+(d-1)[2\triangle\sigma+(2-d)\sigma_{,\alpha}\sigma^{,\alpha}]\}, \tag{A7}$$

$$\tilde{k}=e^{\sigma}[k-(d-1)\sigma_{,\mu}n^\mu], \tag{A8}$$

one gets, from Eq. (A6),

$$W(\tilde{\gamma})-W(\gamma)=\frac{1}{24\pi}\left[\int_{\mathcal{M}_\alpha}[R\sigma-(\nabla\sigma)^2]+\int_{\partial\mathcal{M}_\alpha}(2k\sigma+3\sigma_{,\mu}n^\mu)\right]+\frac{1}{12}\left(\frac{1}{\alpha}-\alpha\right)\sigma(x_s). \tag{A9}$$

This is the desired conformal transformation of the effective action where $\sigma(x_s)$ is the value of the conformal factor in the point of conical singularity. If the manifold has a number of conical singularities in points x_s with different deficits $2\pi(1-\alpha_s)$, then the last term in the right-hand side of Eq. (A9) must be replaced by the corresponding sum over all x_s. If the manifold does not have conical singularities the last term in Eq. (A9) vanishes ($\alpha=1$). Equation (A9) can be also represented in another equivalent form which sometimes is more convenient

$$\begin{aligned}W(\tilde{\gamma})-W(\gamma)={}&\frac{1}{48\pi}\int_{\mathcal{M}_\alpha}d^2x\,\sigma(\tilde{\gamma}^{1/2}\tilde{R}+\gamma^{1/2}R)\\&+\frac{1}{24\pi}\int_{\partial\mathcal{M}_\alpha}dx\,\sigma(\tilde{h}^{1/2}\tilde{k}+h^{1/2}k)\\&-\frac{1}{8\pi}\int_{\partial\mathcal{M}_\alpha}dx(\tilde{h}^{1/2}\tilde{k}-h^{1/2}k)\\&+\frac{1}{12}\left(\frac{1}{\alpha}-\alpha\right)\sigma(x_s).\end{aligned} \tag{A10}$$

Here,

$$h^{1/2}k-\tilde{h}^{1/2}\tilde{k}=h^{1/2}n^\alpha\partial_\alpha\sigma$$

and the conformal factor σ should be understood as a solution of the equation

$$-2\gamma^{1/2}\Box\sigma=\gamma^{1/2}R-\tilde{\gamma}^{1/2}\tilde{R}.$$

APPENDIX B: EFFECTIVE ACTION AND FREE ENERGY OF A SCALAR FIELD IN TWO DIMENSIONS

Let us consider a conformal massless scalar field ϕ on a two-dimensional manifold. The two-dimensional metric is supposed to be independent on the Euclidean time. It can be represented in the form

$$ds^2=\exp[2\sigma(x)]\{d\tau^2+dx^2\},\quad 0\le\tau\le\beta,\quad x_0\le x\le x_1. \tag{B1}$$

The conformal scalar field ϕ satisfies the equation

$$\triangle\phi=\exp[-2\sigma(x)]\left\{\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial x^2}\right\}\phi=0. \tag{B2}$$

For simplicity, we consider the problem with the Dirichlet boundary conditions $\phi(x_0)=\phi(x_1)=0$.

Using the conformal transformation of the effective action (see Appendix A), we can reduce the problem of calculation of the effective action on the manifold Eq. (B1) to a calculation of the effective action on a cylinder Q with period in Euclidean time β and length $L=x_1-x_0$. The one-loop effective action on a cylinder $W_1^Q(\beta,L)$ can be written in the form

$$W_1^Q(\beta,L)=\frac{1}{2}\ln\det(-\mu^2\triangle)$$
$$=-\frac{1}{2}\zeta'(0)+\frac{1}{2}\zeta(0)\ln\mu^2$$
$$=-\frac{1}{2}\left[\frac{\partial}{\partial z}\sum_{\lambda}(\mu^2\lambda)^{-z}\right]_{z=0}.$$

Here, μ is an arbitrary parameter with a dimensionality of length and the generalized ζ function $\zeta(z)=\Sigma_\lambda[\mu^2\lambda]^{-z}$ represents the sum over all eigenvalues λ of the operator $-\triangle$. Although, the effective action is determined up to the rescaling of the parameter μ, all the physical observables are unambiguously defined. For the Dirichlet boundary conditions the substitution of the eigenvalues $\lambda_{mn}=(2\pi/\beta)^2n^2+(\pi/L)^2m^2$ of the Laplace operator on the cylinder leads to the relation

$$W_1^Q(\beta,L)$$
$$=-\frac{1}{2}\left\{\frac{\partial}{\partial z}\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\left[\mu^2\left(\frac{4\pi^2}{\beta^2}n^2+\frac{\pi^2}{L^2}m^2\right)\right]^{-z}\right\}_{z=0}$$
$$=-\frac{1}{2}\left\{\frac{\partial}{\partial z}\prod_{m=1}^{\infty}\prod_{n=-\infty}^{\infty}\left(\frac{2\pi\mu}{\beta}n\right)^{-2z}\right.$$
$$\left.\times\left(1+\frac{\beta^2}{4L^2}\frac{m^2}{n^2}\right)^{-z}\right\}_{z=0}. \tag{B3}$$

Applying the formula

$$\prod_{n=1}^{\infty}\left(1+\frac{a^2}{n^2}\right)=\frac{\sinh\pi a}{\pi a} \tag{B4}$$

and representing other infinite sums and products in terms of the Riemann ζ function, we eventually have

$$W_1^Q(\beta,L)=\beta\mathcal{F}-\frac{\pi\beta}{24L}, \tag{B5}$$

where

$$\beta\mathcal{F}=\sum_{n=1}^{\infty}\ln\left(1-\exp\left[-\beta\frac{\pi}{L}n\right]\right). \tag{B6}$$

We demonstrate now that $\mathcal{F}$ coincides with the thermodynamical free energy of a gas of scalar particles in the volume L. In statistical mechanics the free energy $\mathcal{F}$ of a quantum system is defined by a relation

$$\exp[-\beta\mathcal{F}]=\mathrm{Tr}\exp[-\beta{:}\hat{H}{:}]. \tag{B7}$$

If we choose the basis functions to be eigenfunctions of the Hamiltonian $\hat{H}=\sqrt{-\partial_x^2}$, the free energy can be expressed in terms of a sum over all dynamical degrees of freedom

$$\beta\mathcal{F}=\sum_n\ln(1-e^{-\beta\omega_n}), \tag{B8}$$

where β is an inverse temperature and ω_n are the energy levels of the quantum system. Thus, we are to know only the spectrum of the system to calculate the free energy. One can easily solve the Eq. (B2) and find the energy levels of the system

$$\omega_n=\frac{\pi}{L}n,\quad L=x_1-x_0.$$

Note that the mode with $n=0$ should be eliminated from the summation in Eq. (B8), since its amplitude is fixed by the Dirichlet boundary conditions and, hence, it is not normalizable and is not a dynamical degree of freedom. (For the Neumann boundary conditions zero modes will contribute to the free energy.)

Thus, for the Dirichlet boundary conditions the free energy $\mathcal{F}$ reads

$$\mathcal{F}=\frac{1}{\beta}\sum_{n=1}^{\infty}\ln\left(1-\exp\left[-\beta\frac{\pi}{L}n\right]\right),$$

which coincides with Eq. (B6).

Now, let us calculate $\mathcal{F}$ in the high-temperature limit, i.e., when the length of the cylinder L is much larger than its perimeter β. In this limit the distance between the levels is less than temperature $\pi/L\ll 1/\beta$ and the sum over n can be estimated using the Euler-McLourain formula

$$\sum_{n=1}^{\infty}f(n)=\int_0^{\infty}dx f(x)-\int_0^1 dx f(x)+\frac{1}{2}f(1)+\sum_{k=1}^{\infty}c_k f^{(k)}(1).$$

Here, the coefficients c_k can be expressed in terms of Bernouli numbers

$$c_k=(-1)^k\frac{B_{k+1}}{(k+1)!}$$

and the function $f(x)$ is supposed to decrease at infinity together with all its derivatives. Substituting here the function $f(x)=\ln[1-\exp(-sx)]$ and taking into account the relation

$$\ln\Gamma(z)=(z-\tfrac{1}{2})\ln(z)-z+\frac{1}{2}\ln(2\pi)$$
$$+\sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)(2m-1)z^{2m-1}},$$
$$|\arg z|<\pi,$$

one can prove that

$$\sum_{n=1}^{\infty}\ln(1-\exp[-sn])=-\frac{\pi^2}{6s}-\frac{1}{2}\ln\left[\frac{s}{2\pi}\right]+\frac{1}{24}s+o(s). \tag{B9}$$

For the free energy it leads to a formula

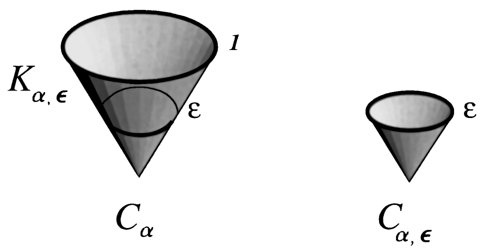

FIG. 7. Cones.

$$\beta\mathcal{F}=-\frac{\pi L}{6\beta}-\frac{1}{2}\ln\frac{\beta}{2L}+\frac{\pi\beta}{24L}+o\left(\frac{\beta}{L}\right), \tag{B10}$$

and hence the effective action reads

$$W_1^Q=-\frac{\pi L}{6\beta}-\frac{1}{2}\ln\frac{\beta}{2L}+o\left(\frac{\beta}{L}\right). \tag{B11}$$

It can be shown that $o(\beta/L)$ is nonanalytical in its argument and tends to zero extremely fast when $\beta\leqslant L$.

Note that by construction $\beta\mathcal{F}$ for a conformal fields is conformally invariant, since the spectrum is conformally invariant. This property distinguishes it from an Euclidean effective action W_1 which transforms inhomogeneously under the conformal transformations because of the conformal anomaly. Note that the renormalized effective action $W_1^Q(\beta,L)$ and $\beta\mathcal{F}$ differ only by the term linear in β [67,68].

APPENDIX C: CASIMIR EFFECT AND FIELD FLUCTUATIONS NEAR THE BRICK-WALL BOUNDARY

In this appendix we present a more detailed discussion of the field fluctuations on the boundary near the horizon and their relation with the Casimir effect which inevitably arises in the brick-wall approach. Instead of the black hole background, and consider the quantum field in the Rindler space at the inverse temperature $2\pi\alpha$ measured at the point $x=1$, and put $\mu=1$. This simplification is justified by the fact that we are interested in the effects which happen very close to the horizon where the space is similar to a cone (6.3).

Assume that the brick wall is at the point $x=\epsilon$ in coordinates (6.3). The brick-wall effective action in this case is the action on the part $K_{\alpha,\epsilon}$ of the cone C_α, see Fig. 7.

Then, as follows from Eqs. (5.6), (5.7), and (9.9), the analogue of the Eq. (9.10) for the cone

$$W_1^{\rm BW}(\alpha,\epsilon)=W_1[K_{\alpha,\epsilon}]$$
$$=W_1[C_\alpha]-W_1[C_{\alpha,\epsilon}]+W_1^{\rm Cas}(2\pi\alpha,\alpha,\epsilon), \tag{C1}$$

$$W_1^{\rm Cas}(2\pi\alpha,\alpha,\epsilon)=-\frac{1}{2}\ln\frac{\pi\alpha}{\ln\epsilon^{-1}}.$$

Our aim now is to understand how the presence of the Casimir term $W_1^{\rm Cas}(2\pi\alpha,\alpha,\epsilon)$ is related with the quantum fluctuations near the point $x=\epsilon$. This can be done by analyzing the path integral representation for the partition function on C_α:

$$Z_1[C_\alpha]=e^{-W_1[C_\alpha]}$$
$$=\int[D\phi]e^{-I[\phi]}$$
$$=\int[D\phi]\exp\left(-\frac{1}{2}\int\phi_{,\mu}\phi^{,\mu}\right). \tag{C2}$$

Here, one can divide the variables into three groups

$$Z_1[C_\alpha]=\int[D\phi_1][D\psi][D\phi_2]e^{-I[\phi]}, \tag{C3}$$

where ϕ_1 and ϕ_2 are the fields in the domains $x<\epsilon$ and $x>\epsilon$, respectively, and $\psi=\phi(x=\epsilon)$. In each of the regions one can change the fields as

$$\phi_k=\phi_k'+\chi_k, \tag{C4}$$

$$\triangle\chi_k=0,\quad \chi_k(x=\epsilon)=\psi,\quad k=1,2,\quad \chi_2(x=1)=0. \tag{C5}$$

The new variables ϕ_k' satisfy the Dirichlet conditions on the boundaries of their domains. Using this fact and that the fields χ_k are harmonic, one can represent the classical action in the way

$$I[\phi_1+\phi_2]=I[\phi_1']+I[\phi_2']+\mathcal{W}[\psi], \tag{C6}$$

where $\mathcal{W}[\psi]=I[\chi_1]+I[\chi_2]$ for $\chi_1(x=\epsilon)=\chi_2(x=\epsilon)=\psi$. The partition function (C3) is represented now in the form where contributions from the fields ϕ_1, ϕ_2, and χ are completely factorized

$$Z_1[C_\alpha]=\int[D\phi_1']e^{-I[\phi_1']}\int[D\psi]e^{-\int\mathcal{W}[\psi]}\int[D\phi_2']e^{-I[\phi_2']}$$
$$=Z[C_{\alpha,\epsilon}]Z[K_{\alpha,\epsilon}]\int[D\psi]e^{-\int\mathcal{W}[\psi]}. \tag{C7}$$

The first multiplier in Eq. (C7) is the partition function on a cone of the small radius ϵ, the second one is the partition function on the space $K_{\alpha,\epsilon}$, which is determined by the brick-wall action $W_1^{\rm BW}(\alpha,\epsilon)$

$$Z[K_{\alpha,\epsilon}]=e^{-W_1[K_{\alpha,\epsilon}]}=e^{-W_1^{\rm BW}(\alpha,\epsilon)}. \tag{C8}$$

The left integral over ψ describes the quantum fluctuations of the field in the point $x=\epsilon$. Let us show that it reproduces explicitly the Casimir term in the effective action. Indeed, Eqs. (C5) have the solutions

$$\chi_1(x,\tau)=\sqrt{\frac{1}{\pi\alpha}}\sum_{n=1}^{\infty}\left(\psi_n^{(1)}\cos\frac{n\tau}{\alpha}+\psi_n^{(2)}\sin\frac{n\tau}{\alpha}\right)\left(\frac{x}{\epsilon}\right)^{n/\alpha}, \tag{C9}$$

$$\chi_2(x,\tau)=\sqrt{\frac{1}{\pi\alpha}}\sum_{n=1}^{\infty}\left(\psi_n^{(1)}\cos\frac{n\tau}{\alpha}+\psi_n^{(2)}\sin\frac{n\tau}{\alpha}\right)$$
$$\times\left(\frac{\epsilon}{x}\right)^{n/\alpha}\frac{1-x^{2n/\alpha}}{1-\epsilon^{2n/\alpha}}+\frac{\psi_0}{\sqrt{2\pi\alpha}}\ln x/\epsilon, \tag{C10}$$

where $\psi_n^{(k)},\psi_0$ are the Fourier coefficients of the field ψ on the boundary:

$$\psi(\tau)=\sqrt{\frac{1}{\pi\alpha}}\sum_{n=1}^{\infty}\left(\psi_n^{(1)}\cos\frac{n\tau}{\alpha}+\psi_n^{(2)}\sin\frac{n\tau}{\alpha}\right)+\frac{\psi_0}{\sqrt{2\pi\alpha}} \tag{C11}$$

defined with respect to the orthonormal basis on the circle $0\leqslant\tau\leqslant 2\pi\alpha$. This gives the action up to the terms of $O(\epsilon)$ in the form

$$\mathcal{W}[\psi]=I[\chi_1]+I[\chi_2]$$
$$=\frac{1}{\alpha}\sum_{n=1}^{\infty}[(\psi_n^{(1)})^2+(\psi_n^{(2)})^2]+\left(2\ln\frac{1}{\epsilon}\right)^{-1}\psi_0^2+O(\epsilon). \tag{C12}$$

The integral over ψ has the Gaussian form and can be evaluated exactly. The integration measure can be written up to a normalization numerical coefficient as

$$[D\psi]=\epsilon^{1/2}d\psi_0\prod_{n=1}^{\infty}\epsilon^{1/2}d\psi_n^{(1)}\prod_{n=1}^{\infty}\epsilon^{1/2}d\psi_n^{(1)}, \tag{C13}$$

where the multiplier $\epsilon^{1/2}$ is the heritage of the definition of the covariant measure which includes the factor $g^{1/4}$ at $x=\epsilon$. Thus, the result of the integration over fields ψ looks as

$$\int[D\psi]e^{-\int\mathcal{W}[\psi]}=\mathcal{N}\left(\epsilon\ln\frac{1}{\epsilon}\right)^{1/2}\exp\left(\sum_{n=1}^{\infty}\ln(\alpha\epsilon)\right) \tag{C14}$$

($\mathcal{N}$ is the numerical constant) which after regularization of the infinite sum with the help of the Riemann zeta function $\zeta_R(z)$

$$\sum_{n=1}^{\infty}=\lim_{z\to 0}\sum_{n=1}^{\infty}n^{-z}=\zeta_R(0)=-\frac{1}{2}$$

gives the Casimir term

$$\int[D\psi]e^{-\int\mathcal{W}[\psi]}=\mathcal{N}\left(\frac{\ln\epsilon^{-1}}{\alpha}\right)^{1/2}=\mathcal{N}e^{W_1^{\rm Cas}(\alpha,\epsilon)}. \tag{C15}$$

The Eqs. (C7) and (C15) result in the formula

$$e^{-W_1[C_\alpha]}=Z_1[C_{\alpha,\epsilon}]Z[K_{\alpha,\epsilon}]e^{W_1^{\rm Cas}(\alpha,\epsilon)}$$
$$=\exp\{-(W[C_{\alpha,\epsilon}]+W[K_{\alpha,\epsilon}]-W_1^{\rm Cas}[\alpha,\epsilon])\} \tag{C16}$$

which, obviously, reproduces the relation (C1) between the brick-wall action $W_1^{\rm BW}$ and action $W_1[C_\alpha]$ on the cone, which we obtained earlier by the conformal transformation.

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$$\int_{M_B^\alpha} f(r)R(r)\gamma^{1/2}d^2x = \alpha\int_{M_B} f(r)R(r)\gamma^{1/2}d^2x + 4\pi(1-\alpha)f(r_h). \qquad \text{(C17)}$$

We use this relation for $f(r)=r^2$, $r_h=r_+$. The first term in the right-hand side is calculated on the regular manifold $M_B = M_B^\alpha|_{\alpha=1}$, while the second term is present because of the conical singularity.

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