Euclidean approach to the entropy for a scalar field in Rindler-like space-times

Sergio Zerbini,* Guido Cognola,† and Luciano Vanzo‡
Dipartimento di Fisica, Università di Trento and Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Trento, Italy
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The off-shell entropy for a massless scalar field in a D-dimensional Rindler-like space-time is investigated within the conical Euclidean approach in the manifold $C_\beta\times\mathcal{M}^N$, C_β being the two-dimensional cone, making use of the ζ-function regularization. Because of the presence of conical singularities, it is shown that the relation between the ζ function and the heat kernel is nontrivial and, as first pointed out by Cheeger, requires a separation between small and large eigenvalues of the Laplace operator. As a consequence, in the massless case, the (naive) nonexistence of the Mellin transform is bypassed by Cheeger's analytical continuation of the ζ function on the manifold with conical singularities. Furthermore, the continuous spectrum leads to the introduction of smeared traces. In general, it is pointed out that the presence of the divergences may depend on the smearing function and they arise in removing the smearing cutoff. With a simple choice of the smearing function, horizon divergences in the thermodynamical quantities are recovered and these are similar to the divergences found by means of off-shell methods such as the brick-wall model, the optical conformal transformation techniques, or the canonical path-integral method. [S0556-2821(96)05316-7]

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I. INTRODUCTION

As is well known there exist several methods for calculating the semiclassical entropy (tree-level contribution) for a stationary black hole (see, for example, [1]). In Einstein theory, for a nonrotating four-dimensional black hole, all the methods lead to the celebrated Bekenstein-Hawking classical entropy $4\pi GM^2$ [2–4]. The thermodynamical origin of this quantity is well known and recently in a series of papers (see, for example, [5] and references therein) this fact has been stressed; namely, the Bekenstein-Hawking entropy can be defined by the response of the free energy of the black hole to the change of the equilibrium (Unruh-Hawking) temperature. This temperature depends on the mass of the black hole and may be determined by requiring the smoothness of the related Euclidean solution [4]. This is an example of an on-shell computation. In the derivation of the above result, one usually neglects quantum fluctuation effects. If one takes quantum effects into account, one can show that the on-shell one-loop contribution is finite (see, for example, Ref. [5]).

The situation drastically changes if one tries to investigate the issue of the black-hole entropy within a statistical-mechanical approach, i.e., by counting the quantum states of the black hole. In this case, in order to evaluate the entropy, one is forced to work off shell, namely, at a temperature different from the Unruh-Hawking one. The first off-shell computation of the black-hole entropy appeared in 't Hooft's seminal paper [6], where the black hole degrees of freedom identified with those of a quantum gas of scalar particles propagating outside, but very near the horizon at a temperature β^{-1}. The statistical-mechanical quantities were found to be divergent and regularized by Dirichlet boundary conditions imposed at a small distance from the black-hole horizon (the so-called brick-wall model).

In a generic off-shell procedure, these divergences are not totally unexpected. In fact their physical origin may be described by the following simple considerations. The equivalence principle implies that a system in thermal equilibrium has a local Tolman temperature given by $T(x)=T/\sqrt{|g_{00}(x)|}$, T being the temperature measured at the spatial infinity. Thus, the asymptotic high-temperature expansion for the free energy of a massless quantum gas on a D-dimensional static space-time may be written as

$$F_T\simeq T^D\int |g_{00}(\vec{x})|^{-D/2}\sqrt{|g_{D-1}(\vec{x})|}dx^{D-1},$$

where $g_{D-1}=\det\{g_{ij}\}$ $(i,j=1,\dots,D-1)$. In the presence of horizons, the integrand have nonsummable singularities and horizon divergences appear. As a consequence also the entropy is divergent. The nature of these divergences depends on the zeros and the poles of g_{00} and g_{D-1} respectively. In general, for extreme black holes, where g_{00} has higher-order zeros, the divergences are much more severe than the divergences in the nonextremal case (see, for example, [7]).

These considerations suggest the use of another off-shell method, based on conformal transformation techniques, which consists in mapping the original metric onto the optical one $\overline{g_{\mu\nu}}=g_{\mu\nu}/g_{00}$, (see Refs. [8–10]). Related methods that lead to optical manifolds have been considered in Refs. [11–13]. The conformal optical transformation method has been used in the case of fields in four-dimensional black hole space-time [14,15] and also for massive scalar fields in D-dimensional Rindler-like space-times [16]. These are space-times of the form $\mathbb{R}\times\mathbb{R}^+\times\mathcal{M}^N$, with the metric

$$ds^2=-\frac{b^2\rho^2}{r_H^2}dx_0^2+d\rho^2+d\sigma_N^2,\quad N=D-2\,, \tag{1}$$

*Electronic address: zerbini@science.unitn.it
†Electronic address: cognola@science.unitn.it
‡Electronic address: vanzo@science.unitn.it

where r_H is a dimensional constant, b a constant factor and $d\sigma_N^2$ the spatial metric related to the smooth manifold $\mathcal{M}^N$. If $\mathcal{M}^N=\mathbb{R}^N$, then $b=1$, $r_H=1/a$, a being the constant acceleration, the manifold is noncompact and one has to deal just with the D-dimensional Rindler space-time. If $\mathcal{M}^N=S^N$, one can show that one is dealing with a space-time that approximates, near the horizon and in the large mass limit, a D-dimensional black hole (see, for example, [16]). In this case $b=(D-3)/2$ and r_H is the horizon radius depending on the mass of the black hole.

In the case of canonical horizons [this means that $g_{00}(\vec{x})$ has simple zeros only] and in order to treat finite-temperature effects, an alternative off-shell approach has been proposed [17–21]. It consists in performing the Wick rotation $x_0=-i\tau$ and working in the Euclidean continuation of the space-time, with the imaginary time compactified to an arbitrary interval of length β and with the fields periodic in τ with period β, which is interpreted as the inverse temperature. For an arbitrary choice of β, such a static manifold has a conical-like singularity. As already mentioned, only in the absence of such a singularity there exist equilibrium states with finite stress-energy tensor. This happens only for $\beta=\beta_H$, the Unruh-Hawking temperature [4]. Furthermore, the Bekenstein-Hawking entropy can be derived within this approach [22,17]. For the metric of our simplified model, Eq. (1), one has

$$\beta_H=\frac{2\pi r_H}{b}. \tag{2}$$

The situation looks quite different in the computation of the quantum corrections to the entropy. In fact in this case one is really forced to consider an arbitrary β, thus working off shell again. This proposal seems highly nontrivial, since only in an ultrastatic space-time the imaginary time formalism has been shown to be equivalent to the canonical formalism of the finite temperature quantum field theory. The use of the trace of the heat kernel plus standard proper-time regularization on manifolds with conical singularities leads to a free energy, which is ultraviolet divergent and with a dependence on temperature different from the one expected for a D-dimensional space. In fact the leading term goes as T^2 independently on the dimensions [23]. Let us summarize this crucial issue, which seems to have been overlooked in the recent papers.

It is well known (see Sec. II) that one-loop effects can be described by ζ-function regularization. The ζ function related to a free massless scalar field on $\mathcal{M}^D$ can be obtained by means of the Mellin transform of $K(t|-\Delta_D) = \mathrm{Tr}\exp(t\Delta_D)$, Δ_D being the Laplace operator. However, in the noncompact manifold $\mathcal{M}^D=C_\beta\times\mathcal{M}^N$, in order to give meaning to the trace of the heat kernel, one has to use a smearing function ϕ. A simple choice is given by the product of the characteristic function $\chi(\Omega)$ ($\Omega\subset\mathbb{R}^N$, compact) and a cutoff function $\theta(\Lambda-\rho)$ regularizing the infinite conical volume. One has [24–27]

$$K(t|-\Delta_D)(\phi)=\frac{V_D}{(4\pi t)^{D/2}}+\frac{1}{12}\left(\frac{2\pi}{\beta}-\frac{\beta}{2\pi}\right)\frac{V_N}{(4\pi t)^{N/2}},$$

V_N being the volume of Ω and $V_D=\beta\Lambda^2V_N/2$. As a result, the naive Mellin transform of this heat-kernel trace does not exist or is zero if interpreted in the sense of distributions. If a mass term is included, then the latter equation has to be multiplied by the factor $\exp(-tm^2)$ and the global ζ function may be defined via the Mellin transform [26], obtaining in this way a well-defined quantity, apart from the volume divergences associated with the noncompactness of the manifold. The surprising thing is that, in contrast with the other methods mentioned above, such a partition function seems not to have any horizon divergence. It has also a dependence on β, which does not depend on the dimensions of the Rindler space one is considering and, besides, it vanishes in the limit $m\to 0$. Of course, the contribution computed in this way is only the finite part of the partition function, since ordinary ultraviolet divergences are present (formally one is dealing with a ''zero-temperature'' field theory on a nontrivial background) and they have been accounted for by means of the ζ-function regularization. If one makes use of another regularization, for example the proper-time regularization, it turns out that such ultraviolet divergences are not confined to the vacuum sector as in the usual finite-temperature theory on ultrastatic space-times, but they appear also in the nontrivial part of the free energy in such a way that they give a contribution to the entropy, even if it is evaluated at the Hawking temperature [18–21]. In the limit $m\to 0$ only the leading divergent term remains and this has been interpreted as the analogy of horizon divergence.

In this paper, in order to try to elucidate this issue, we will present an off-shell Euclidean ζ-function regularization approach applicable directly in the massless case and we will make the comparison between this Euclidean conical method and the optical conformal transformation methods (the brick-wall method gives the same result as the latter). To avoid the risk of creating confusion, other kinds of regularizations shall not be taken into account. Our main aim will be the construction of the ζ function for a massless scalar field in $C_\beta\times\mathcal{M}^N$, starting from the local one, which can be evaluated by using an analytical procedure suggested by Cheeger [24]. As mentioned above, all global quantities, for example free energy or entropy, which are related to the trace of some operator, require a smearing function in order to be defined. The horizon divergences appear in the smearing removal. It turns out that the change of the smearing prescription may modify the final result. However, we will show that the results obtained in this way are compatible with the ones obtained by using the optical conformal method, including the correct dimensional behavior of the free energy on β. We also would like to mention that the techniques presented in this paper may be useful in investigating quantum fields in space-times with spatial conical singularity (see, for example, [27,28] and references therein).

The contents of the paper are the following. In Sec. II, the general formalism is summarized and the partition function, as well as the other related quantities, are introduced. In Sec. III the local ζ function for a massless scalar field in a Rindler space is constructed according to the Cheeger method and the global (smeared) ζ function is computed and then generalized to any Rindler-like space-times. In Sec. IV the thermodynamical quantities in these kind of spaces are derived and their properties analyzed. The application to the four-dimensional black hole, in the large mass limit approximation, is presented in Sec. V. Choosing a suitable smearing function, horizon divergences of the first quantum corrections to the free energy are recovered. The conclusions are reported in Sec. VI. The paper ends with an Appendix devoted to the analytical extension of a series that frequently appears in the formulas and that plays an important role in the Cheeger method.

II. GENERAL FORMALISM

To start with we recall the formalism we shall use in the following in order to discuss the finite temperature effects within this conical singularity approach. We may consider, as a prototype of the quantum correction (quantum degrees of freedom), a scalar field on a D-dimensional Rindler-like space-time. Its related Euclidean metric reads

$$ds^2=\frac{b^2\rho^2}{r_H^2}d\tau^2+d\rho^2+d\sigma_N^2, \quad x=(\tau,\rho,\vec{x}),$$

where τ is the imaginary time, $\rho\geqslant 0$ the radial coordinate and $\vec{x}$ the transverse coordinates. As explained in the Introduction, finite temperature effects are assumed to arise when τ is compactified $0\leqslant\tau\leqslant\beta$, β being the inverse of the temperature. For arbitrary β the manifold $\mathcal{M}^D$ has the topology of $C_\beta\times\mathcal{M}_N$, C_β being the two-dimensional cone, $(\tau,\rho)\in C_\beta$, $\vec{x}\in\mathcal{M}_N$. From now on, we put $r_H=1$ and $\tau\to b\tau$.

The one-loop partition function depends on β and is given by

$$Z_\beta=\int d[\phi]\exp\left(-\frac{1}{2}\int \phi L_D\phi d^Dx\right), \tag{3}$$

where ϕ is a scalar density of weight $-1/2$, which obeys periodic boundary conditions $\phi(0,\vec{x})=\phi(\beta,\vec{x})$ and L_D is the Laplace-like operator on $C_\beta\times\mathcal{M}_N$. In our case, it has the form

$$L_D=-\Delta_D+\xi R+m^2=-\Delta_\beta+L_N=-\Delta_\beta-\Delta_N+\xi R+m^2. \tag{4}$$

Here Δ_D, Δ_N, and Δ_β are the Laplace-Beltrami operators on $\mathcal{M}^D$, $\mathcal{M}^N$, and C_β respectively, ξ is an arbitrary parameter, m the mass and R the scalar curvature of the manifold, which is assumed to be a constant.

In the one-loop or external field approximation the importance of the ζ-function regularization as a powerful tool to deal with the ambiguities (ultraviolet divergences) present in the relativistic quantum field theory is well known (see, for example, [29]). It permits us to give a meaning, in the sense of analytic continuation, to the determinant of a differential operator that, as a product of eigenvalues, is formally divergent. One has [30]

$$\ln Z_\beta=-\tfrac{1}{2}\ln\det L_D=\tfrac{1}{2}\zeta'_\beta(0|L_D)+\tfrac{1}{2}\zeta_\beta(0|L_D)\ln\mu^2,$$

where $\zeta_\beta(s|L_D)$ is the ζ-function related to L_D, $\zeta'_\beta(0|L_D)$ its derivative with respect to s, and μ^2 a renormalization scale. The analytically continued ζ function is regular at $s=0$ and thus its derivative is well defined.

When the manifold is smooth and compact the spectrum is discrete and one has

$$\zeta_\beta(s|L_D)=\sum_i \lambda_i^{-2s},$$

λ_i^2 being the eigenvalues of L_D. As a result, one can make use of the relationship between the ζ function and the heat-kernel trace via the the Mellin transform and its inverse. For $\mathrm{Re}\,s>D/2$, one can write

$$\zeta_\beta(s|L_D)=\mathrm{Tr}L_D^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_\beta(t|L_D)dt, \tag{5}$$

$$K_\beta(t|L_D)=\frac{1}{2\pi i}\int_{\mathrm{Re}\,s>D/2} t^{-s}\Gamma(s)\zeta_\beta(s|L_D)ds, \tag{6}$$

where $K_\beta(t|L_D)=\mathrm{Tr}\exp(-tL_D)$ is the heat operator. The previous relations are valid also in the presence of zero modes with the trivial replacement $K_\beta(t|L_D)\to K_\beta(t|L_D)-P_0$, P_0 being the projector onto the zero modes. We may call this the global approach. Moreover one may follow a local approach, starting from local quantities like the heat kernel and the related Mellin transform local ζ function $\zeta_\beta(s;x|L_D)$. Then one may introduce an effective Lagrangian density

$$\mathcal{L}(x)=\frac{1}{2}\zeta'_\beta(0;x|L_D)+\frac{1}{2}\zeta_\beta(0;x|L_D)\ln\mu^2,$$

obtaining, in this way,

$$\ln Z_\beta=\frac{1}{2}\int [\zeta'_\beta(0;x|L_D)+\zeta_\beta(0;x|L_D,0)\ln\mu^2]dV_D. \tag{7}$$

Normally the two approaches give the same results. In the presence of conical singularities and in the massless case, we have seen that the global approach cannot be used, so we shall make use of the local one. In the presence of conical singularities and in the noncompact case (continuous spectrum), some care has to be used in the implementation of the relationship between heat kernel and local ζ function. With regard to this, we shall show in the next section that a separation of lower eigenvalues from the higher ones, together with a suitable analytic continuation, is necessary [24]. Furthermore the local ζ function turns out to be a nonlocal summable function. For this reason one can take the distributional characters of the local ζ function into account by introducing a smearing in terms of a function with compact support. In this way, in order to treat global quantities, one has to deal with smeared traces [31].

Once the (smeared) partition function is given by Eq. (7), we assume the validity of the usual thermodynamical relations, thus the free energy can be computed by means of

$$F_\beta=-\frac{1}{\beta}\ln Z_\beta=-\frac{\zeta'_\beta(0|L_D)}{2\beta}-\frac{\zeta_\beta(0|L_D)}{2\beta}\ln\mu^2, \tag{8}$$

and the entropy and the internal energy read

$$S_\beta=\beta^2\partial_\beta F_\beta, \quad U_\beta=\frac{S_\beta}{\beta}+F_\beta. \tag{9}$$

All these quantities are evaluated off shell. However the only admissible equilibrium thermal state is the one corresponding to the Unruh-Hawking temperature $\beta=\beta_H$ [in our model, see Eq. (2)]. Thus, strictly speaking, one has to make the comparison of the different off-shell approaches only at $\beta=\beta_H$.

III. ζ-FUNCTION REGULARIZATION IN A SPACE WITH CONICAL SINGULARITIES

As we mentioned in the Introduction, the evaluation of the partition function using the ζ-function regularization requires some care. Here we shall evaluate the kernel of the ζ function and then we will give a meaning to the global ζ function by means of smearing. As has been stressed by Cheeger [24], it is crucial to treat small and large eigenvalues separately.

The spectral properties of the Laplace operator on the cone are well known and in fact, a complete set of normalized eigenfunctions for $L_\beta=-\Delta_\beta$ is easily found to be

$$\psi(\tau,\rho)=\frac{1}{\sqrt{2\pi}}e^{i\nu_l\tau}J_{\nu_l}(\lambda\rho), \quad \nu_l=\frac{2\pi l}{\beta}, \quad l\in\mathbb{Z}, \tag{10}$$

together with its complex conjugate (double degeneration). Here λ^2 $(\lambda\geqslant 0)$ is the eigenvalue corresponding to ψ and ψ^*, while J_ν is the regular Bessel function. This choice of the eigenfunctions correspond to a positive elliptic self-adjoint operator (the Friedrichs extension [32,33]).

Now, using the standard separation of variables, it is easy to get the spectrum and the eigenfunctions of the operator $L_D=-\Delta_\beta+L_N$ on the Rindler-like space-time $\mathcal{M}^D=C_\beta\times\mathcal{M}^N$, L_N being a Laplace-type operator on $\mathcal{M}_N$ including (eventually) mass and scalar curvature coupling term. Indicating by $f_\alpha(\vec{x})$ and λ_α^2 the eigenvectors and the eigenfunctions of L_N, respectively, one has $\Psi(x)=\psi(\tau,\rho)f_\alpha(\vec{x})$ and $\lambda^2+\lambda_\alpha^2$ for the eigenvectors and the eigenfunctions of L_D. Thus, for the diagonal kernel of a operator $F_\beta(L_D)$ one has

$$F_\beta(x|L_D)=\frac{1}{\beta}\sum_\alpha\left[\int_0^\infty F(\lambda^2+\lambda_\alpha^2)J_0^2(\lambda\rho)\lambda\, d\lambda + 2\sum_{l=1}^\infty\int_0^\infty F(\lambda^2+\lambda_\alpha^2)J_{\nu_l}^2(\lambda\rho)\lambda d\lambda\right]. \tag{11}$$

As it stands, such an expression is only formal, since the series and the integral could not be convergent.

A. A special case: massless scalar field in Rindler space

For the sake of simplicity and for illustrative purposes, let us start to consider a massless scalar field on Euclidean Rindler space $C_\beta\times\mathbb{R}^N$. We suppose $N\geqslant 1$, but all results on the pure cone $(N=0)$ can be obtained as limit cases. For this case $L_N=-\Delta_N$ has a continuous spectrum $\lambda_{\vec{k}}^2=k^2$ and so the sum over α reduces to an integral over $\vec{k}\in\mathbb{R}^N$ and its spectral data are well known: namely,

$$f_{\vec{k}}=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{N/2}}, \quad \lambda_{\vec{k}}^2=k^2=\vec{k}\cdot\vec{k}.$$

We are interested in the local ζ function, so we choose $F(L_D)=L_D^{-s}$ and, using Eq. (11), one formally has

$$\zeta_\beta(s;x|L_D)=\frac{2(4\pi)^{-N/2}}{\beta\Gamma\left(\frac{N}{2}\right)}\int_0^\infty dk\, k^{N-1}\times\left[\int_0^\infty(\lambda^2+k^2)^{-s}J_0^2(\lambda\rho)\lambda\, d\lambda + 2\sum_{l=1}^\infty\int_0^\infty(\lambda^2+k^2)^{-s}J_{\nu_l}^2(\lambda\rho)\lambda\, d\lambda\right]. \tag{12}$$

Recalling the asymptotic behavior of Bessel functions one can easily see that both the integrations over k and λ can be performed in any term of the latter equation if s is restricted in the range

$$\frac{N+1}{2}<\text{Re}s<\frac{N+2+2\nu_l}{2}.$$

In fact one has

$$\frac{2(4\pi)^{-N/2}}{\beta\Gamma\left(\frac{N}{2}\right)}\int_0^\infty dk\, k^{N-1}\int_0^\infty(\lambda^2+k^2)^{-s}J_{\nu_l}^2(\lambda\rho)\lambda\, d\lambda = \frac{\rho^{2s-D}}{2\beta(4\pi)^{N/2}\Gamma(s)}\frac{\Gamma\left(s-\frac{N+1}{2}\right)\Gamma\left(\nu_l-\left(s-\frac{N}{2}\right)+1\right)}{\sqrt{\pi}\Gamma\left(\nu_l+s-\frac{N}{2}\right)}. \tag{13}$$

To get the ζ function, now one has to sum over l. As we shall show in the Appendix, the series is convergent for $\text{Re}s>N/2+1$. This range does not overlap with the previous one for $\nu_l=0$ $(l=0)$. This means that there are no values of s for which Eq. (12) is a finite quantity. The solution of this convergence obstruction has been suggested by Cheeger [24]. It simply consists in a separate treatment of the lower and the higher eigenvalues (in this particular case $\nu_0=0$ and $\nu_l>0$, $l\geqslant 1$). Only after the analytic continuation is performed, one may define the ζ function by summing the two contributions obtained in this way. Of course, such a definition of ζ function has all the requested properties and coincides with the usual one when the manifold is smooth.

So, following Cheeger, in Eq. (12) we first isolate the term $l=0$ and define [see Eq. (13)], for $1/2+N/2<\text{Re}s<1+N/2$,

$$\zeta_<(s;x|L_D)$$
$$=\frac{\rho^{2s-D}}{\beta(4\pi)^{N/2}\Gamma(s)}\frac{\Gamma[s-(N+1)/2]\Gamma(1-s+N/2)}{2\sqrt{\pi}\Gamma(s-N/2)}$$
$$=-\frac{\rho^{2s-D}}{\beta(4\pi)^{N/2}\Gamma(s)}\frac{\Gamma[s-(N+1)/2]G_{2\pi}(s-N/2)}{\sqrt{\pi}}. \quad (14)$$

Then we consider all the other terms, perform the integration as in Eq. (13) and the summation over $l\geqslant 1$. In this second case we have to restrict to $1+N/2<\mathrm{Re}s<1+\nu_1+N/2$. The result reads

$$\zeta_>(s;x|L_D)t=\frac{\rho^{2s-D}}{\beta(4\pi)^{N/2}\Gamma(s)}\times\frac{\Gamma[s-(N+1)/2]G_\beta(s-N/2)}{\sqrt{\pi}}. \quad (15)$$

We have put

$$G_\beta(s)=\sum_{l=1}^{\infty}\frac{\Gamma(\nu_l-s+1)}{\Gamma(\nu_l+s)},\quad G_{2\pi}=-\frac{\Gamma(1-s)}{2\Gamma(s)},$$

the series being convergent for $\mathrm{Re}s>1$. As we shall show in the Appendix, the analytic continuation of $G_\beta(s)$ is a meromorphic function with only a simple pole at $s=1$. This means that both Eqs. (14) and (15) can be analytically continued to the whole complex s plane and, by definition

$$\zeta_\beta(s;x|L_D)=\zeta_<(s;x|L_D)+\zeta_>(s;x|L_D)=\frac{\rho^{2s-D}}{\beta(4\pi)^{N/2}\Gamma(s)}I_\beta\left(s-\frac{N}{2}\right), \quad (16)$$

where

$$I_\beta(s)=\frac{\Gamma(s-1/2)}{\sqrt{\pi}}[G_\beta(s)-G_{2\pi}(s)].$$

The properties of G_β, as well as of I_β, will be studied in the Appendix. An important property is that I_β, as well as G_β, has only a simple pole at $s=1$.

Note that in spite of the definition (16), in the use of the inverse Mellin transform one has to consider $\zeta_<$ and $\zeta_>$ again separately and the original ranges of convergence. That is

$$K_<(t;x|L_D)=\frac{1}{2\pi i}\int_{1/2+N/2<\mathrm{Re}s<1+N/2}t^{-s}\Gamma(s)\zeta_<(s;x|L_D)ds, \quad (17)$$

$$K_>(t;x|L_D)=\frac{1}{2\pi i}\int_{1+N/2<\mathrm{Re}s<1+\nu_1+N/2}t^{-s}\Gamma(s)\zeta_>(s;x|L_D)ds, \quad (18)$$

$$\zeta_<(s;x|L_D)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_<(t;x|L_D)dt,\quad \frac{1}{2}+\frac{N}{2}<\mathrm{Re}s<1+\frac{N}{2}, \quad (19)$$

$$\zeta_>(s;x|L_D)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_>(t;x|L_D)dt,\quad 1+\frac{N}{2}<\mathrm{Re}s<1+\nu_1+\frac{N}{2}, \quad (20)$$

and, by definition,

$$K_\beta(t;x|L_D)\equiv K_<(t;x|L_D)+K_>(t;x|L_D)=\frac{2\pi}{\beta}\frac{1}{(4\pi t)^{D/2}}+\frac{(4\pi)^{-N/2}}{2\pi i\beta}\times\int_{\mathrm{Re}s>1+N/2}t^{-s}\rho^{2s-D}I_\beta\left(s-\frac{N}{2}\right)ds. \quad (21)$$

On the right-hand side of the latter equation we immediately recognize the kernel $K_{2\pi}(t;x|L_D)$ and so the integral represents the difference $K_\beta(t;x|L_D)-2\pi/\beta K_{2\pi}(t;x|L_D)$. Similar expressions are valid for all quantities. This can be seen by observing that Eq. (11) can be written in the form

$$F_\beta(x|L_D)-\frac{2\pi}{\beta}F_{2\pi}(x|L_D)=\frac{2}{\beta}\sum_\alpha\left\{\sum_{l=1}^{\infty}\int_0^\infty F(\lambda^2+\lambda^2\alpha)\times[J^2_{\nu_l}(\lambda\rho)-J^2_l(\lambda\rho)]\lambda\,d\lambda\right\}.$$

The advantage is that the low eigenvalue $\nu_0=0$ is absent on the right-hand side of this expression. As a result

$$\zeta_\beta(s;x|L_D)-\frac{2\pi}{\beta}\zeta_{2\pi}(s;x|L_D)=\frac{4(4\pi)^{-N/2}}{\Gamma\left(\frac{N}{2}\right)}\int_{-\infty}^{\infty}dk\,k^{N-1}\sum_{l=1}^{\infty}\int_0^\infty(\lambda^2+k^2)^{-s}\times[J^2_{\nu_l}(\lambda\rho)-J^2_l(\lambda\rho)]\lambda\,d\lambda.$$

Now the right-hand side of the latter equation is well defined for $1+N/2<\mathrm{Re}s<1+\nu_1+N/2$. After integration one has

$$\zeta_\beta(s;x|L_D)-\frac{2\pi}{\beta}\zeta_{2\pi}(s;x|L_D)=\frac{\rho^{2s-D}}{\beta(4\pi)^{N/2}\Gamma(s)}I_\beta\left(s-\frac{N}{2}\right),$$

which is identical to the previous definition of ζ function and this means that, for this particular case, the Cheeger analytical procedure gives $\zeta_{2\pi}(s;x|L_D)=0$ (note that formally $\zeta_{2\pi}$ is a divergent integral whatever s is).

Heat kernel and local ζ function are related by

$$\zeta_\beta(s;x|L_D)-\frac{2\pi}{\beta}\zeta_{2\pi}(s;x|L_D)$$
$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left[K_\beta(t;x|L_D)-\frac{2\pi}{\beta}K_{2\pi}(t;x|L_D)\right]dt, \tag{22}$$

$$K_\beta(t;x|L_D)-\frac{2\pi}{\beta}K_{2\pi}(t;x|L_D)$$
$$=\frac{1}{2\pi i}\int_{\mathrm{Re}s>1+N/2}t^{-s}\Gamma(s)\times\left[\zeta_\beta(s;x|L_D)-\frac{2\pi}{\beta}\zeta_{2\pi}(s;x|L_D)\right]ds. \tag{23}$$

Note that for $\beta=2\pi$ the conical singularity disappears and the manifold becomes $\mathbb{R}^D$. Thus, $\zeta_{2\pi}$ and $K_{2\pi}$ are trivial. Furthermore, by making use of Eq. (21) and taking the analytical properties of $I_\beta(s)$ discussed in the Appendix into account, one gets the asymptotics of the heat kernel: namely,

$$K_\beta(t;x|L_D)\simeq\frac{1}{(4\pi t)^{D/2}}+E_t(\rho). \tag{24}$$

where $E_t(\rho)$ is an exponentially small term in t. This local asymptotics is in agreement with the results of Refs. [24–27].

We conclude this section introducing the global quantities. Strictly speaking, only the distributional trace has a mathematical meaning, since the local ζ function above has nonintegrable singularities in ρ (see, for example, [31]). As a consequence one has to introduce a smearing by means of a suitable function $\phi(\rho)$ with compact support not containing the origin, thus defining

$$\zeta_\beta(s|L_D)(\phi)=\beta\int dV_N\int_0^\infty\phi(\rho)\zeta_\beta(s;x|L_D)\rho\,d\rho. \tag{25}$$

For the smeared trace we get

$$\zeta_\beta(s|L_D)(\phi)=\frac{V_N}{(4\pi)^{N/2}}\frac{I_\beta(s-N/2)\hat\phi(2s-N)}{\Gamma(s)},$$
$$\hat\phi(s)=\int_0^\infty\rho^{s-1}\phi(\rho)d\rho, \tag{26}$$

$\hat\phi$ being an analytic function since the integral in Eq. (25) exists for all s by definition. As a smearing function we may simply choose $\phi(\rho)=\theta(\Lambda-\rho)\theta(\rho-\varepsilon)$ $(\Lambda>\varepsilon)$, which is convergent to 1 in the limits $\Lambda\to\infty$ and $\varepsilon\to0$. Thus, we have

$$\hat\phi(s)=\frac{\Lambda^s-\varepsilon^s}{s}$$

and for the smeared ζ function

$$\zeta_\beta(s|L_D)(\phi)=\frac{V_N}{(4\pi)^{N/2}}\frac{I_\beta(s-N/2)(\Lambda^{2s-N}-\varepsilon^{2s-N})}{\Gamma(s)(2s-N)},$$
$$\times\zeta_\beta(0|L_D)(\phi)=0. \tag{27}$$

B. The general case: scalar fields in Rindler-like spaces

Here we will give some results concerning the more general case $\mathcal{M}^D=C_\beta\times\mathcal{M}^N$, $\mathcal{M}^N$ being an arbitrary smooth manifold without boundary. In such conditions all known results concerning heat-kernel and ζ function for L_N on $\mathcal{M}^N$, which we suppose to be known, are applicable. In particular we note that the kernels are related by means of Mellin transforms, the analogues of Eqs. (5) and (6). Furthermore, the heat kernel has the usual asymptotic expansion

$$K(t;\vec{x}|L_N)\simeq\sum_r A_r(\vec{x}|L_N)t^{r-N/2},$$

while the local ζ function has the meromorphic structure (theorem of Seeley)

$$\Gamma(s)\zeta(s;\vec{x}|L_N)=\sum_r\frac{A_r(\vec{x}|L_N)}{s+r-N/2}+\text{the analytical part}, \tag{28}$$

the spectral coefficients $A_r(\vec{x}|L_N)$ being computable functions (for a review see [34]). Here we suppose zero modes to be absent, but of course one can take them into account with simple modifications of the formulas.

Now let us try to derive the meromorphic structure of $\zeta_\beta(s;\vec{x}|L_D)$ on $\mathcal{M}^D$. To this aim we use the factorization property of the heat kernel,

$$K_\beta(t;x|L_D)=K(t;\tau,\rho|L_\beta)K(t;\vec{x}|L_N), \tag{29}$$

in which the heat kernels of the Laplace-like operators on $\mathcal{M}^D$, C_β, and $\mathcal{M}^N$, respectively, appear. By taking the Mellin transform of Eq. (29) one usually gets the Dikii-Gelfand representation for the ζ-function, which easily permits us to read off the meromorphic structure. However, as we have shown in the previous section, in the presence of conical singularities we have to separate low and high eigenvalues in order to have a well defined Mellin transform. So we set

$$\zeta_<(s;x|L_D)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_<(t;\tau,\rho|L_\beta)K(t;\vec{x}|L_N)dt,$$

$$\zeta_>(s;x|L_D)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_>(t;\tau,\rho|L_\beta)K(t;\vec{x}|L_N)dt,$$

where $K_<(t;\tau,\rho|L_\beta)$ and $K_>(t;\tau,\rho|L_\beta)$ are related to the corresponding ζ functions by mean of Eqs. (17)–(20), but with $N=0$ (pure cone). Now we make use of the Mellin-Parseval identity

$$\int_0^\infty f(t)g(t)dt=\frac{1}{2\pi i}\int_{\mathrm{Re}z=c}\hat f(z)\hat g(1-z)dz, \tag{30}$$

where c is in the common strip of analyticity of the Mellin transforms $\hat{f}(z)$ and $\hat{g}(1-z)$. After some calculations paying attention to the range of convergence we get

$$\zeta_<(s;x|L_D)=\frac{1}{2\pi i\Gamma(s)}\int_{c_0}\Gamma(z)\zeta_<(z;\tau,\rho|L_\beta)\Gamma(s-z)\times\zeta(s-z;x|L_N)dz, \tag{31}$$

$$\zeta_>(s;x|L_D)=\frac{1}{2\pi i\Gamma(s)}\int_{c_1}\Gamma(z)\zeta_>(z;\tau,\rho|L_\beta)\Gamma(s-z)\times\zeta(s-z;x|L_N)dz, \tag{32}$$

where $\frac{1}{2}<c_0<1$ and $1<c_1<\text{Re}s-N/2$. These are the Dikii-Gelfand representations for $\zeta_<$ and $\zeta_>$ which are valid for $\text{Re}s>1+N/2$. Since they are well defined in the same range we can directly write the ζ function as $\zeta=\zeta_<+\zeta_>$. Then, a representation valid for $\text{Re}s>1+N/2$ reads

$$\zeta_\beta(s;x|L_D)=\frac{\zeta(s-1;\vec{x}|L_N)}{2\beta(s-1)}+\frac{1}{2\pi i\beta\Gamma(s)}\times\int_{\text{Re}z=c_1}\rho^{2z-2}I_\beta(z)\Gamma(s-z)\times\zeta(s-z;x|L_N)dz, \tag{33}$$

where in Eq. (31) we have shifted the contour integral on the right, taking into account that the integrand function has a simple pole at $z=1$ [see Eq. (14)]. We incidentally observe that the first term on the right-hand side of the latter equation is just $(2\pi/\beta)\zeta_{2\pi}(s;x|L_D)$. So the integral on the right represents $\zeta_\beta(s;x|L_D)-(2\pi/\beta)\zeta_{2\pi}(s;x|L_D)$, in agreement with the second point of view that we discussed in the previous section.

Now, in order to perform the integral, in Eq. (33) we shift the contour on the right and using Eq. (28) we obtain

$$\zeta_\beta(s;x|L_D)=\frac{\zeta(s-1;\vec{x}|L_N)}{2\beta(s-1)}+\frac{1}{\beta\Gamma(s)}\sum_{r=0}^{P}A_r(\vec{x}|L_N)I_\beta\times\left(s+r-\frac{N}{2}\right)\rho^{2s+2r-D}+O(\rho^{2s+2P-D}), \tag{34}$$

P being an arbitrary large integer. By means of this equation and of inverse Mellin transform, one comes back to the local heat-kernel asymptotic

$$K_\beta(t;x|L_D)\sim\frac{1}{4\pi t}\sum_{r=0}^{\infty}A_r(\vec{x}|L_N)t^{n-N/2}\sim\frac{1}{4\pi t}K(t;\vec{x}|L_N). \tag{35}$$

By integrating Eq. (34) on the manifold with the smeared function $\phi(\rho)=\theta(\Lambda-\rho)\theta(\rho-\varepsilon)$ one has

$$\zeta_\beta(s|L_D)(\phi)=\zeta_{2\pi}(s|L_D)+\frac{1}{\Gamma(s)}\sum_{r=0}^{P}A_r(L_N)I_\beta\times\left(s+r-\frac{N}{2}\right)\hat{\phi}(2s+2r-N)+f(s;\Lambda,\varepsilon), \tag{36}$$

where $\zeta_{2\pi}(s|L_D)=\zeta(s-1|L_N)\hat{\phi}(2)/[2(s-1)]$ and $f(s;\Lambda,\varepsilon)$ is an analytic function in s going to 0 as $\varepsilon\to 0$.

Now we may write down the meromorphic structure of global ζ function, which we can directly read off by looking at Eq. (36) and recalling that $I_\beta(s)$ has only a simple pole at $s=1$. The result reads

$$\Gamma(s)\zeta_\beta(s|L_D)(\phi)\sim\sum_{r=0}^{\infty}\frac{A_r(L_N)[\hat{\phi}(2)+(\beta/2\pi-1)\hat{\phi}(2s+2r-N)]}{2(s+r-D/2)}.$$

Taking the limit for $s\to 0$ of the latter equation we easily get $\zeta_\beta(0|L_D)=0$ for odd D, while, for even D,

$$\zeta_\beta(0|L_D)=\frac{\beta}{4\pi}A_{D/2}(L_N)\hat{\phi}(2),$$

and this is just the integral of the $D/2$ coefficient in the asymptotic expansion (35). It has to be noted that $\zeta_\beta(0|L_D)$ is linear in β (or vanishing) and so the μ dependence in Eq. (8), which reflects the ζ-function ultraviolet renormalization, does not contribute to the entropy, and this is again in agreement with the conformal transformation method.

IV. STATISTICAL MECHANICS IN RINDLER-LIKE SPACE-TIMES

Now, making use of the expression for the ζ function we have derived in the previous section, Eq. (36), we can study the statistical mechanics for scalar fields in a Rindler-like space-time by means of Eq. (8). Taking the derivative of Eq. (36) and the limit $s\to 0$ we obtain

$$F_\beta=-\frac{1}{2\beta}\zeta'_{2\pi}(0|L_D)-\frac{1}{2\beta}\sum_{r=0}^{\infty}A_r(L_N)I_\beta\left(r-\frac{N}{2}\right)\hat{\phi}(2r-N)-\frac{\zeta_\beta(0|L_D)}{2\beta}\ln\mu^2-\frac{f(0;\Lambda,\varepsilon)}{2\beta}.$$

As usual, it is convenient to distinguish between odd and even dimensional cases, respectively: i.e.,

$$F_\beta=-\frac{1}{2\beta}\sum_{r=0}^{N-1/2}\frac{A_r(L_N)I_\beta(r-N/2)}{(N-2r)\varepsilon^{N-2r}}-\frac{1}{2\beta}\zeta'_{2\pi}(0|L_D)+O(\Lambda^2), \tag{37}$$

$$F_\beta=-\frac{1}{2\beta}\sum_{r=0}^{N/2-1}\frac{A_r(L_N)I_\beta(r-N/2)}{(N-2r)\varepsilon^{N-2r}}-\frac{A_{N/2}(L_N)I_\beta(0)}{2\beta}\ln\frac{\Lambda^2}{\varepsilon^2}-\frac{1}{2\beta}\zeta'_{2\pi}(0|L_D)+O(\Lambda^2). \tag{38}$$

The last term $O(\Lambda^2)$ contains also the dependence on the scale parameter μ. In any case such a term is independent on β and does not give contributions to the entropy.

Some comments are in order. We observe that the free energy in the vicinity of the horizon has a number of divergences depending on the dimension D and in the even dimensional case also a logarithmic divergence appears, proportional to $A_{N/2}$, in agreement with results obtained by other methods [16]. There is a finite part linear in the temperature and proportional to $\zeta'_{2\pi}$. Besides, the leading term has the expected β^{-D} behavior. This is a nontrivial result and it is a consequence of our local approach, which requires the analytical continuation investigated in the Appendix. In fact, from Eqs. (A8) and (A9) we have

$$F_\alpha \sim -\frac{(-1)^{(D-1/2)}\zeta'_R(1-D)}{\sqrt{\pi}(4\pi)^{D/2}\Gamma[(D+1)/2]}\frac{V_N}{N\varepsilon^N}\alpha^D, \quad D=3,5,7,\ldots,$$

$$F_\alpha \sim -\frac{\Gamma[(1-D)/2]\zeta_R(1-D)}{\sqrt{\pi}(4\pi)^{D/2}}\frac{V_N}{N\varepsilon^N}\alpha^D, \quad D=4,6,\ldots.$$

Here ζ_R is the usual Riemann ζ function and $\alpha=2\pi/\beta$. Note that here the equilibrium temperature corresponds to $\alpha=1$.

Using Eq. (9) or the equivalent relation

$$S_\alpha=-2\pi\frac{\partial F_\alpha}{\partial\alpha} \tag{39}$$

and Eqs. (37) and (38), one can compute the entropy in any D-dimensional Rindler-like space-time. For the sake of simplicity, here we shall deal with the four-dimensional case only. One easily gets

$$F_\alpha=-\frac{A(\alpha^2-1)(\alpha^2+11)}{180(4\pi)^2\varepsilon^2}+\frac{A_1(L_2)(\alpha^2-1)}{48\pi}\ln\frac{\Lambda^2}{\varepsilon^2}-\frac{\alpha}{4\pi}\zeta'_{2\pi}(0|L_4)+O(\Lambda^2), \tag{40}$$

$A=V_2$ being the transverse area and finally, at the equilibrium temperature $\alpha=1$:

$$S_{\alpha=1}=\frac{A}{60\pi\varepsilon^2}+\frac{A_1(L_2)}{12}\ln\frac{\Lambda^2}{\varepsilon^2}+\frac{1}{2}\zeta'_{2\pi}(0|L_4)+O(\Lambda^2).$$

The results that we have obtained in this section are valid for a scalar field in a general Rindler-like space-time. For a massless scalar field in a Rindler space-time, $\zeta_{2\pi}(s|L_D)$, as well as all coefficients $A_r(L_N)$, but A_0 vanishes. Therefore, previous formulas reduce to

$$F_\beta=-\frac{I_\beta(0)}{2\beta}\ln\frac{\Lambda}{\varepsilon}=-\frac{1}{24\pi}\ln\frac{\Lambda}{\varepsilon}(\alpha^2-1), \quad D=2,$$

$$F_\beta=-\frac{I^\beta(-N/2)}{2\beta(4\pi)^{N/2}}\frac{V_N}{N\varepsilon^N}, \quad D\geqslant 3,$$

which can be directly derived from Eq. (27). Finally, for $D=2$ and $D=4$ the entropies read, respectively,

$$S_{\alpha=1}=\frac{1}{6}\ln\frac{\Lambda}{\varepsilon} \quad D=2, \quad S_{\alpha=1}=\frac{A}{60\pi\varepsilon^2} \quad D=4. \tag{41}$$

The first reproduces the well-known two-dimensional result, while the latter is compatible with the same quantity calculated by other methods, the only difference being the numerical factor in the denominator.

In this paper we are mainly interested in entropy. However, in the following we shall briefly discuss the renormalization of the internal energy. We note that we have made use of an analytical regularization. We have at our disposal a renormalization prescription. In this approach it is quite natural to require the internal energy to be finite (vanishing) when the conical singularity is absent, namely when $\beta=2\pi$. This can be accomplished by making use of the same statistical mechanical identities among the renormalized quantities and assuming (for example, $D=2$)

$$U^R_\alpha=U_\beta-U_{\alpha=1}=\frac{1}{24\pi}\ln\frac{\Lambda}{\varepsilon}[\alpha^2-1].$$

This prescription automatically gives

$$S^R_\alpha=S_\alpha=\frac{1}{6}\ln\frac{\Lambda}{\varepsilon}.$$

Note that $U_{2\pi}$ depends on the horizon cutoff. The corresponding free energy reads

$$F^R_\alpha=\frac{1}{24\pi}\ln\frac{\Lambda}{\varepsilon}[\alpha^2-1]-\frac{1}{12\pi}\ln\frac{\Lambda}{\varepsilon}\alpha^2, \tag{42}$$

which is not vanishing for $\beta=2\pi$. The same analysis can be extended to the higher dimensional cases.

We conclude this section with few remarks. With regard to other off-shell computations of the entropy and free energy for a black hole, we recall that the horizon divergences can be obtained, for example, within the path integral approach, making use of the high-temperature approximation [8,13]. This gives the correct leading term in α^4, proportional to the optical volume and this result is in agreement with our expression of the free energy in the four-dimensional case. However, in this case there is also a disagreement with the computations based on the other off-shell methods, occurring in the α^2 terms, and this leads to the anomalous numerical coefficient in Eq. (41) for the expression of the entropy.

V. *D*-DIMENSIONAL BLACK HOLE NEAR THE HORIZON AND IN THE LARGE MASS LIMIT

Here we consider the case in which $\mathcal{M}^D=C_\beta\times S^N$. To justify this choice from a physical viewpoint, first of all we show that, near the horizon and in the large black-hole mass, a (Euclidean) D-dimensional black hole with vanishing cosmological constant may be approximated by a manifold of this kind and so the statistical mechanics can be investigated by using the formulas of previous sections.

We recall that the static metric describing a D-dimensional Schwarzschild black hole (we assume $D>3$ and vanishing cosmological constant) read [35]

$$ds^2=-\left[1-\left(\frac{r_H}{r}\right)^{D-3}\right]dx_0^2+\left[1-\left(\frac{r_H}{r}\right)^{D-3}\right]^{-1}dr^2 +r^2\,d\Omega_{D-2},$$

where we are using polar coordinates, r being the radial one and $d\Omega_{D-2}$ the $D-2$-dimensional spherical unit metric. The horizon radius is given by

$$r_H=\left[\frac{2\pi^{(D-3)/2}MG_D}{(D-2)\Gamma[(D-1)/2]}\right]^{1/(D-3)},$$

M being the mass of the black hole and G_D the generalized Newton constant. The associated Hawking temperature reads $\beta_H=4\pi r_H/(D-3)$. From now on, we put $r_H=1$.

To study the black hole near the horizon it is convenient to redefine the radial Schwarzschild coordinates $x_0=x_0'/b$ and $r=r(\rho)$ by means of the implicit relation

$$\left(\frac{b\rho^2}{2}\right)^{1/(D-3)}=e^{r-1}\exp\int\frac{dr}{r^{D-3}-1},\quad \rho^2\sim\frac{1-r^{3-D}}{b^2},$$

where $b=(D-3)/2$. In the new set of coordinates we have

$$ds^2=-\frac{1-r^{3-D}(\rho)}{b^2}dx_0'^2+\frac{1-r^{3-D}(\rho)}{b^2\rho^2}d\rho^2 +r^2(\rho)d\Omega_{D-2}, \tag{43}$$

and finally, near the horizon,

$$ds^2=-\rho^2dx_0'^2+d\rho^2+d\Omega_N,$$

which is the metric of a Rindler-like space $C_\beta\times S^N$. As a consequence we can use all results developed in previous sections. In particular, the ζ function can be computed making use of Eq. (33), since the ζ function for the Laplace operator on S^N is well known (see, for example, [36,37]). More simply, we can directly derive the free energy by using Eqs. (37) and (38).

For its physical interest now we shall investigate in more detail the case $D=4$, a toy model for the four-dimensional eternal black hole. For this case we have

$$\Gamma(z)\zeta(z|L_2)=2\sum_{k=0}^{\infty}\frac{(-a^2)^k}{k!}\Gamma(z+k)\zeta_H\left(2z+2k-1;\frac{1}{2}\right),$$

where $L_2=-\Delta_{S^2}+\frac{1}{4}+a^2$ and $\zeta_H(s;q)$ is the Hurwitz ζ-function. All the spectral coefficients may be evaluated from the above expression computing the residues at the simple poles $z=1-r$, $(r=0,1,2,\dots)$. As a result $A_0=1$ and

$$A_r(L_2)=(-1)^r\left[\frac{a^{2r}}{r!}-2\sum_{j=0}^{r-1}\frac{a^{2j}\zeta_H(2j-2r+1;1/2)}{j!(r-j-1)!}\right],\quad r\geq 1.$$

The free energy is given by Eq. (40) and reads

$$F_\alpha=-\frac{A(\alpha^2-1)(\alpha^2+11)}{180(4\pi)^2\varepsilon^2}+\left(\frac{1}{12}-a^2\right)\frac{(\alpha^2-1)}{48\pi}\ln\frac{\Lambda^2}{\varepsilon^2} -\frac{\alpha}{4\pi}\zeta'_{2\pi}(0|L_4)+O(\Lambda^2),$$

where $A=4\pi r_H^2$ is the horizon area. Also in this case, we may require that the internal energy has to be finite at the Hawking temperature. This can be realized by adding the infinite constant $-U_{2\pi}$. However, this prescription does not modify the entropy. For a massless scalar field one has $a^2=-1/4$ and so the (renormalized) entropy at the equilibrium temperature $\alpha=1$ (which means $\beta_H=2\pi r_H/b=8\pi MG$) is

$$S_{\alpha=1}=\frac{A}{60\pi\varepsilon^2}-\frac{1}{36}\ln\frac{\Lambda^2}{\varepsilon^2}+\frac{1}{2}\zeta'_{\mathbb{R}^2\times S^2}(0|L_4)+O(\Lambda^2).$$

The appearance of the logarithmic horizon divergences [38,15], which is absent in the massless Rindler case, should be noted. Higher dimensional cases can be analyzed on the same lines, without any difficulties.

VI. CONCLUSIONS

In this paper the entropy for a massless scalar field in a D-dimensional Rindler-like space-time has been investigated by means of the off-shell conical Euclidean method based on a local ζ-function regularization. The degrees of freedom of the black hole have been assumed to be equivalent to those of a massless scalar quantum field on $C_\beta\times\mathcal{M}^N$, C_β being the two-dimensional cone. The period β of the imaginary compactified time has been interpreted as the inverse of the temperature measured at infinity. One of the advantages of this approach is the determination of the unique equilibrium temperature, the Unruh-Hawking temperature, by the requirement of the absence of conical singularities ($\beta=2\pi$). Within this approach, the Bekenstein-Hawking entropy can also be obtained, but again without a statistical interpretation. With regard to this issue, the formal partition function related to the determinant of a Laplace-like operator on $C_\beta\times\mathcal{M}^N$, evaluated off shell ($\beta\neq 2\pi$), in order to permit the computation of the entropy by means of statistical formulas, has been regularized according to the ζ-function method. This has posed the mathematical problem of defining properly the related ζ function.

A suitable analytical procedure first suggested by Cheeger has been used in order to implement the usual relationship between local ζ function and heat kernel, as well as the corresponding traces, for which a smearing function has been introduced in order to define them. The so-called horizon divergences of this entropy evaluated at the equilibrium temperature, which are also present if one is dealing with other off-shell techniques, are recovered with a natural choice of the smearing function. We have obtained agreement with other methods (the conformal transformation method, the brick-wall model, and the canonical approach), the only difference being the numerical coefficients of the divergences, even though the structural form of the divergent terms is the same.

Another by-product of our conical Euclidean approach has been the dimensionally correct leading behavior of the free energy in β, a result that the global heat-kernel method completely misses. In the usual conical approach, the one we have called the global approach, one needs a mass as infrared cutoff and has to use proper-time regularization, namely an ultraviolet regularization different from the ζ function. The leading divergence survives in the limit $m\to 0$, but one gets again a β behavior independent on the dimensions of the manifold [21]. Furthermore, within this approach these ''ultraviolet'' divergences are interpreted as horizon divergences. As a result, even though the interpretation of the divergences is different, the conclusions are similar. With regard to this issue, one should try to investigate the limit $m\to 0$ and the infinite volume limit by starting, *ab initio*, with a truncate cone and imposing suitable boundary conditions. Then one should study the massless limit and the infinite cone limit in order to better understand the existence of an infrared phenomena. Our local approach has implicitly assumed the infinite cone limit. We stress again that the Cheeger method permits us to study the massless case directly in the infinite cone case.

As far as the horizon divergences of the off-shell quantities are concerned, we have little to add to the considerations that have recently appeared in the literature; a detailed discussion can be found in Ref. [5]. There it has been shown, working with two-dimensional models, that all the observables related to a black hole at Hawking temperature can be evaluated in terms of on-shell finite quantities and a subtraction procedure between the on-shell and off-shell quantities has been proposed, the divergences of the former being removed by the related quantities in the Rindler space-time. Another proposal to deal with such divergences, consisting in the implementation of the 't Hooft approach by means of Pauli-Villars regularization, has been recently introduced in Ref. [39] and it has been used in a two-dimensional model in Ref. [40], where a comparison between the Frolov-Fursaev-Zelnikov scheme [5] and the latter can be found. The absence of the on-shell entropy divergences has been also claimed in Ref. [41].

We conclude with some remarks. The Rindler case could be the key example in order to better understand the horizon divergences. Here it is well established that the internal energy must be finite (actually vanishing) if and only if $\beta=\beta_H$. As a consequence, the related entropy is divergent at the same equilibrium temperature. It has been shown that such statistical-mechanical entropy coincides with the entropy of entanglement obtained from the density matrix describing the vacuum state of the field (scalar or spinor) as observed from one side of a boundary in Minkowski space-time [42,43,18,44,21].

However, the entropy of entanglement, although formally divergent, might be operationally finite [45]. With regard to the black-hole case, the fluctuations of the horizon [6] as well as the quantum evaporation [46] might provide again a mechanism for the absence of the entropy divergences. Finally, we have to mention that recently several attempts to clarify the microscopic origin of black-hole entropy have appeared within the string theory, which seems, at the moment, a promising theory capable of offering a solution to this important issue (see, for example, [47] and references therein).

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APPENDIX: PROPERTIES OF THE $G_\beta(s)$ FUNCTION

Here we make the analytic continuation of the function

$$G_\beta(s)=\sum_{l=1}^{\infty}\frac{\Gamma(\nu_l-s+1)}{\Gamma(\nu_l+s)},\quad \nu_l=\alpha l,\quad \alpha(\beta)=\frac{2\pi}{\beta},$$

which is convergent for $\mathrm{Re}s>1$. In fact, for $\nu\to\infty$ one has the asymptotic expansion

$$\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)}\sim \nu^{1-2s}\sum_{j=0}^{\infty} c_j(s)\,\nu^{-2j},$$

where $c_j(s)$ are easily computable using the known expansion of $\Gamma(z)$, that is

$$\Gamma(z)\sim\sqrt{\frac{2\pi}{z}}e^{-z+z\ln z+B(z)},\quad B(z)=\sum_{j=0}^{\infty}\frac{B_{2j}z^{1-2j}}{2j(2j-1)},$$

B_j being the Bernoulli numbers. It it easy to see that since the function $\Gamma(\nu-s+1)/[\Gamma(\nu+s)]$ for any $s=-n/2$ $(n=-1,0,1,2,\dots)$ is effectively a polynomial of order ν^{n+1}, $c_j(-n/2)$ has to vanish for all $j>(n+1)/2$.

The first coefficients read $c_0(s)=1$,

$$c_1(s)=\frac{s(s-1/2)(s-1)}{3},$$

$$c_2(s)=\frac{s(s^2-1/4)(s^2-1)(s-6/5)}{18}. \tag{A1}$$

It has to be noted that $G_\beta(s)$ is certainly analytic in the strip $1<\mathrm{Re}s<1+\nu_1$ and, as we shall see later, it has a simple pole at $s=1$ with residue equal to $1/2\alpha$. In order to make the analytic continuation of $G_\beta(s)$ we define

$$f_n(\nu,s)=\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)}-\sum_{j=0}^{[n/2]+1} c_j(s)\,\nu^{1-2s-2j}$$
$$\sim c_{[n/2]+2}(s)\,\nu^{-(2s+2[n/2]+3)},$$

where $[n/2]$ represents the integer part of $n/2$. For $\mathrm{Re}s>1$ we have

$$G_\beta(s)=\sum_{j=0}^{[n/2]+1}\alpha^{1-2s-2j}c_j(s)\zeta_R(2s+2j-1)$$
$$+\sum_{k=1}^{\infty} f_n(\nu_k,s). \tag{A2}$$

Now, the right-hand side of the latter equation has meaning for $\mathrm{Re}s>-1-[n/2]$ and so we have obtained the analytic continuation we were looking for. The function $f_n(\nu,s)$ is in general unknown, but it is vanishing for all $s=1/2,0,-1/2,-1,\dots,-n/2$, since in this case the function $\Gamma(\nu+1+n/2)/[\Gamma(\nu-n/2)]$ is a polynomial. Then for $n=-1,0,1,2,3,\dots$, we obtain

$$G_\beta(-n/2)=\sum_{j=0}^{[n/2]}\alpha^{n+1-2j}c_j(-n/2)\zeta_R(2j-n-1)+\alpha^{n-1-2[n/2]}c_{[n/2]+1}(s)\zeta_R\times\left(2s+2\left[\frac{n}{2}\right]+1\right)\Bigg|_{s=-n/2}. \quad \text{(A3)}$$

Using Eqs. (A1), (A2), and (A3) we have

$$\text{Res}G_\beta(s)|_{s=1}=\frac{\beta}{4\pi},\quad G_\beta(0)=\frac{1}{12}\left(\frac{1}{\alpha}-\alpha\right),\quad \text{(A4)}$$

$$G_\beta(-1)=\frac{1}{120}\left(\alpha^3+10\alpha-\frac{11}{\alpha}\right). \quad \text{(A5)}$$

Recalling that $\zeta_R(0)=-1/2$ and $\zeta_R(-2j)=0$ for any $j\in\mathbb{N}$, from Eq. (A3) we also get

$$G_\beta(-n/2)=-\frac{1}{2}c_{(n+1)/2}(-n/2)=-\frac{\Gamma(1+n/2)}{2\Gamma(-n/2)},\quad n=-1,1,3,5,\ldots\ . \quad \text{(A6)}$$

The latter expression has been derived from the identity

$$\frac{\Gamma(\nu+n/2+1)}{\Gamma(\nu-n/2)}=\left[\nu^2-\left(\frac{1}{2}\right)^2\right]\left[\nu^2-\left(\frac{3}{2}\right)^2\right]\cdots\left[\nu^2-\left(\frac{n}{2}\right)^2\right],$$

valid for any odd $n=-1,1,3,5,\ldots$. The identity

$$G_{2\pi}(s)=-\frac{\Gamma(1-s)}{2\Gamma(s)},$$

also holds. From the latter equation and Eq. (A6) we have

$$G_\beta(-n/2)-G_{2\pi}(-n/2)=0\ ,\quad n=-1,1,3,5,\ldots\ .$$

In the paper we frequently meet the function

$$I_\beta(s)=\frac{\Gamma(s-1/2)}{\sqrt{\pi}}[G_\beta(s)-G_{2\pi}(s)], \quad \text{(A7)}$$

which has a simple pole at $s=1$. We have

$$\text{Res}I_\beta(s)|_{s=1}=\frac{1}{2}\left(\frac{\beta}{2\pi}-1\right),$$

$$I_\beta(0)=-2G_\beta(0)=\frac{1}{6}\left(\frac{\beta}{2\pi}-\frac{2\pi}{\beta}\right),$$

and by definition $I_{2\pi}(s)=0$. We also need the behavior of I_β with respect to β at $s=-N/2$. >From Eqs. (46) and (50) for even N we immediately have

$$I_\beta(-N/2)=\frac{\Gamma[-(N+1)/2]}{\sqrt{\pi}}[G_\beta(-N/2)-G_{2\pi}(-N/2)]\sim\frac{\Gamma[-(N+1)/2]\zeta_R(-N-1)}{\sqrt{\pi}}\alpha^{N+1},\quad N=0,2,4,\ \ldots, \quad \text{(A8)}$$

while for odd N, using Eq. (A2) we obtain

$$I_\beta(-N/2)=\frac{(-1)^{(N+1)/2}}{2\sqrt{\pi}\Gamma[(N+3)/2]}[G'_\beta(-N/2)-G'_{2\pi}(-N/2)]\sim\frac{(-1)^{(N+1)/2}\zeta'_R(-N-1)}{2\sqrt{\pi}\Gamma[(N+3)/2]}\alpha^{N+1},\quad N=1,3,5,\ \ldots\ . \quad \text{(A9)}$$

In the evaluation of the latter expansion, we have considered only the first term on the right-hand side of Eq. (A2), since the derivatives at $s=-N/2$ of the functions $f_N(\nu_k,s)$ give contributions of the order α^{-2}.

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