One loop graviton self-energy in a locally de Sitter background

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The graviton tadpole has recently been computed at two loops in a locally de Sitter background. We apply intermediate results of this work to exhibit the graviton self-energy at one loop. This quantity is interesting both to check the accuracy of the first calculation and to understand the relaxation effect it reveals. In the former context we show that the self-energy obeys the appropriate Ward identity. We also show that its flat space limit agrees with the flat space result obtained by Capper in what should be the same gauge. [S0556-2821(96)06212-1]

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I. INTRODUCTION

We have suggested that inflation ended in the early universe because the quantum gravitational back reaction slowly generated negative vacuum energy which eventually screened a not unnaturally small, positive cosmological constant [1,2]. This is an attractive scenario for solving the problem of the cosmological constant because (1) it operates in the far infrared where general relativity can be used reliably as a quantum theory of gravitation, (2) it introduces no new light quanta which would embarrass low energy phenomenology, (3) it has the potential to make unique predictions because gravity is the only phenomenologically viable theory which possesses the essential feature of massless quanta whose self-interactions are not conformally invariant, and (4) the weakness of gravitational interactions makes the process slow enough to account for a long period of inflation.

If this proposal is correct there will be far-reaching consequences for theories of the very early universe. Scarcely less significant, in the long run, is the fact that contact will finally have been made between observed reality and the hitherto murky realm of quantum gravity.

We have recently done a calculation which establishes the validity of our scenario for at least as long as perturbation theory remains reliable. The quantity we computed is the expectation value of the invariant element, starting from a homogeneous and isotropic, locally de Sitter, free vacuum on the manifold $T^3 \times \mathcal{R}$:

$$\langle \Omega | g_{\mu\nu}(t,\vec{x}) dx^\mu dx^\nu | \Omega \rangle = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x}. \quad (1.1)$$

The rate of spacetime expansion is measured using the coordinate invariant effective Hubble constant:

$$H_{\text{eff}}(t) \equiv \frac{1}{a(t)} \frac{da(t)}{dt}. \quad (1.2)$$

One loop tadpoles make no contribution because they are ultralocal whereas infrared effects derive from the causal and coherent superposition of interactions throughout the past lightcone. The first secular effect comes from the two loop diagrams shown in Fig. 1. At the end of a very long calculation we obtain the following result [3,4]:

$$H_{\text{eff}}(t) = H \left\{ 1 - \left(\frac{\kappa H}{4\pi} \right)^4 \left[\frac{1}{6} (Ht)^2 + O(Ht) \right] - O(\kappa^6) \right\}, \quad (1.3)$$

where $H \equiv \sqrt{(1/3)\Lambda}$ is the Hubble constant at the onset of inflation and $\kappa^2 \equiv 16\pi G$ is the usual loop counting parameter of perturbative quantum gravity. We have also been able to show that the ℓ loop contribution to the bracketed term can be no stronger than $-\#(\kappa H)^{2\ell}(Ht)^\ell$.

It is not easy to compute at two loops even for scalar field theories on flat space, and truth can sometimes remain well hidden amidst the forest of indices which characterize any calculation in quantum gravity. In fact only one other two loop result has been obtained for quantum gravity, and this was limited to the ultraviolet divergent part of the standard, in-out effective action for zero cosmological constant [5]. To study the ultraviolet one can use asymptotic expansions in which the effects of spacetime curvature are segregated from what is basically a calculation in flat space. The infrared does not allow this simplification; we had to obtain the full propagators on a curved background and integrate them against the appropriate interaction vertices over a large invariant spacetime volume. There was an additional complication in having to use Schwinger's formalism [6] to obtain a true expectation value rather than an in-out matrix element. One naturally
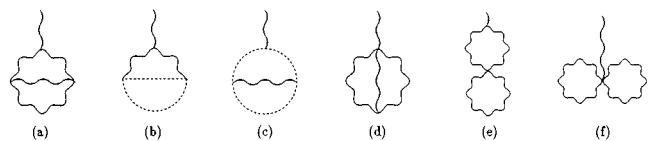

FIG. 1. Two-loop contributions to the background. Gravitons reside on wavy lines and ghosts on segmented lines.

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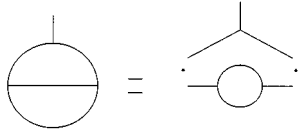†Electronic address: woodard@phys.ufl.edu

FIG. 2. Representation of how a two loop tadpole comes from contracting the one loop self-energy through propagators, into the outer vertex.

wonders, therefore, about the accuracy of a result such as the one we are reporting. This concern is heightened by the fact that so much of the relevant formalism has only recently been developed.

The possibility for dramatic checks on the consistency of the formalism and on our proficiency in applying it is provided by the manner in which we computed the two most complicated diagrams, 1(a) and 1(b). In order to economize on the size of intermediate expressions we evaluated the lower loops first and then contracted them into the two upper propagators and the final vertex as represented diagramatically in Fig. 2. A consequence is that we can extract the one loop graviton self-energy. This quantity can be subjected to two powerful tests: the flat space limit ($H \rightarrow 0$ with κ and t held fixed) and the Ward identity. The Ward identity checks our gauge fixing procedure, our solution for the ghost and graviton propagators, our three-point vertices, and the automated reduction procedures through which we contracted propagators into vertices and acted the various derivatives. In addition to providing a largely complementary check on all these things, the flat space limit tests the overall proportionality constant.

In Sec. II of this paper we define the self-energy, explain how it was calculated, and give our result for it. In Sec. III we define the flat space limit, compute it for the one loop graviton self-energy, and show that it agrees with the flat space result obtained earlier by Cappen [7]. In Sec. IV we derive the Ward identity for our gauge and describe the procedure used to check it. Our conclusions are discussed in Sec. V.

II. THE ONE LOOP SELF-ENERGY

The self-energy of a quantum field is usually defined in momentum space. This is not convenient for our problem because the curved background prevents the free theory from being diagonal in a Fourier basis. However, it is simple to translate the usual prescription into a position space version which we can use. Consider an uncharged scalar field $\phi(x)$ which has physical mass m, field strength Z, and zero vacuum expectation value. In a flat, spacelike metric background we would write the full propagator as
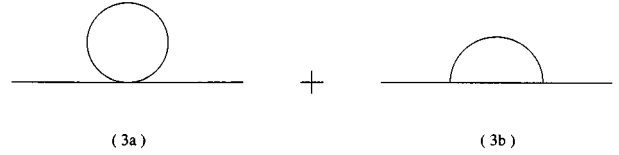

FIG. 3. One loop contributions to the scalar self-energy.

$$\langle \text{out} | T[\widetilde{\phi}(p) \widetilde{\phi}(k)] | \text{in} \rangle = \frac{-iZ(2\pi)^4 \delta^4(p+k)}{p^2+m^2+\Sigma(p^2)-i\epsilon}, \quad (2.1)$$

where $\widetilde{\phi}(p)$ is the Fourier transform:

$$\widetilde{\phi}(p) \equiv \int d^4x \, e^{ip \cdot x} \phi(x). \quad (2.2)$$

This means that the effective action is

$$\Gamma[\phi] = -\frac{1}{2Z} \int \frac{d^4p}{(2\pi)^4} \widetilde{\phi}^*(p)[p^2+m^2+\Sigma(p^2)]\widetilde{\phi}(p)$$
$$+ O(\phi^3) \quad (2.3a)$$

$$= \frac{1}{2Z} \int d^4x \, \phi(x)[\square - m^2]\phi(x)$$

$$- \frac{1}{2Z} \int d^4x_1 d^4x_2 \phi(x_1)\Sigma(x_1;x_2)\phi(x_2) + O(\phi^3), \quad (2.3b)$$

where the position space self-energy is

$$\Sigma(x_1;x_2) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x_2-x_1)} \Sigma(p^2). \quad (2.4)$$

It is instructive to give the one loop expansion of the self-energy for a general scalar field whose classical action is $S[\phi]$. We define the position space n-point vertex as

$$\mathcal{V}_n(x_1,\ldots,x_n) \equiv \frac{\delta^n S[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\bigg|_{\phi=0}. \quad (2.5)$$

In a local theory these vertices consist of a finite number of the various derivatives times a product of δ functions:

$$\mathcal{V}_n(x_1,\ldots,x_n) = V_n(x_1;\partial_1,\ldots,\partial_n)\delta^4(x_1-x_2)\cdots\delta^4(x_1-x_n). \quad (2.6)$$

Note that we allow the vertex operator V_n to depend upon position. Figure 3 gives the diagrams that contribute at one loop. In our notation the result is

$$-i\Sigma(x_1';x_2'') = -\frac{1}{2} \int d^4x_2' d^4x_3' \mathcal{V}_3(x_1',x_2',x_3') \int d^4x_2'' d^4x_3'' i\Delta(x_2';x_2'') i\Delta(x_3';x_3'') \mathcal{V}_3(x_1'',x_2'',x_3'')$$

$$+ \frac{i}{2} \int d^4x_2' \int d^4x_2'' \mathcal{V}_4(x_1',x_2',x_1'',x_2'') i\Delta(x_2';x_2'') + \cdots. \quad (2.7)$$

For a local theory we can do the integrations to obtain the form

$$\Sigma(x';x'') = -\frac{i}{2}\, V_3(x';\partial'_1,\partial'_2,\partial'_3)i\Delta_2(x';x'')i\Delta_3(x';x'')V_3(x'';\partial''_1,\partial''_2,\partial''_3) - \frac{1}{2}\, V_4(x';\partial'_1,\partial'_2,\partial''_1,\partial''_2)i\Delta_2(x';x'')\delta^4(x'-x'').$$
$$(2.8)$$

The ∂'_1 and ∂''_1 derivatives act outward. The other derivatives act on the propagator whose subscript matches their own; for example, ∂'_3 acts on the first argument of $i\Delta_3(x';x'')$. It is sometimes convenient to partially integrate the outer derivatives, ∂'_1 and ∂''_1. In this case they go to minus themselves and they act on all x''s or x'''s, respectively, in the expression.

It is remarkable that at one loop the position space self-energy involves no integrations. This is why it exists at all in the in-out formalism for quantum general relativity in a de Sitter background. The volume factors in the interaction vertices of this theory grow so rapidly that in-out matrix elements are generally infrared divergent if they contain even a single integration [1,8]. Of course the higher loop contributions do contain such integrations, so we cannot speak of an in-out self-energy beyond one loop. A related point is that only the first term of Eq. (2.8) is nonzero for $x'\neq x''$. The infrared properties of the one loop self-energy are entirely controlled by the first term, Fig. 3(b), and it is only the analog of this first term that we shall study in quantum general relativity. Note as well that the first term of Eq. (2.8) is completely well defined for x' and x'' away from coincidence.

The invariant Lagrangian of general relativity is

$$\mathcal{L} = \frac{1}{16\pi G}\,(R-2\Lambda)\sqrt{-g} + (\text{counterterms}), \quad (2.9)$$

where G is Newton's constant and Λ is the cosmological constant. Our classical background has the homogeneous and isotropic form (1.1) with scale factor:

$$a_{\text{class}}(t) = e^{Ht}. \quad (2.10)$$

It is simplest to perform the calculation in conformally flat coordinates, for which the invariant element of the background is

$$-dt^2 + a_{\text{class}}^2(t)d\vec{x}\cdot d\vec{x} = \Omega^2(-du^2 + d\vec{x}\cdot d\vec{x}), \quad (2.11\text{a})$$

$$\Omega \equiv \frac{1}{Hu} = \exp(Ht). \quad (2.11\text{b})$$

Note the temporal inversion and the fact that the onset of inflation at $t=0$ corresponds to $u = H^{-1}$. Since the infinite future is at $u = 0^+$, and since the spatial coordinates fall within the region, $-\frac{1}{2}H^{-1} < x^i \leq \frac{1}{2}H^{-1}$, the range of conformal coordinates is rather small. This is why a conformally invariant field—whose dynamics are locally the same as in flat space, except for ultraviolet regularization—cannot induce a big infrared effect.

Perturbation theory is organized most conveniently in terms of a "pseudograviton" field, $\psi_{\mu\nu}$, obtained by conformally rescaling the metric:

$$g_{\mu\nu} \equiv \Omega^2 \widetilde{g}_{\mu\nu} \equiv \Omega^2(\eta_{\mu\nu} + \kappa\psi_{\mu\nu}). \quad (2.12)$$

As usual, pseudograviton indices are raised and lowered with the Lorentz metric,[1] and the loop counting parameter is $\kappa^2 \equiv 16\pi G$. After some judicious partial integrations the invariant part of the bare Lagrangian takes the form [9]

$$\mathcal{L}_{\text{inv}} = \sqrt{-\widetilde{g}}\,\widetilde{g}^{\alpha\beta}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}\left[\frac{1}{2}\,\psi_{\alpha\rho,\mu}\psi_{\nu\sigma,\beta} - \frac{1}{2}\,\psi_{\alpha\beta,\rho}\psi_{\sigma\mu,\nu}\right.$$
$$\left. + \frac{1}{4}\,\psi_{\alpha\beta,\rho}\psi_{\mu\nu,\sigma} - \frac{1}{4}\,\psi_{\alpha\rho,\mu}\psi_{\beta\sigma,\nu}\right]\Omega^2$$
$$- \frac{1}{2}\,\sqrt{-\widetilde{g}}\,\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}\psi_{\rho\sigma,\mu}\psi_\nu{}^\alpha(\Omega^2)_{,\alpha}. \quad (2.13)$$

Note that each interaction term contains at least one ordinary derivative. This occurs because the dimension three coupling is canceled by the undifferentiated terms from the covariant derivatives of the dimension five coupling. Such a cancellation—for which there is no scalar field or flat space analog—is essential for classical stability [10] against growth of zero modes. An interesting consequence is that the leading infrared effects cancel as well in the quantum theory. However, the two couplings do not agree at subleading order, and there is still a very strong quantum effect.

Gauge fixing is accomplished through the addition of $-1/2\,\eta^{\mu\nu}F_\mu F_\nu$ where [9]

$$F_\mu \equiv \left[\psi^\rho{}_{\mu,\rho} - \frac{1}{2}\,\psi^\rho{}_{\rho,\mu} + 2\psi^\rho{}_\mu(\ln\Omega)_{,\rho}\right]\Omega. \quad (2.14)$$

The associated ghost Lagrangian is [9]

$$\mathcal{L}_{\text{ghost}} = -\Omega^2\overline{\omega}^{\mu,\nu}[\widetilde{g}_{\rho\mu}\partial_\nu + \widetilde{g}_{\rho\nu}\partial_\mu + \widetilde{g}_{\mu\nu,\rho} + 2\widetilde{g}_{\mu\nu}(\ln\Omega)_{,\rho}]\omega^\rho$$
$$+ (\Omega^2\overline{\omega}^\mu)_{,\mu}\eta^{\rho\sigma}\left[\widetilde{g}_{\nu\rho}\partial_\sigma + \frac{1}{2}\,\widetilde{g}_{\rho\sigma,\nu} + \widetilde{g}_{\rho\sigma}(\ln\Omega)_{,\nu}\right]\omega^\nu. \quad (2.15)$$

The zeroth order action results in the free field expansion [11]

$$\psi_{\mu\nu}(u,\vec{x}) = (\text{zero modes}) + H^3\sum_{\lambda,\,\vec{k}\neq 0}\{\Psi_{\mu\nu}(u,\vec{x};\vec{k},\lambda)a(\vec{k},\lambda)$$
$$+ \Psi^*_{\mu\nu}(u,\vec{x};\vec{k},\lambda)a^\dagger(\vec{k},\lambda)\}. \quad (2.16)$$

The spatial polarizations consists of "A" modes:

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[1]Note, however, that $\widetilde{g}^{\mu\nu}$ is the full matrix inverse of $\widetilde{g}_{\mu\nu}$ and has the usual geometric series expansion:

$$\widetilde{g}_{\mu\nu} = \eta^{\mu\nu} - \kappa\psi^{\mu\nu} + \kappa^2\psi^{\mu\rho}\psi_\rho{}^\nu - \cdots.$$

$$\Psi_{\mu\nu}(u,\vec{x};\vec{k},\lambda) = \frac{Hu}{\sqrt{2k}} \left(1 + \frac{i}{ku}\right) \exp\left[ik\left(u - \frac{1}{H}\right) + i\vec{k}\cdot\vec{x}\right] \epsilon_{\mu\nu}(\vec{k},\lambda), \quad \forall\lambda\in A,$$

(2.17a)

while the space-time and purely temporal polarizations are associated, respectively, with ''B'' and ''C'' modes:

$$\Psi_{\mu\nu}(u,\vec{x};\vec{k},\lambda) = \frac{Hu}{\sqrt{2k}} \exp\left[ik\left(u - \frac{1}{H}\right) + i\vec{k}\cdot\vec{x}\right] \epsilon_{\mu\nu}(\vec{k},\lambda), \quad \forall\lambda\in B,C.$$

(2.17b)

In Lehmann-Symanzik-Zimmermann (LSZ) reduction one would integrate against and contract into $\Psi_{\mu\nu}(u,\vec{x};\vec{k},\lambda)$ to insert and ''in''-coming graviton of momentum $\vec{k}$ and polarization λ; the conjugate would be used to extract an ''out''-going graviton with the same quantum numbers. The zero modes evolve as free particles with time dependences 1 and u^3 for the A modes, and u and u^2 for the B and C modes. Since causality decouples the zero modes shortly after the onset of inflation, they play no role in screening and we shall not trouble with the them further.

We define $|0\rangle$ as the Heisenberg state annihilated by $a(\vec{k},\lambda)$—and the analogous ghost operators—at the onset of inflation. We can use this condition and expansion (2.16) to express the free pseudograviton propagator as a mode sum [8]:

$$i[_{\mu\nu}\Delta_{\rho\sigma}](x;x') \equiv \langle 0|T\{\psi_{\mu\nu}(x)\psi_{\rho\sigma}(x')\}|0\rangle_{\text{free}}$$

(2.18a)

$$= H^3 \sum_{\lambda,\vec{k}\neq 0} \{\theta(u'-u)\Psi_{\mu\nu}\Psi_{\rho\sigma}'^* + \theta(u-u')\Psi_{\mu\nu}^*\Psi_{\rho\sigma}'\}e^{-\epsilon\|\vec{k}\|}.$$

(2.18b)

Note that the convergence factor $e^{-\epsilon\|\vec{k}\|}$ serves as an ultraviolet mode cutoff. Although the resulting regularization is very convenient for this calculation, its failure to respect general coordinate invariance necessitates the use of noninvariant counterterms. These are analogous to the photon mass which must be added to QED when using a momentum cutoff. Just as in QED, these noninvariant counterterms do not affect long distance phenomena.

Because the propagator is only needed for small conformal coordinate separations, $\Delta x \equiv \|\vec{x}'-\vec{x}\|$ and $\Delta u \equiv u'-u$, the sum over momenta is well approximated as an integral. When this is done the pseudograviton and ghost propagators become [8]

$$i[_{\mu\nu}\Delta_{\rho\sigma}](x;x')$$

$$\approx \frac{H^2}{8\pi^2}\left\{ \frac{2u'u}{\Delta x^2-\Delta u^2+2i\epsilon|\Delta u|+\epsilon^2}[2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}] \right.$$

$$\left. - \ln[H^2(\Delta x^2-\Delta u^2+2i\epsilon|\Delta u|+\epsilon^2)] \times [2\bar{\eta}_{\mu(\rho}\bar{\eta}_{\sigma)\nu} - 2\bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}]\right\},$$

(2.19a)

$$i[_{\mu}\Delta_{\nu}](x;x') \approx \frac{H^2}{8\pi^2}\left\{ \frac{2u'u}{\Delta x^2-\Delta u^2+2i\epsilon|\Delta u|+\epsilon^2}\eta_{\mu\nu} - \ln[H^2(\Delta x^2-\Delta u^2+2i\epsilon|\Delta u|+\epsilon^2)]\bar{\eta}_{\mu\nu}\right\}.$$

(2.19b)

Parenthesized indices are symmetrized and a bar above a Lorentz metric or a Kronecker δ symbol means that the zero component is projected out, e.g., $\bar{\eta}_{\mu\nu}\equiv\eta_{\mu\nu}+\delta_\mu^0\delta_\nu^0$. The decoupling between functional dependence upon spacetime and tensor indices—and the simplicity of each—greatly facilitates calculations.

We find the cubic self-interactions by expanding expression (2.13) to third order in the pseudograviton field. The result is $\kappa\Omega^2$ times

$$-\frac{1}{2u}\,\psi\psi_{,\mu}\psi^{\mu\nu}t_\nu + \frac{1}{u}\,\psi^{\rho\sigma}\psi_{\rho\sigma,\mu}\psi^{\mu\nu}t_\nu + \frac{1}{u}\,\psi_{\rho\sigma}\psi^\rho\psi^{\sigma\nu}t_\nu + \frac{1}{4}\,\psi\psi^{\rho\sigma,\mu}\psi_{\mu\rho,\sigma} - \psi^{\rho\sigma}\psi_\rho^{,\mu,\nu}\psi_{\mu\nu,\sigma} - \frac{1}{2}\,\psi^{\rho\sigma}\psi_\rho^{\mu\nu}\psi_{\sigma\nu,\mu}$$

$$-\frac{1}{4}\,\psi\psi_{,\rho}\psi^{\rho\sigma}{}_{,\sigma} + \frac{1}{2}\,\psi^{\rho\sigma}\psi_{\rho\sigma,\mu}\psi^{\mu\nu}{}_{,\nu} + \frac{1}{2}\,\psi^{\rho\sigma}\psi_{,\rho}\psi_{\sigma\mu}{}^{,\mu} + \frac{1}{2}\,\psi^{\rho\sigma}\psi^{,\mu}\psi_{\mu\rho,\sigma} + \frac{1}{8}\,\psi\psi^{,\mu}\psi_{,\mu} - \frac{1}{2}\,\psi^{\rho\sigma}\psi_{\rho\sigma,\mu}\psi^{,\mu} - \frac{1}{4}\,\psi^{\rho\sigma}\psi_{,\rho}\psi_{,\sigma}$$

$$-\frac{1}{8}\,\psi\psi^{\rho\sigma,\mu}\psi_{\rho\sigma,\mu} + \frac{1}{2}\,\psi^{\rho\sigma}\psi_{\rho\mu,\nu}\psi_\sigma^{\mu,\nu} + \frac{1}{4}\,\psi^{\rho\sigma}\psi_{\mu\nu,\rho}\psi^{\mu\nu}{}_{,\sigma},$$

(2.20)

where $\psi\equiv\psi^\mu{}_\mu$ and $t_\nu\equiv\eta_{0\nu}$. All but the first three terms should agree with the flat space expansion when $\psi_{\mu\nu}$ is regarded as the graviton field and $\Omega=1$. Of course this allows us to check them against published results [12] and they do check.[2]

The vertex operators will be fully symmetrized if we define them by functional differentiation as in Eq. (2.5). This is not efficient for computing the self-energy because only one of the vertices needs to be symmetrized on its internal lines. Fully symmetrizing both vertex operators causes each distinct pairing to appear twice in the self-energy, which is why the symmetry

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[2]Note, however, that some earlier results are given for a timelike metric, so our field is minus theirs.

TABLE I. Cubic partial vertex operators with No. 1 distinguished. Each term should be multiplied by $\kappa\Omega^2$.

No.	Partial vertex	No.	Partial vertex
1	$-\frac{1}{2u}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\partial_2^{(\alpha_3}t^{\beta_3)}$	22	$\frac{1}{2}\eta^{\alpha_1(\alpha_2}\eta^{\beta_2)\beta_1}\partial_2^{(\alpha_3}\partial_3^{\beta_3)}$
2	$-\frac{1}{2u}\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}\partial_3^{(\alpha_1}t^{\beta_1)}$	23	$\frac{1}{2}\eta^{\alpha_2(\alpha_3}\eta^{\beta_3)\beta_2}\partial_3^{(\alpha_1}\partial_1^{\beta_1)}$
3	$-\frac{1}{2u}\eta^{\alpha_3\beta_3}\eta^{\alpha_1\beta_1}\partial_1^{(\alpha_2}t^{\beta_2)}$	24	$\frac{1}{2}\eta^{\alpha_3(\alpha_1}\eta^{\beta_1)\beta_3}\partial_1^{(\alpha_2}\partial_2^{\beta_2)}$
4	$\frac{1}{u}\eta^{\alpha_1(\alpha_2}\eta^{\beta_2)\beta_1}\partial_2^{(\alpha_3}t^{\beta_3)}$	25	$\frac{1}{2}\partial_2^{(\alpha_1}\eta^{\beta_1)(\alpha_3}\partial_3^{\beta_3)}\eta^{\alpha_2\beta_2}$
5	$\frac{1}{u}\eta^{\alpha_2(\alpha_3}\eta^{\beta_3)\beta_2}\partial_3^{(\alpha_1}t^{\beta_1)}$	26	$\frac{1}{2}\partial_3^{(\alpha_2}\eta^{\beta_2)(\alpha_1}\partial_1^{\beta_1)}\eta^{\alpha_3\beta_3}$
6	$\frac{1}{u}\eta^{\alpha_3(\alpha_1}\eta^{\beta_1)\beta_3}\partial_1^{(\alpha_2}t^{\beta_2)}$	27	$\frac{1}{2}\partial_1^{(\alpha_3}\eta^{\beta_3)(\alpha_2}\partial_2^{\beta_2)}\eta^{\alpha_1\beta_1}$
7	$\frac{1}{u}t^{(\alpha_3}\eta^{\beta_3)(\alpha_1}\partial_2^{\beta_1)}\eta^{\alpha_2\beta_2}$	28	$\frac{1}{2}\partial_2^{(\alpha_1}\eta^{\beta_1)(\alpha_2}\partial_3^{\beta_2)}\eta^{\alpha_3\beta_3}$
8	$\frac{1}{u}t^{(\alpha_1}\eta^{\beta_1)(\alpha_2}\partial_3^{\beta_2)}\eta^{\alpha_3\beta_3}$	29	$\frac{1}{2}\partial_3^{(\alpha_2}\eta^{\beta_2)(\alpha_3}\partial_1^{\beta_3)}\eta^{\alpha_1\beta_1}$
9	$\frac{1}{u}t^{(\alpha_2}\eta^{\beta_2)(\alpha_3}\partial_1^{\beta_3)}\eta^{\alpha_1\beta_1}$	30	$\frac{1}{2}\partial_1^{(\alpha_3}\eta^{\beta_3)(\alpha_1}\partial_2^{\beta_1)}\eta^{\alpha_2\beta_2}$
10	$\frac{1}{4}\eta^{\alpha_1\beta_1}\partial_3^{(\alpha_2}\eta^{\beta_2)(\alpha_3}\partial_2^{\beta_3)}$	31	$\frac{1}{8}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}\partial_2\cdot\partial_3$
11	$\frac{1}{4}\eta^{\alpha_2\beta_2}\partial_1^{(\alpha_3}\eta^{\beta_3)(\alpha_1}\partial_3^{\beta_1)}$	32	$\frac{1}{4}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}\partial_3\cdot\partial_1$
12	$\frac{1}{4}\eta^{\alpha_3\beta_3}\partial_2^{(\alpha_1}\eta^{\beta_1)(\alpha_2}\partial_1^{\beta_2)}$	33	$-\frac{1}{2}\eta^{\alpha_1(\alpha_2}\eta^{\beta_2)\beta_1}\eta^{\alpha_3\beta_3}\partial_2\cdot\partial_3$
13	$-\partial_3^{(\alpha_1}\eta^{\beta_1)(\alpha_2}\eta^{\beta_2)(\alpha_3}\partial_2^{\beta_3)}$	34	$-\frac{1}{2}\eta^{\alpha_2(\alpha_3}\eta^{\beta_3)\beta_2}\eta^{\alpha_1\beta_1}\partial_3\cdot\partial_1$
14	$-\partial_1^{(\alpha_2}\eta^{\beta_2)(\alpha_3}\eta^{\beta_3)(\alpha_1}\partial_3^{\beta_1)}$	35	$-\frac{1}{2}\eta^{\alpha_3(\alpha_1}\eta^{\beta_1)\beta_3}\eta^{\alpha_2\beta_2}\partial_1\cdot\partial_2$
15	$-\partial_2^{(\alpha_3}\eta^{\beta_3)(\alpha_1}\eta^{\beta_1)(\alpha_2}\partial_1^{\beta_2)}$	36	$-\frac{1}{4}\partial_2^{(\alpha_1}\partial_3^{\beta_1)}\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}$
16	$-\frac{1}{2}\partial_3^{(\alpha_2}\eta^{\beta_2)(\alpha_1}\eta^{\beta_1)(\alpha_3}\partial_2^{\beta_3)}$	37	$-\frac{1}{2}\partial_3^{(\alpha_2}\partial_1^{\beta_2)}\eta^{\alpha_3\beta_3}\eta^{\alpha_1\beta_1}$
17	$-\frac{1}{2}\partial_1^{(\alpha_3}\eta^{\beta_3)(\alpha_2}\eta^{\beta_2)(\alpha_1}\partial_3^{\beta_1)}$	38	$-\frac{1}{8}\eta^{\alpha_1\beta_1}\eta^{\alpha_2(\alpha_3}\eta^{\beta_3)\beta_2}\partial_2\cdot\partial_3$
18	$-\frac{1}{2}\partial_2^{(\alpha_1}\eta^{\beta_1)(\alpha_3}\eta^{\beta_3)(\alpha_2}\partial_1^{\beta_2)}$	39	$-\frac{1}{4}\eta^{\alpha_2\beta_2}\eta^{\alpha_3(\alpha_1}\eta^{\beta_1)\beta_3}\partial_3\cdot\partial_1$
19	$-\frac{1}{4}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\partial_2^{(\alpha_3}\partial_3^{\beta_3)}$	40	$\frac{1}{2}\eta^{\alpha_1)(\alpha_2}\eta^{\beta_2)(\alpha_3}\eta^{\beta_3)(\beta_1}\partial_2\cdot\partial_3$
20	$-\frac{1}{4}\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}\partial_3^{(\alpha_1}\partial_1^{\beta_1)}$	41	$\eta^{\alpha_1)(\alpha_2}\eta^{\beta_2)(\alpha_3}\eta^{\beta_3)(\beta_1}\partial_3\cdot\partial_1$
21	$-\frac{1}{4}\eta^{\alpha_3\beta_3}\eta^{\alpha_1\beta_1}\partial_1^{(\alpha_2}\partial_2^{\beta_2)}$	42	$\frac{1}{4}\partial_2^{(\alpha_1}\partial_3^{\beta_1)}\eta^{\alpha_2(\alpha_3}\eta^{\beta_3)\beta_2}$
		43	$\frac{1}{2}\partial_3^{(\alpha_2}\partial_1^{\beta_2)}\eta^{\alpha_3(\alpha_1}\eta^{\beta_1)\beta_3}$

factor for this diagram is $\frac{1}{2}$. The large number of distinct cubic self-interaction terms (2.20) means that this is not an efficient strategy for quantum gravity. The fully symmetrized vertex operator contains 75 separate terms [4] whereas permuting over only the three possible choices for the outer line results in just 43 terms. One must sum over both ways of pairing the internal lines but there is still a saving of almost 50%—which is important in summing over 75^2 eightfold contractions of 20-index objects.

To obtain the partially symmetrized vertex operators from a given cubic self-interaction one merely assigns the three legs any of the six possible ways and then permutes cyclically. For example, the first term in Eq. (2.20) gives

$$-\frac{1}{2u}\,\psi\psi_\mu\psi^{\mu\nu}t_\nu \rightarrow -\frac{1}{2u}\,\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\partial_2^{(\alpha_3}t^{\beta_3)}\quad\text{(initial assignment)}$$

$$\rightarrow -\frac{1}{2u}\,\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\partial_2^{(\alpha_3}t^{\beta_3)}-\frac{1}{2u}\,\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}\partial_3^{(\alpha_1}t^{\beta_1)}-\frac{1}{2u}\,\eta^{\alpha_3\beta_3}\eta^{\eta_1\beta_1}\partial_1^{(\alpha_2}t^{\beta_2)}\quad\text{(cyclic permutation).}$$

The various partial vertex operators are given in Table I. We consider line No. 1 to be the distinguished one, and we have used symmetries among the remaining two fields to reduce the number of vertices whenever possible.

The ghost-antighost-pseudograviton interactions can be read off from Eq. (2.15). There are only ten interactions and they are $\kappa\Omega^2$ times

$$-\psi_{\mu\nu}\varpi^{\mu,\rho}\omega^\nu{}_{,\rho}-\psi_{\mu\nu}\varpi^{\rho,\mu}\omega^\nu{}_{,\rho}-\psi_{\mu\nu,\sigma}\varpi^{\mu,\nu}\omega^\sigma-\frac{2}{u}\,\psi_{\mu\nu}\varpi^{\mu,\nu}\omega^\sigma t_\sigma+\psi_{\mu\nu}\varpi^\rho{}_{,\rho}\omega^{\mu,\nu}+\frac{1}{2}\,\psi_{,\sigma}\varpi^\rho{}_{,\rho}\omega^\sigma+\frac{1}{u}\,\psi\varpi^\rho{}_{,\rho}\omega^\sigma t_\sigma$$

$$+\frac{2}{u}\,\psi_{\mu\nu}\varpi^\rho\omega^{\mu,\nu}t_\rho+\frac{1}{u}\,\psi_{,\sigma}\varpi^\rho\omega^\sigma t_\rho+\frac{2}{u^2}\,\psi\varpi^\rho\omega^\sigma t_\rho t_\sigma.\tag{2.21}$$

There is no issue of symmetrization in finding the associated vertex operators because each of the three fields is different. The result is presented in Table II.

It is simple to write the graviton self-energy as a sum over contractions of the various vertex operators between the two internal propagators:

TABLE II. Ghost-pseudograviton vertex operators. Each term should be multiplied by $\kappa\Omega^2$.

No.	Vertex operator	No.	Vertex operator
1	$-\eta^{\alpha_2(\alpha_1}\eta^{\beta_1)\alpha_3}\partial_2\cdot\partial_3$	6	$\frac{1}{2}\eta^{\alpha_1\beta_1}\partial_2^{\alpha_2}\partial_1^{\alpha_3}$
2	$-\eta^{\alpha_3(\alpha_1}\partial_2^{\beta_1)}\partial_3^{\alpha_2}$	7	$\frac{1}{u}\eta^{\alpha_1\beta_1}\partial_2^{\alpha_2}t^{\alpha_3}$
3	$-\eta^{\alpha_2(\alpha_1}\partial_2^{\beta_1)}\partial_1^{\alpha_3}$	8	$\frac{2}{u}\eta^{\alpha_3(\alpha_1}\partial_3^{\beta_1)}t^{\alpha_2}$
4	$-\frac{2}{u}\eta^{\alpha_2(\alpha_1}\partial_2^{\beta_1)}t^{\alpha_3}$	9	$\frac{1}{u}\eta^{\alpha_1\beta_1}\partial_1^{\alpha_3}t^{\alpha_2}$
5	$\eta^{\alpha_3(\alpha_1}\partial_3^{\beta_1)}\partial_2^{\alpha_2}$	10	$(2/u^2)\eta^{\alpha_1\beta_1}t^{\alpha_2}t^{\alpha_3}$

$$[^{\alpha_1\beta_1}\Sigma^{\rho_1\sigma_1}](x';x'') = -i\sum_{i,j=1}^{43} V_i^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x';\partial_1',\partial_2',\partial_3')\,i[_{\alpha_2\beta_2}\Delta_{\rho_2\sigma_2}](x';x'')\,i[_{\alpha_3\beta_3}\Delta_{\rho_3\sigma_3}](x';x'')$$

$$\times\{V_j^{\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3}(x'';\partial_1'',\partial_2'',\partial_3'') + V_j^{\rho_1\sigma_1\rho_3\sigma_3\rho_2\sigma_2}(x'';\partial_1'',\partial_3'',\partial_2'')\}$$

$$+i\sum_{i,j=1}^{10} V_i^{\alpha_1\beta_1\alpha_2\alpha_3}(x';\partial_1',\partial_2',\partial_3')\,i[_{\alpha_2}\Delta_{\rho_2}](x';x'')\,i[_{\alpha_3}\Delta_{\rho_3}](x';x'')\,V_j^{\rho_1\sigma_1\rho_3\rho_2}(x'';\partial_1'',\partial_3'',\partial_2'').$$

$$(2.22)$$

Since we have not symmetrized the pseudograviton vertex operators on lines 2 and 3 it is necessary to include both a symmetric loop [Fig. 4(a)] and an asymmetric loop [Fig. 4(b)]. The second double sum gives the ghost loop [Fig. 4(c)]. The distinction between the ψ^3 and $\psi\bar\omega\omega$ vertex operators does not require a separate symbol; the different number of indices suffices. Where each derivative acts is denoted by primes and subscripts. For example, the derivative ∂_2' in the first ψ^3 vertex operator acts on the first argument of the propagator, $[_{\alpha_2\beta_2}\Delta_{\rho_2\sigma_2}](x';x'')$. The derivatives ∂_1' and ∂_1'' act on the left and right outer legs. In computing the two loop tadpole we left them as free operators to act on the upper propagators when they were attached. For computing the self-energy we have of course partially integrated them to act on all the x''s and x'''s in the expression.

The entire calculation was performed by computer using the symbolic manipulation program MATHEMATICA [13]. The first step was to contract each pair of vertex operators into the internal propagators. This was done using Mertig's package FEYNCALC [14], and the result was written onto a file. The next step was acting the internal derivatives (∂_2', ∂_2'', ∂_3', and ∂_3''), the results of which were also stored for each pair of vertex operators. Selected vertex pairs were computed by hand to check the procedure. At this stage the results for all vertex pairs were summed, and the total was checked for symmetry under interchanging the two external legs.

The result we obtained for the self-energy operator is a very long sum of four-index objects times scalar functions. The four-index objects are constructed from $\eta^{\mu\nu}$,

$x^\mu\equiv(x'-x'')^\mu$, $t^\mu\equiv\delta_0^\mu$, $\partial_1'^\mu$ and $\partial_1''^\mu$. The scalar functions can depend upon $x^2\equiv x^\mu x^\nu\eta_{\mu\nu}$, u', u'', $x\cdot\partial_1'$, $x\cdot\partial_1''$, $t\cdot\partial_1'$, $t\cdot\partial_1''$, and $\partial_1'\cdot\partial_1''$. Each term can contain at most one factor of each of the external derivatives, either free or contracted. It can be shown that with this requirement there are 79 distinct four-index objects and ten possibilities for contracted derivatives [4]. To make the expression more manageable, prior to attaching the outer vertex and propagators, we extracted the scalar coefficient of each allowed combination of four-index object and contracted external derivative operators. It is from this data that we later computed the self-energy by partially integrating the external derivatives to act back on x' and x'', respectively. Since the data actually contained x' and x'' through x, u', and u'', we used the rules

$$-\partial_{1\mu}' = -\frac{\partial}{\partial x^\mu} + t_\mu\frac{\partial}{\partial u'},\qquad(2.23a)$$

$$-\partial_{1\mu}'' = \frac{\partial}{\partial x^\mu} + t_\mu\frac{\partial}{\partial u''}.\qquad(2.23b)$$

The result after this is done consists of a sum of four-index tensors times scalar functions. The four-index tensors can depend only upon $\eta^{\mu\nu}$, x^μ, and t^μ, while the scalar functions depend only upon x^2, u', and u''. The 21 possible four-index objects are listed in Table III. Note that we have dispensed with the now-irrelevant line subscript 1. Note also that the
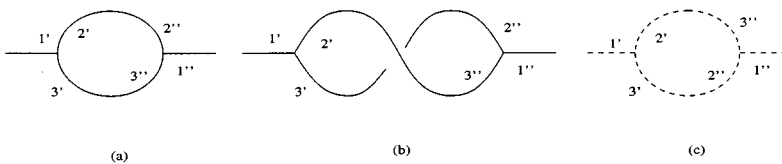

FIG. 4. One-loop contributions to the graviton self-energy.

ONE LOOP GRAVITON SELF-ENERGY IN A LOCALLY DE ...

TABLE III. Tensor factors in the self-energy.

No.	Tensor	No.	Tensor	No.	Tensor
1	$\eta^{\alpha\beta}\eta^{\rho\sigma}$	8	$x^{\alpha}x^{\beta}\eta^{\rho\sigma}$	15	$t^{(\alpha}x^{\beta)}t^{\rho}t^{\sigma}$
2	$\eta^{\alpha(\rho}\eta^{\sigma)\beta}$	9	$t^{(\alpha}\eta^{\beta)(\rho}t^{\sigma)}$	16	$t^{\alpha}t^{\beta}x^{\rho}x^{\sigma}$
3	$\eta^{\alpha\beta}t^{\rho}t^{\sigma}$	10	$t^{(\alpha}\eta^{\beta)(\rho}x^{\sigma)}$	17	$x^{\alpha}x^{\beta}t^{\rho}t^{\sigma}$
4	$t^{\alpha}t^{\beta}\eta^{\rho\sigma}$	11	$x^{(\alpha}\eta^{\beta)(\rho}t^{\sigma)}$	18	$t^{(\alpha}x^{\beta)}t^{(\rho}x^{\sigma)}$
5	$\eta^{\alpha\beta}t^{(\rho}x^{\sigma)}$	12	$x^{(\alpha}\eta^{\beta)(\rho}x^{\sigma)}$	19	$t^{(\alpha}x^{\beta)}x^{\rho}x^{\sigma}$
6	$t^{(\alpha}x^{\beta)}\eta^{\rho\sigma}$	13	$t^{\alpha}t^{\beta}t^{\rho}t^{\sigma}$	20	$x^{\alpha}x^{\beta}t^{(\rho}x^{\sigma)}$
7	$\eta^{\alpha\beta}x^{\rho}x^{\sigma}$	14	$t^{\alpha}t^{\beta}t^{(\rho}x^{\sigma)}$	21	$x^{\alpha}x^{\beta}x^{\rho}x^{\sigma}$

reflection symmetry relates the coefficients of pairs 3 and 4, 5 and 6, 7 and 8, 10 and 11, 14 and 15, 16 and 17, and 19 and 20.

It is worth working out an example to illustrate the method. To save space during the contractions we define the normal and log propagators as

$$i\Delta_N \equiv \frac{H^2}{8\pi^2}\frac{2u'u''}{x^2+i\epsilon}, \quad i\Delta_L \equiv \frac{H^2}{8\pi^2}\ln[H^2(x^2+i\epsilon)]. \quad (2.24)$$

Now consider the $i=j=41$ term from the symmetric graviton loop [Fig. 4(a)]:

$$[^{\alpha_1\beta_1}\Sigma^{\rho_1\sigma_1}_{41-41}](x';x'') = -iV^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}_{41}(x';\partial'_1,\partial'_2,\partial'_3)i[_{\alpha_2\beta_2}\Delta_{\rho_2\sigma_2}](x';x'')$$

$$\times i[_{\alpha_3\beta_3}\Delta_{\rho_3\sigma_3}](x';x'')V^{\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3}_{41}(x'';\partial''_1,\partial''_2,\partial''_3) \quad (2.25a)$$

$$= -i\kappa^2\Omega'^2\Omega''^2\partial'_1\cdot\partial'_3\partial''_1\cdot\partial''_3\eta^{\alpha_1)(\alpha_2}\eta^{\beta_2)(\alpha_3}\eta^{\beta_3)(\beta_1}\eta^{\rho_1)(\rho_2}\eta^{\sigma_2)(\rho_3}\eta^{\sigma_3)(\sigma_1}$$

$$\times\{i\Delta_{2N}[2\eta_{\alpha_2(\rho_2}\eta_{\sigma_2)\beta_2}-\eta_{\alpha_2\beta_2}\eta_{\rho_2\sigma_2}]-i\Delta_{2L}[2\overline{\eta}_{\alpha_2(\rho_2}\overline{\eta}_{\sigma_2)\beta_2}-2\overline{\eta}_{\alpha_2\beta_2}\overline{\eta}_{\rho_2\sigma_2}]\}$$

$$\times\{i\Delta_{3N}[2\eta_{\alpha_3(\rho_3}\eta_{\sigma_3)\beta_3}-\eta_{\alpha_3\beta_3}\eta_{\rho_3\sigma_3}]-i\Delta_{3L}[2\overline{\eta}_{\alpha_3(\rho_3}\overline{\eta}_{\sigma_3)\beta_3}-2\overline{\eta}_{\alpha_3\beta_3}\overline{\eta}_{\rho_3\sigma_3}]\} \quad (2.25b)$$

$$= -i\kappa^2\Omega'^2\Omega''^2\partial'_1\cdot\partial'_3\partial''_1\cdot\partial''_3\Big\{i\Delta_{2N}i\Delta_{3N}[2\eta^{\alpha_1\beta_1}\eta^{\rho_1\sigma_1}+2\eta^{\alpha_1(\rho_1}\eta^{\sigma_1)\beta_1}]$$

$$+(i\Delta_{2N}i\Delta_{3L}+i\Delta_{2L}i\Delta_{3N})[-3\eta^{\alpha_1\beta_1}\eta^{\rho_1\sigma_1}+\eta^{\alpha_1(\rho_1}\eta^{\sigma_1)\beta_1}-3\eta^{\alpha_1\beta_1}t^{\rho_1}t^{\sigma_1}$$

$$-3t^{\alpha_1}t^{\beta_1}\eta^{\rho_1\sigma_1}+4t^{(\alpha_1}\eta^{\beta_1)(\rho_1}t^{\sigma_1)}]+i\Delta_{2L}i\Delta_{3L}[5\eta^{\alpha_1\beta_1}\eta^{\rho_1\sigma_1}$$

$$-3\eta^{\alpha_1(\rho_1}\eta^{\sigma_1)\beta_1}+5\eta^{\alpha_1\beta_1}t^{\rho_1}t^{\sigma_1}+5t^{\alpha_1}t^{\beta_1}\eta^{\rho_1\sigma_1}-6t^{(\alpha_1}\eta^{\beta_1)(\rho_1}t^{\sigma_1)}+2t^{\alpha_1}t^{\beta_1}t^{\rho_1}t^{\sigma_1}]\Big\}. \quad (2.25c)$$

We now label the various tensor factors according to the scheme of Table III as $T_1\equiv\eta^{\alpha_1\beta_1}\eta^{\rho_1\sigma_1}$, etc., and act the inner derivatives:

$$[^{\alpha_1\beta_1}\Sigma^{\rho_1\sigma_1}_{41-41}](x';x'') = -i\kappa^2\Omega'^2\Omega''^2\partial'_1\cdot\partial'_3\partial''_1\cdot\partial''_3\{i\Delta_{2N}i\Delta_{3N}[2T_1+2T_2]+(i\Delta_{2N}i\Delta_{3L}+i\Delta_{2L}i\Delta_{3N})[-3T_1+T_2-3T_3-3T_4$$

$$+4T_9]+i\Delta_{2L}i\Delta_{3L}[5T_1-3T_2+5T_3+5T_4-6T_9+2T_{13}]\} \quad (2.26a)$$

$$= \frac{-i\kappa^2}{2^6\pi^4}\Bigg\{\left(-\frac{32x\cdot\partial'_1x\cdot\partial''_1}{x^8}+\frac{8\partial'_1\cdot\partial''_1}{x^6}+\frac{8x\cdot\partial'_1t\cdot\partial''_1}{u''x^6}-\frac{8t\cdot\partial'_1x\cdot\partial''_1}{u'x^6}+\frac{4t\cdot\partial'_1t\cdot\partial''_1}{u'u''x^4}\right)[2T_1+2T_2]$$

$$+\left[\left(-\frac{16x\cdot\partial'_1x\cdot\partial''_1}{u'u''x^6}+\frac{4\partial'_1\cdot\partial''_1}{u'u''x^4}+\frac{4x\cdot\partial'_1t\cdot\partial''_1}{u'u''^2x^4}-\frac{4t\cdot\partial'_1x\cdot\partial''_1}{u'^2u''x^4}+\frac{2t\cdot\partial'_1t\cdot\partial''_1}{u'^2u''^2x^2}\right)\ln(H^2x^2)\right.$$

$$+\left.\left(-\frac{4\partial'_1\cdot\partial''_1}{u'u''x^4}+\frac{8x\cdot\partial'_1x\cdot\partial''_1}{u'u''x^6}\right)\right][-3T_1+T_2-3T_3-3T_4+4T_9]$$

$$+\left(-\frac{2\partial'_1\cdot\partial''_1}{u'^2u''^2x^2}+\frac{4x\cdot\partial'_1x\cdot\partial''_1}{u'^2u''^2x^4}\right)\ln(H^2x^2)[5T_1-3T_2+5T_3+5T_4-6T_9+2T_{13}]\Bigg\}. \quad (2.26b)$$

This is the point at which the outer vertex and propagators would be attached in computing the two loop tadpole. However, to extract the one loop self we partially integrate the outer derivatives to act them back on x' and x'' using Eq. (2.23). The result is

$$[^{\alpha_1\beta_1}\Sigma^{\rho_1\sigma_1}_{41-41}](x';x'') = \frac{-i\kappa^2}{2^6\pi^4}\left\{\left(\frac{192}{x^8} + \frac{32}{u'u''x^6} + \frac{4}{u'^2u''^2x^4}\right)[2T_1+2T_2] + \left(-\frac{32}{u'u''x^6} - \frac{8\Delta u^2}{u'^3u''^3x^4} - \frac{28}{u'^2u''^2x^4} + \frac{4\ln(H^2x^2)}{u'^2u''^2x^4}\right.\right.$$

$$\left. + \frac{8\ln(H^2x^2)}{u'^3u''^3x^2}\right)[-3T_1+T_2-3T_3-3T_4+4T_9] + \left(-\frac{8\Delta u^2}{u'^3u''^3x^4} - \frac{16}{u'^2u''^2x^4} + \frac{8\ln(H^2x^2)}{u'^3u''^3x^2}\right)$$

$$\left. \times[5T_1-3T_2+5T_3+5T_4-6T_9+2T_{13}]\right\}. \tag{2.27}$$

Note that $\Delta u \equiv u'' - u'$ and that we have suppressed the factors of $i\epsilon$ which go with x^2's. The final results for the entire self-energy are given in Tables IV(a) and IV(b).

III. THE FLAT SPACE LIMIT

Consideration of the classical background (2.11) reveals that flat space can be recovered by setting the conformal time to

$$u = \frac{1}{H} - t \tag{3.1}$$

and then taking the Hubble constant H to zero [11]. Note that in this limit the scale factor Ω becomes unity, as does the ratio of products of equal numbers of conformal times. Note finally that the difference of two conformal times is just minus the same difference of flat space times:

$$x^0 \equiv u' - u'' = \left(\frac{1}{H} - t'\right) - \left(\frac{1}{H} - t''\right) = t'' - t'. \tag{3.2}$$

This means that our quantity $x^2 \equiv (x'-x'')^\mu (x'-x'')^\nu \eta_{\mu\nu}$ goes to the usual Lorentz invariant interval, which we shall continue to call x^2.

The preceding facts make it very simple to take the self-energy's flat space limit. Consider, for example, the no-log coefficient of tensor factor No. 7, which we can read from Table IV(a). A simple calculation gives the limit

$$-\frac{1280}{x^{10}} + \frac{1152u'}{u''x^{10}} - \frac{704u''}{u'x^{10}} + \frac{8}{u'^2x^8} + \frac{192}{u''^2x^8} - \frac{144}{u'u''x^8}$$

$$+ \frac{32}{u'u''^3x^6} - \frac{24}{u'^2u''^2x^6} + \frac{64}{u'^3u''x^6} \rightarrow -\frac{1280}{x^{10}} + \frac{1152}{x^{10}}$$

$$- \frac{704}{x^{10}} = -\frac{832}{x^{10}}. \tag{3.3}$$

The logarithm terms of Table IV(b) cancel completely, and the only nonzero contributions from the nonlog terms of Table IV(a) come from the coefficients of tensors Nos. 1, 2, 7, 8, 12, and 21. The answer is

$$[^{\alpha\beta}\Sigma^{\rho\sigma}_{\text{flat}}](x';x'') = \frac{-i\kappa^2}{2^6\pi^4}\left\{\frac{288}{x^8}\eta^{\alpha\beta}\eta^{\rho\sigma} + \frac{720}{x^8}\eta^{\alpha(\rho}\eta^{\sigma)\beta}\right.$$

$$- \frac{832}{x^{10}}[\eta^{\alpha\beta}x^\rho x^\sigma + x^\alpha x^\beta \eta^{\rho\sigma}]$$

$$\left. - \frac{3712}{x^{10}}x^{(\alpha}\eta^{\beta)\,(\rho}x^{\sigma)} + \frac{5376}{x^{12}}x^\alpha x^\beta x^\rho x^\sigma\right\}. \tag{3.4}$$

It is useful to recast this expression in the form of a derivative operator acting on $1/x^4$. The key identities are

$$\frac{1}{x^8} = \frac{1}{192}\partial^4\left(\frac{1}{x^4}\right), \tag{3.5a}$$

$$\frac{x^\mu x^\nu}{x^{10}} = \left[\frac{1}{384}\partial^\mu\partial^\nu\partial^2 + \frac{1}{1536}\eta^{\mu\nu}\partial^4\right]\left(\frac{1}{x^4}\right), \tag{3.5b}$$

$$\frac{x^\mu x^\nu x^\rho x^\sigma}{x^{12}} = \left[\frac{1}{1920}\partial^\mu\partial^\nu\partial^\rho\partial^\sigma + \frac{1}{640}\eta^{(\mu\nu}\partial^\rho\partial^{\sigma)}\partial^2\right.$$

$$\left. + \frac{1}{5120}\eta^{(\mu\nu}\eta^{\rho\sigma)}\partial^4\right]\left(\frac{1}{x^4}\right). \tag{3.5c}$$

Our final result for the flat space limit of the one loop self-energy is therefore

$$[^{\alpha\beta}\Sigma^{\rho\sigma}_{\text{flat}}](x';x'') = \frac{-i\kappa^2}{60(2\pi)^4}\left\{\frac{23}{2}\eta^{\alpha\beta}\eta^{\rho\sigma}\partial^4\right.$$

$$+ \frac{61}{2}\eta^{\alpha(\rho}\eta^{\sigma)\beta}\partial^4 - \frac{23}{2}[\eta^{\alpha\beta}\partial^\rho\partial^\sigma$$

$$+ \eta^{\rho\sigma}\partial^\alpha\partial^\beta]\partial^2 - 61\partial^{(\alpha}\eta^{\beta)\,(\rho}\partial^{\sigma)}\partial^2$$

$$\left. + 42\partial^\alpha\partial^\beta\partial^\rho\partial^\sigma\right\}\left(\frac{1}{x^4}\right). \tag{3.6}$$

It remains to compare Eq. (3.6) with the flat space result obtained for the $\Lambda=0$ theory by Capper [7]. He defined the graviton field as[3]

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[3] Capper also used a timelike metric, he associates the inverse factors of 2π in Fourier transforms with what he calls the propagators, and what he calls the self-energy is $-i$ times what we call the self-energy. We have translated all these conventions into our notation to avoid confusion.

TABLE IV. (a) Tensor coefficients which are free of logarithms. Multiply each term by $-i\kappa^2/(2^6\pi^4)$ times the appropriate tensor factor from Table III to obtain the contribution to the self-energy. (b) Tensor coefficients which contain logarithms. Multiply each term by $-i\kappa^2\ln(H^2x^2)/(2^6\pi^4)$ times the appropriate tensor factor from Table III to obtain the contribution to the self-energy.

No.	Coefficient
	(a)
1	$\dfrac{1056}{x^8} - \dfrac{96u'^2}{u''^2x^8} - \dfrac{288u'}{u''x^8} - \dfrac{288u''}{u'x^8} - \dfrac{96u''^2}{u'^2x^8} - \dfrac{48}{u'^2x^6} - \dfrac{64u'}{u''^3x^6} - \dfrac{48}{u''^2x^6} + \dfrac{132}{u'u''x^6}$ $- \dfrac{64u''}{u'^3x^6} - \dfrac{40}{u'u''^3x^4} + \dfrac{41}{u'^2u''^2x^4} - \dfrac{40}{u'^3u''x^4}$
2	$\dfrac{1808}{x^8} - \dfrac{544u'}{u''x^8} - \dfrac{544u''}{u'x^8} - \dfrac{176}{u'^2x^6} + \dfrac{32u'}{u''^3x^6} - \dfrac{176}{u''^2x^6} + \dfrac{272}{u'u''x^6} + \dfrac{32u''}{u'^3x^6} + \dfrac{24}{u'u''^3x^4} + \dfrac{28}{u'^2u''^2x^4} + \dfrac{24}{u'^3u''x^4}$
3	$-\dfrac{112}{u'^2x^6} + \dfrac{32u'}{u''^3x^6} - \dfrac{16}{u''^2x^6} - \dfrac{640}{u'u''x^6} - \dfrac{8}{u'u''^3x^4} + \dfrac{102}{u'^2u''^2x^4} - \dfrac{80}{u'^3u''x^4}$
4	$-\dfrac{16}{u'^2x^6} - \dfrac{112}{u''^2x^6} - \dfrac{640}{u'u''x^6} + \dfrac{32u''}{u'^3x^6} - \dfrac{80}{u'u''^3x^4} + \dfrac{102}{u'^2u''^2x^4} - \dfrac{8}{u'^3u''x^4}$
5	$\dfrac{1344}{u'x^8} - \dfrac{576u'}{u''^2x^8} + \dfrac{928}{u''x^8} + \dfrac{96u''}{u'^2x^8} + \dfrac{64}{u'^3x^6} - \dfrac{64}{u''^3x^6} + \dfrac{64}{u'u''^2x^6} + \dfrac{168}{u'^2u''x^6} - \dfrac{24}{u'^3u''^2x^4}$
6	$-\dfrac{928}{u'x^8} - \dfrac{96u'}{u''^2x^8} - \dfrac{1344}{u''x^8} + \dfrac{576u''}{u'^2x^8} + \dfrac{64}{u'^3x^6} - \dfrac{64}{u''^3x^6} - \dfrac{168}{u'u''^2x^6} - \dfrac{64}{u'^2u''x^6} + \dfrac{24}{u'^2u''^3x^4}$
7	$-\dfrac{1280}{x^{10}} + \dfrac{1152u'}{u''x^{10}} - \dfrac{704u''}{u'x^{10}} + \dfrac{8}{u'^2x^8} + \dfrac{192}{u''^2x^8} - \dfrac{144}{u'u''x^8} + \dfrac{32}{u'u''^3x^6} - \dfrac{24}{u'^2u''^2x^6} + \dfrac{64}{u'^3u''x^6}$
8	$-\dfrac{1280}{x^{10}} - \dfrac{704u'}{u''x^{10}} + \dfrac{1152u''}{u'x^{10}} + \dfrac{192}{u'^2x^8} + \dfrac{8}{u''^2x^8} - \dfrac{144}{u'u''x^8} + \dfrac{64}{u'u''^3x^6} - \dfrac{24}{u'^2u''^2x^6} + \dfrac{32}{u'^3u''x^6}$
9	$\dfrac{96}{u'^2x^6} + \dfrac{96}{u''^2x^6} - \dfrac{464}{u'u''x^6} + \dfrac{56}{u'u''^3x^4} + \dfrac{8}{u'^2u''^2x^4} + \dfrac{56}{u'^3u''x^4}$
10	$\dfrac{224}{u'x^8} - \dfrac{1024}{u''x^8} + \dfrac{128u''}{u'^2x^8} - \dfrac{64}{u'^3x^6} + \dfrac{64}{u''^3x^6} - \dfrac{360}{u'u''^2x^6} + \dfrac{120}{u'^2u''x^6} - \dfrac{8}{u'^2u''^3x^4}$
11	$\dfrac{1024}{u'x^8} - \dfrac{128u'}{u''^2x^8} - \dfrac{224}{u''x^8} - \dfrac{64}{u'^3x^6} + \dfrac{64}{u''^3x^6} + \dfrac{120}{u'u''^2x^6} + \dfrac{360}{u'^2u''x^6} + \dfrac{8}{u'^3u''^2x^4}$
12	$-\dfrac{6528}{x^{10}} + \dfrac{1408u'}{u''x^{10}} + \dfrac{1408u''}{u'x^{10}} + \dfrac{160}{u'^2x^8} + \dfrac{160}{u''^2x^8} - \dfrac{416}{u'u''x^8} - \dfrac{64}{u'u''^3x^6} - \dfrac{24}{u'^2u''^2x^6} - \dfrac{64}{u'^3u''x^6}$
13	$\dfrac{40}{u'u''^3x^4} + \dfrac{180}{u'^2u''^2x^4} + \dfrac{40}{u'^3u''x^4}$
14	$-\dfrac{32}{u'^3x^6} - \dfrac{640}{u'u''^2x^6} - \dfrac{400}{u'^2u''x^6} - \dfrac{8}{u'^2u''^3x^4} + \dfrac{8}{u'^3u''^2x^4}$
15	$\dfrac{32}{u''^3x^6} + \dfrac{400}{u'u''^2x^6} + \dfrac{640}{u'^2u''x^6} - \dfrac{8}{u'^2u''^3x^4} + \dfrac{8}{u'^3u''^2x^4}$
16	$\dfrac{240}{u'^2x^8} + \dfrac{992}{u'u''x^8} + \dfrac{32}{u'u''^3x^6} + \dfrac{56}{u'^2u''^2x^6}$
17	$\dfrac{240}{u''^2x^8} + \dfrac{992}{u'u''x^8} + \dfrac{56}{u'^2u''^2x^6} + \dfrac{32}{u'^3u''x^6}$
18	$-\dfrac{960}{u'^2x^8} - \dfrac{960}{u''^2x^8} + \dfrac{96}{u'u''x^8} - \dfrac{64}{u'u''^3x^6} + \dfrac{96}{u'^2u''^2x^6} - \dfrac{64}{u'^3u''x^6}$
19	$\dfrac{3200}{u''x^{10}} + \dfrac{576}{u'u''^2x^8} - \dfrac{224}{u'^2u''x^8}$
20	$-\dfrac{3200}{u'x^{10}} + \dfrac{224}{u'u''^2x^8} - \dfrac{576}{u'^2u''x^8}$
21	$\dfrac{5376}{x^{12}}$

TABLE IV. (*Continued*).

(b)

1	$-\dfrac{32}{u'^2 x^6} + \dfrac{16u'}{u''^3 x^6} - \dfrac{32}{u''^2 x^6} + \dfrac{32}{u'u''x^6} + \dfrac{16u''}{u'^3 x^6} + \dfrac{8}{u'u''^3 x^4} + \dfrac{24}{u'^2 u''^2 x^4} + \dfrac{8}{u'^3 u''x^4} + \dfrac{12}{u'^3 u''^3 x^2}$
2	$-\dfrac{384}{x^8} + \dfrac{192u'}{u''x^8} + \dfrac{192u''}{u'x^8} + \dfrac{112}{u'^2 x^6} - \dfrac{32u'}{u''^3 x^6} + \dfrac{112}{u''^2 x^6} - \dfrac{128}{u'u''x^6} - \dfrac{32u''}{u'^3 x^6} - \dfrac{8}{u'u''^3 x^4} - \dfrac{36}{u'^2 u''^2 x^4}$
	$-\dfrac{8}{u'^3 u''x^4} - \dfrac{4}{u'^3 u''^3 x^2}$
3	$\dfrac{32}{u'^2 x^6} + \dfrac{8}{u'u''^3 x^4} - \dfrac{20}{u'^2 u''^2 x^4} + \dfrac{48}{u'^3 u''x^4} - \dfrac{4}{u'^3 u''^3 x^2}$
4	$\dfrac{32}{u''^2 x^6} + \dfrac{48}{u'u''^3 x^4} - \dfrac{20}{u'^2 u''^2 x^4} + \dfrac{8}{u'^3 u''x^4} - \dfrac{4}{u'^3 u''^3 x^2}$
5	$-\dfrac{192}{u'x^8} + \dfrac{192}{u''x^8} - \dfrac{32}{u'^3 x^6} + \dfrac{32}{u''^3 x^6} - \dfrac{16}{u'u''^2 x^6} - \dfrac{16}{u'^2 u''x^6} + \dfrac{24}{u'^3 u''^2 x^4}$
6	$-\dfrac{192}{u'x^8} + \dfrac{192}{u''x^8} + \dfrac{32}{u'^3 x^6} + \dfrac{32}{u''^3 x^6} + \dfrac{16}{u'u''^2 x^6} + \dfrac{16}{u'^2 u''x^6} - \dfrac{24}{u'^2 u''^3 x^4}$
7	$-\dfrac{16}{u'u''^3 x^6} - \dfrac{16}{u'^2 u''^2 x^6} - \dfrac{32}{u'^3 u''x^6}$
8	$-\dfrac{32}{u'u''^3 x^6} - \dfrac{16}{u'^2 u''^2 x^6} - \dfrac{16}{u'^3 u''x^6}$
9	$\dfrac{64}{u'^2 x^6} + \dfrac{64}{u''^2 x^6} - \dfrac{64}{u'u''x^6} - \dfrac{24}{u'u''^3 x^4} - \dfrac{32}{u'^2 u''^2 x^4} - \dfrac{24}{u'^3 u''x^4} + \dfrac{8}{u'^3 u''^3 x^2}$
10	$\dfrac{384}{u'x^8} - \dfrac{192}{u''x^8} - \dfrac{192u''}{u'^2 x^8} + \dfrac{32}{u'^3 x^6} - \dfrac{64}{u''^3 x^6} + \dfrac{64}{u'u''^2 x^6} + \dfrac{8}{u'^2 u''^3 x^4}$
11	$\dfrac{192}{u'x^8} + \dfrac{192u'}{u''^2 x^8} - \dfrac{384}{u''x^8} + \dfrac{64}{u'^3 x^6} - \dfrac{32}{u''^3 x^6} - \dfrac{64}{u'^2 u''x^6} - \dfrac{8}{u'^3 u''^2 x^4}$
12	$\dfrac{3072}{x^{10}} - \dfrac{1536u'}{u''x^{10}} - \dfrac{1536u''}{u'x^{10}} - \dfrac{288}{u'^2 x^8} - \dfrac{288}{u''^2 x^8} + \dfrac{384}{u'u''x^8} + \dfrac{32}{u'u''^3 x^6} + \dfrac{80}{u'^2 u''^2 x^6} + \dfrac{32}{u'^3 u''x^6}$
13	$-\dfrac{16}{u'u''^3 x^4} + \dfrac{80}{u'^2 u''^2 x^4} - \dfrac{16}{u'^3 u''x^4}$
14	$\dfrac{128}{u'u''^2 x^6} - \dfrac{192}{u'^2 u''x^6} + \dfrac{8}{u'^2 u''^3 x^4} - \dfrac{8}{u'^3 u''^2 x^4}$
15	$\dfrac{192}{u'u''^2 x^6} - \dfrac{128}{u'^2 u''x^6} + \dfrac{8}{u'^2 u''^3 x^4} - \dfrac{8}{u'^3 u''^2 x^4}$
16	$\dfrac{192}{u'^2 x^8} - \dfrac{384}{u'u''x^8} - \dfrac{32}{u'u''^3 x^6} - \dfrac{16}{u'^2 u''^2 x^6}$
17	$\dfrac{192}{u''^2 x^8} - \dfrac{384}{u'u''x^8} - \dfrac{16}{u'^2 u''^2 x^6} - \dfrac{32}{u'^3 u''x^6}$
18	$\dfrac{192}{u'^2 x^8} + \dfrac{192}{u''^2 x^8} - \dfrac{1152}{u'u''x^8} + \dfrac{32}{u'u''^3 x^6} + \dfrac{32}{u'^3 u''x^6}$
19	$\dfrac{1536}{u'x^{10}} - \dfrac{1536}{u''x^{10}} - \dfrac{288}{u'u''^2 x^8} + \dfrac{96}{u'^2 u''x^8}$
20	$\dfrac{1536}{u'x^{10}} - \dfrac{1536}{u''x^{10}} - \dfrac{96}{u'u''^2 x^8} + \dfrac{288}{u'^2 u''x^8}$
21	0

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{2}\kappa\phi_{\mu\nu} \qquad (3.7)$$

and he used a general gauge-fixing term of the form

$$\mathcal{L}_B = -(\alpha\phi^{\mu\rho}{}_{,\rho} + \beta\phi^{,\mu})\eta_{\mu\nu}(\alpha\phi^{\nu\sigma}{}_{,\sigma} + \beta\phi^{,\nu}). \quad (3.8)$$

Comparison with our gauge-fixing term (2.14) implies that our flat space limit should agree with his result for $\alpha = 1$ and $\beta = -\frac{1}{2}$. These are certainly the values for which his graviton and ghost propagators agree with the flat space limits of Eqs. (2.19a) and (2.19b), respectively.

Of course Capper worked in momentum space, using dimensional regularization in D spacetime dimensions to define the divergent loop integral. To compare with Eq. (3.6) we first Fourier transform to position space and then evaluate the result—which is well defined for $x^2 \neq 0$—at $D=4$:

$$[^{\alpha\beta}\Sigma^{\rho\sigma}_{\text{Capper}}](x';x'') = \lim_{D\to 4} \frac{i}{(2\pi)^D} \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x} T^{\alpha\beta\rho\sigma}(p^2).$$
(3.9)

Cappers's result for $T^{\alpha\beta\rho\sigma}(p^2)$ has the form

$$T^{\alpha\beta\rho\sigma}(p^2) = \kappa^2 \{ T_1(D) p^\alpha p^\beta p^\rho p^\sigma + T_2(D) \eta^{\alpha\beta}\eta^{\rho\sigma} p^4$$
$$+ 2 T_3(D) \eta^{\alpha(\rho}\eta^{\sigma)\beta} p^4 + T_4(D)$$
$$\times [\eta^{\alpha\beta} p^\rho p^\sigma + p^\alpha p^\beta \eta^{\rho\sigma}] p^2$$
$$+ 4 T_5(D) p^{(\alpha}\eta^{\beta)\,(\rho} p^{\sigma)} p^2 \} I(p^2).$$
(3.10)

Although Capper's published paper [7] quotes only the pole terms, he was kind enough to communicate the general result to us some time ago [15]. The coefficient functions are

$$T_1(D) = \frac{9}{16} D^4 - \frac{21}{16} D^3 - \frac{9}{8} D^2,$$
(3.11a)

$$T_2(D) = \frac{1}{D-2} \left(\frac{9}{16} D^5 - \frac{39}{16} D^4 - \frac{25}{8} D^3 \right.$$
$$\left. + \frac{123}{8} D^2 + \frac{33}{4} D - 8 \right),$$
(3.11b)

$$T_3(D) = - T_5(D) = \frac{1}{D-2} \left(\frac{1}{4} D^4 + \frac{17}{16} D^3 \right.$$
$$\left. - \frac{97}{16} D^2 - \frac{17}{8} D + 4 \right),$$
(3.11c)

$$T_4(D) = \frac{1}{D-2} \left(-\frac{9}{16} D^5 + \frac{43}{16} D^4 + \frac{15}{8} D^3 \right.$$
$$\left. - \frac{119}{8} D^2 - \frac{25}{4} D + 8 \right),$$
(3.11d)

and Capper defines his ''basic integral'' as

$$I(p^2) \equiv \frac{1}{4(D^2-1)} \int \frac{d^D k}{[k^2 - i\epsilon][(p-k)^2 - i\epsilon]}.$$
(3.12)

The familiar relation

$$\frac{1}{k^2 - i\epsilon} = \frac{i\Gamma(D-2/2)}{4\pi^{D/2}} \int d^D y \frac{e^{-ik\cdot y}}{[y^2 + i\epsilon]^{D/2-1}}$$
(3.13)

allows us to reexpress Capper's basic integral in the form

$$I(p^2) = - \frac{\left[\Gamma\left(\dfrac{D-2}{2} \right) \right]^2}{2^{6-D}(D^2-1)} \int d^D y \frac{e^{-ip\cdot y}}{[y^2 + i\epsilon]^{D-2}}.$$
(3.14)

Substitution into Eq. (3.9) reveals complete agreement with our answer (3.6):

$$[^{\alpha\beta}\Sigma^{\rho\sigma}_{\text{Capper}}](x';x'') = \lim_{D\to 4} \frac{-i\kappa^2 \left[\Gamma\left(\dfrac{D-2}{2} \right) \right]^2}{2^6 \pi^D (D^2-1)} \{ T_1(D) \partial^\alpha \partial^\beta \partial^\rho \partial^\sigma + T_2(D) \eta^{\alpha\beta}\eta^{\rho\sigma} \partial^4 + 2 T_3(D) \eta^{\alpha(\rho}\eta^{\sigma)\beta} \partial^4$$

$$+ T_4(D) [\eta^{\alpha\beta}\partial^\rho\partial^\sigma + \partial^\alpha\partial^\beta\eta^{\rho\sigma}]\partial^2 + 4 T_5(D) \partial^{(\alpha}\eta^{\beta)\,(\rho}\partial^{\sigma)}\partial^2 \} \left(\frac{1}{x^2 + i\epsilon} \right)^{D-2}$$
(3.15a)

$$= \frac{-i\kappa^2}{60(2\pi)^4} \left\{ 42\partial^\alpha\partial^\beta\partial^\rho\partial^\sigma + \frac{23}{2} \eta^{\alpha\beta}\eta^{\rho\sigma}\partial^4 + \frac{61}{2} \eta^{\alpha(\rho}\eta^{\sigma)\beta}\partial^4 - \frac{23}{2} [\eta^{\alpha\beta}\partial^\rho\partial^\sigma + \partial^\alpha\partial^\beta\eta^{\rho\sigma}]\partial^2 \right.$$

$$\left. - 61\partial^{(\alpha}\eta^{\beta)\,(\rho}\partial^{\sigma)}\partial^2 \right\} \left(\frac{1}{x^2 + i\epsilon} \right)^2.$$
(3.15b)

IV. THE WARD IDENTITY

The result of the previous section is reassuring because our reduction procedure is the same for all terms. By checking the flat space limit we have, therefore, partially checked even terms which vanish in this limit. However, it is conceivable that an error might also vanish in the flat space limit, and it is important to note that the dominant infrared terms which are of greatest interest to us also go to zero in this limit. So an independent check which is intrinsic to the curved background would be highly desirable. Such a check is provided by the Ward identity. Since this is a consequence

of the theory's gauge invariance—as reflected in the Becchi-Rouet-Stora (BRS) symmetry of the gauge fixed action—it can be checked on the curved background without taking the flat space limit.

The gauge-fixed action is invariant under the BRS transformation

$$\delta_{\text{BRS}}\psi_{\mu\nu} = \left[2\tilde{g}_{\rho(\mu}\partial_{\nu)} + \tilde{g}_{\mu\nu,\rho} + \frac{2}{u} \tilde{g}_{\mu\nu} t_\rho \right] \omega^\rho \delta\zeta,$$
(4.1a)

$$\delta_{\text{BRS}}\bar{\omega}^\mu = - \Omega^{-1} F^\mu \delta\zeta,$$
(4.1b)

$$\delta_{\mathrm{BRS}}\omega_\mu = \kappa\omega_{\mu,\nu}\omega^\nu\delta\zeta, \tag{4.1c}$$

where $\delta\zeta$ is an anticommuting $\mathbb{C}$-number constant. An important consequence is that the BRS transformation of the gauge-fixing function F_μ is proportional to the antighost equation of motion:

$$\delta_{\mathrm{BRS}}F_\mu(x) = \Omega^{-1}\frac{\delta S_{\mathrm{GF}}}{\delta\overline{\omega}^\mu(x)}\,\delta\zeta. \tag{4.2}$$

The various Slavnov-Taylor identities follow from the BRS invariance of the functional formalism. One makes a change of variables that is a BRS transformation and then collects the variation terms. Since a functional integral is independent of the dummy variable of integration, the sum of the variations must vanish. Since the action and the measure factor are BRS invariant, the variations derive entirely from the operator whose in-out matrix element is being computed.

We do not want to compute another Green's function so we seek an operator whose BRS variation involves only the pseudograviton two-point function. The desired object turns out to be the product of the conformally rescaled antighost field and the gauge-fixing function. Its BRS variation is

$$\delta_{\mathrm{BRS}}[\Omega(x')\overline{\omega}_\mu(x')F_\nu(x'')]$$

$$= -\bigg[F_\mu(x')F_\nu(x'') + \Omega(x')\overline{\omega}_\mu(x')$$

$$\times\,\Omega^{-1}(x'')\frac{\delta S_{\mathrm{GF}}}{\delta\overline{\omega}^\nu(x'')}\bigg]\delta\zeta. \tag{4.3}$$

The functional integral of the second term is a δ function:

$$\int[d\psi][d\overline{\omega}][d\omega]\overline{\omega}_\mu(x')\frac{\delta S_{\mathrm{GF}}}{\delta\overline{\omega}^\nu(x'')}\exp[iS_{\mathrm{GF}}[\psi,\overline{\omega},\omega]]$$

$$= -i\int[d\psi][d\overline{\omega}][d\omega]\overline{\omega}_\mu(x')\frac{\delta}{\delta\overline{\omega}^\nu(x'')}$$

$$\times\exp[iS_{\mathrm{GF}}[\psi,\overline{\omega},\omega]] \tag{4.4a}$$

$$= -i\eta_{\mu\nu}\delta^4(x'-x''). \tag{4.4b}$$

If we write the gauge-fixing function as an operator acting on the pseudograviton field,

$$F_\mu(x) = \Omega(x)\bigg[\delta_\mu^{\ p}\partial^\sigma - \frac{1}{2}\partial_\mu\eta^{\rho\sigma} + \frac{2}{u}\delta_u^{\ p}t^\sigma\bigg]\psi_{\rho\sigma}(x) \tag{4.5a}$$

$$\equiv \mathcal{F}_\mu^{\ \rho\sigma}(x)\psi_{\rho\sigma}(x), \tag{4.5b}$$

then the Ward identity can be expressed as

$$\mathcal{F}_\mu^{\ \alpha\beta}(x')\mathcal{F}_\nu^{\ \rho\sigma}(x'')\langle\mathrm{out}|T[\psi_{\alpha\beta}(x')\psi_{\rho\sigma}(x'')]|\mathrm{in}\rangle$$

$$= i\eta_{\mu\nu}\delta^4(x'-x''). \tag{4.6}$$

One of the first things we did after obtaining the pseudograviton and ghost propagators was to verify a somewhat more general version of Eq. (4.6) at tree order [11]. Work at higher orders is embarrassed by the severe infrared divergences of the in-out formalism, which typically prevent ma-

trix elements from existing if they involve even one integration over an interaction vertex [8]. Of course the one loop correction to the in-out two-point function is just the double integral of the one loop self-energy against two external propagators. If we can amputate the external propagators, the one loop self-energy which remains will suffer no infrared divergences because it involves no integrated interaction vertices. However, we cannot amputate the external propagators in Eq. (4.6) so simply on account of the gauge-fixing operators $\mathcal{F}_\mu^{\rho\sigma}$. These do not commute with the pseudograviton kinetic operator—they don't even possess the requisite number of free indices to do so. What we must do instead is to reflect the gauge fixing operators through the external propagators and then amputate the different *scalar* propagators which reside on distinct tensor factors.

Two scalar propagators are of interest. The first is that of a massless, minimally coupled scalar:

$$i\Delta_A(x';x'') = \frac{H^2}{8\pi^2}\bigg\{\frac{2u'u''}{x^2+i\epsilon} - \ln[H^2(x^2+i\epsilon)]\bigg\}. \tag{4.7a}$$

It corresponds to the A modes of Eq. (2.17a), which can harbor physical graviton polarizations. The other scalar propagator is that of a massless, conformally coupled scalar:

$$i\Delta_B(x';x'') = \frac{H^2}{8\pi^2}\frac{2u'u''}{x^2+i\epsilon}. \tag{4.7b}$$

It corresponds to the B and C modes of Eq. (2.17b), which represent constrained and pure gauge degrees or freedom.[4] The corresponding scalar kinetic operators are, respectively,

$$D_A = \Omega^2\bigg(\partial^2 + \frac{2}{u}\partial_0\bigg), \tag{4.8a}$$

$$D_B = \Omega^2\partial^2. \tag{4.8b}$$

We can express the pseudograviton propagator as simple tensor factors times the two scalar propagators [9]:

$$i[_{\alpha\beta}\Delta_{\rho\sigma}](x';x'') = i\Delta_A(x';x'')[_{\alpha\beta}T_{\rho\sigma}^A] + i\Delta_B(x';x'')$$

$$\times\{[_{\alpha\beta}T_{\rho\sigma}^B] + [_{\alpha\beta}T_{\rho\sigma}^C]\}, \tag{4.9a}$$

$$[_{\alpha\beta}T_{\rho\sigma}^A] = 2\,\overline{\eta}_{\alpha(\rho}\overline{\eta}_{\sigma)\beta} - 2\,\overline{\eta}_{\alpha\beta}\overline{\eta}_{\rho\sigma}, \tag{4.9b}$$

$$[_{\alpha\beta}T_{\rho\sigma}^B] = -4t_{(\alpha}\overline{\eta}_{\beta)(\rho}t_{\sigma)}, \tag{4.9c}$$

$$[_{\alpha\beta}T_{\rho\sigma}^C] = (t_\alpha t_\beta + \overline{\eta}_{\alpha\beta})(t_\rho t_\sigma + \overline{\eta}_{\rho\sigma}), \tag{4.9d}$$

where we remind the reader that $t_\mu \equiv \eta_{\mu 0}$ and $\overline{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + t_\mu t_\nu$.

The tensor t_μ and our barring convention permit us to express a free derivative as purely spatial and temporal derivatives:

---

[4]A modes can also be unphysical. In 3+1 dimensions there are six A modes, of which only two are physical, three B modes, and a single C mode. The distinction between B and C propagators becomes apparent in higher dimensions [9,11].

$$\partial_\mu = \overline{\partial}_\mu - t_\mu \partial_0. \qquad (4.10)$$

Spatial derivatives reflect through the scalar propagators the same way they do in flat space:

$$\overline{\partial}'_\mu i\Delta_A(x';x'') = -\overline{\partial}''_\mu i\Delta_A(x';x''), \qquad (4.11a)$$

$$\overline{\partial}'_\mu i\Delta_B(x';x'') = -\overline{\partial}''_\mu i\Delta_B(x';x''). \qquad (4.11b)$$

The reflection identities for temporal derivatives follow from the mode expansions of the various propagators [11]:

$$\partial'_0 i\Delta_A(x';x'') = -\left(\partial''_0 - \frac{2}{u''}\right)i\Delta_B(x';x''), \qquad (4.12a)$$

$$\left(\partial'_0 - \frac{2}{u'}\right)i\Delta_B(x';x'') = -\partial''_0 i\Delta_A(x';x''), \qquad (4.12b)$$

$$\left(\partial'_0 - \frac{1}{u'}\right)i\Delta_B(x';x'') = -\left(\partial''_0 - \frac{1}{u''}\right)i\Delta_B(x';x''). \qquad (4.12c)$$

We reflect the gauge-fixing operator through an external propagator by first contracting into the tensor factors and then exploiting the scalar reflection identities:

$$\Omega'^{-1}\mathcal{F}^{\alpha\beta}_\mu(x')i[_{\alpha\beta}\Delta_{\rho\sigma}](x';x'') = 2[\overline{\eta}_{\mu(\rho}\overline{\partial}'_{\sigma)} - t_\mu\overline{\eta}_{\rho\sigma}\partial'_0]i\Delta_A(x';x'') + 2\left[-t_\mu t_{(\rho}\overline{\partial}'_{\sigma)} - \overline{\eta}_{\mu(\rho}t_{\sigma)}\left(\partial'_0 - \frac{2}{u'}\right)\right.$$

$$\left. + 2t_\mu(t_\rho t_\sigma + \overline{\eta}_{\rho\sigma})\left(\partial'_0 - \frac{1}{u'}\right)\right]i\Delta_B(x';x'') \qquad (4.13a)$$

$$= 2[-\overline{\eta}_{\mu(\rho}\overline{\partial}''_{\sigma)} + \overline{\eta}_{\mu(\rho}t_{\sigma)}\partial''_0]i\Delta_A(x';x'') + 2t_\mu\left[t_{(\rho}\overline{\partial}''_{\sigma)} + \overline{\eta}_{\rho\sigma}\left(\partial''_0 - \frac{2}{u''}\right)\right.$$

$$\left. - (t_\rho t_\sigma + \overline{\eta}_{\rho\sigma})\left(\partial''_0 - \frac{1}{u''}\right)\right]i\Delta_B(x';x'') \qquad (4.13b)$$

$$= -2\overline{\eta}_{\mu(\rho}\partial''_{\sigma)}i\Delta_A(x';x'') + 2t_\mu\left[t_{(\rho}\partial''_{\sigma)} - \frac{1}{u''}\eta_{\rho\sigma}\right]i\Delta_B(x';x''). \qquad (4.13c)$$

We can therefore amputate the external propagator by acting the scalar operator D_A when the free index μ is spatial, and by acting D_B when μ is temporal.

The preceding analysis allows us to pass from Eq. (4.6) to the following identity on the one loop self-energy:

$$\left(\eta_{\mu\alpha}\partial'_\beta - t_\mu\eta_{\alpha\beta}\frac{1}{u'}\right)\left(\eta_{\nu\rho}\partial''_\sigma - t_\nu\eta_{\rho\sigma}\frac{1}{u''}\right)[^{\alpha\beta}\Sigma^{\rho\sigma}](x';x'')$$

$$= 0. \qquad (4.14)$$

Though our derivation was carried out in $3+1$ dimensions it is worth remarking that the result is valid for any dimension. To check it in $3+1$ dimensions we first contract and commute the two tensor-differential operators through the 21 tensors of Table III. The result is a linear combination of five two-index tensors,

$$\eta_{\mu\nu}, \quad t_\mu t_\nu, \quad t_\mu x_\nu, \quad x_\mu t_\nu, \quad x_\mu x_\nu, \qquad (4.15)$$

times various scalar differential operators which act on the appropriate scalar functions of Tables IV(a) and IV(b). Since the coefficient of each tensor must vanish separately we obtain five identities, although those for $t_\mu x_\nu$ and $x_\mu t_\nu$ follow from each other by the diagram's invariance under interchange of the external legs. The various differential operators have been tabulated at the end of this section, according

to the number of the coefficient function upon which they act. Needless to say, the identity is obeyed.

It is worth working out an example of the process through which the two tensor differential operators in Eq. (4.14) are contracted and commuted through the various tensor factors which make up the self-energy. We have chosen for this purpose tensor No. 21 on Table III. Since $x^\mu = (x' - x'')^\mu$ we easily commute derivatives through the tensor factor:

$$\left(\eta_{\mu\alpha}\partial'_\beta - t_\mu\eta_{\alpha\beta}\frac{1}{u'}\right)\left(\eta_{\nu\rho}\partial''_\sigma - t_\nu\eta_{\rho\sigma}\frac{1}{u''}\right)x^\alpha x^\beta x^\rho x^\sigma$$

$$= \partial'_\rho\partial''_\sigma x_\mu x_\nu x^\rho x^\sigma - \frac{1}{u'}\partial''_\rho t_\mu x_\nu x^\rho x^2 - \frac{1}{u''}\partial'_\rho x_\mu t_\nu x^\rho x^2$$

$$- \frac{1}{u'u''}t_\mu t_\nu x^4 \qquad (4.16a)$$

$$= x_\mu x_\nu[x^\rho x^\sigma\partial'_\rho\partial''_\sigma - 7x^\rho\partial'_\rho + 7x^\sigma\partial''_\sigma - 42]$$

$$+ \frac{t_\mu x_\nu}{u'}[-x^2 x^\rho\partial''_\rho + 7x^2] + \frac{x_\mu t_\nu}{u''}[-x^2 x^\rho\partial'_\rho - 7x^2]$$

$$+ t_\mu t_\nu\frac{x^4}{u'u''}. \qquad (4.16b)$$

The coefficient of $t_\mu t_\nu$ at the end of Eq. (4.16b) is already recognizable as No. 21 on Table V(b), and the coefficient of $\eta^{\mu\nu}$ is clearly zero, in agreement with No. 21 on Table V(a). For the rest we recall that the derivatives can be simplified when the functions they act upon depend upon x'^μ and x''^μ only through x^μ, u', and u'':

$$\partial'_\mu \rightarrow \partial_\mu - t_\mu \frac{\partial}{\partial u'}, \qquad (4.17a)$$

$$\partial''_\mu \rightarrow -\partial_\mu - t_\mu \frac{\partial}{\partial u''}. \qquad (4.17b)$$

Since the coefficient functions of Tables IV(a) and IV(b) actually depend upon x^μ only through x^2 a further reduction can eventually be made:

$$\partial_\mu \rightarrow 2x_\mu \frac{\partial}{\partial x^2}. \qquad (4.18)$$

The coefficients of $t_\mu x_\nu$ and $x_\mu t_\nu$ in Eq. (4.16b) are, therefore,

$$\frac{1}{u'}\left[-x^2 x^\rho \partial''_\rho + 7x^2\right] = \frac{1}{u'}\left[2x^4 \frac{\partial}{\partial x^2} + (t\cdot x)x^2 \frac{\partial}{\partial u''} + 7x^2\right], \qquad (4.19a)$$

$$\frac{1}{u''}\left[-x^2 x^\rho \partial'_\rho - 7x^2\right] = \frac{1}{u''}\left[-2x^4 \frac{\partial}{\partial x^2} + (t\cdot x)x^2 \frac{\partial}{\partial u'} - 7x^2\right]. \qquad (4.19b)$$

Upon noting that $t\cdot x = u'' - u'$, we recognize Eqs. (4.19a) and (4.19b) as No. 21 in Tables V(c) and V(d) respectively. We can also write

$$x^\rho x^\sigma \partial'_\rho \partial''_\sigma = x^\rho x^\sigma \left[\partial_\rho - t_\rho \frac{\partial}{\partial u'}\right]\left[-2x_\sigma \frac{\partial}{\partial x^2} - t_\sigma \frac{\partial}{\partial u''}\right] \qquad (4.20a)$$

$$= -4x^4 \frac{\partial^2}{\partial x^4} - 2x^2 \frac{\partial}{\partial x^2} - 2(t\cdot x)\left(\frac{\partial}{\partial u''}\right)$$

$$- \frac{\partial}{\partial u'}\right)x^2 \frac{\partial}{\partial x^2} + (t\cdot x)^2 \frac{\partial}{\partial u'}\frac{\partial}{\partial u''}. \qquad (4.20b)$$

Combining with the other terms in the coefficient of $x_\mu x_\nu$ in Eq. (4.16b) then results in No. 21 of Table V(e).

Note that Capper's flat space Ward identity [7]

$$\eta_{\mu(\alpha}\partial'_{\beta)}\eta_{\nu(\rho}\partial''_{\sigma)}[^{\alpha\beta}\Sigma^{\rho\sigma}_{\text{Capper}}](x';x'') = 0 \qquad (4.21)$$

involves only $\eta_{\mu\nu}$ and $x_\mu x_\nu$. Since the associated scalar functions are just x^2 to the power fixed by dimensional analysis, Capper gets only two scalar relations, each of which is independent of x:

$$T_1(D) + T_2(D) + T_3(D) + 2T_4(D) + 3T_5(D) = 0, \qquad (4.22a)$$

$$T_3(D) + T_5(D) = 0. \qquad (4.22b)$$

[See Eqs. (3.10) and (3.11) for the $T_i(D)$.] The presence of t^μ gives us two more distinct tensors, and the four scalar relations which result can involve a bewildering number of distinct products of x^2, u', u'', and $\ln[H^2 x^2]$.

V. DISCUSSION

We have computed one loop graviton self-energy in a locally de Sitter background and using the integral approximations (2.19a) and (2.19b) for the pseudograviton and ghost propagators. The result consists of 21 independent tensors, given in Table III, times the coefficient functions of Tables IV(a) and IV(b). Not the least of our conclusions is that the result is almost certainly correct, within the integral approximation. It is difficult to doubt this as one witnesses the cancellation of one after another of the hundreds of distinct functional and tensor terms in the Ward identity. Agreement with Capper's flat space result [7] shows that we have even got the sign and the normalization right.

A subtle and interesting point is that the forms (2.19a) and (2.19b) used for the pseudograviton and ghost propagators are only approximations to the exact mode expansions which one obtains on $T^3 \times \mathcal{R}$ [8]. The integral approximations become exact in the flat space limit, so it is obvious why the flat space limit of our result should agree with Capper's work. The fascinating thing is that the Ward identity is also obeyed, exactly and without any need for taking the flat space limit. We saw this as well when checking a somewhat more general Ward identity at tree order [11]. The reason for it seems to be that the integral approximations do invert the pseudograviton and ghost kinetic operators, so they differ from the true propagators only by real, analytic terms which depend upon the choice of vacuum. There can only be mixing with these vacuum dependent terms if the Ward identity involves integrations which can reach the initial or final states. But the one loop identity we checked involves no integrations at all.

Although the one loop self-energy is worthy of study in its own right, our interest derives from its role as an important constituent in the two loop tadpole from which we have lately inferred the quantum gravitational back reaction on inflation [4]. Of course the one loop self-energy cannot completely verify the two loop tadpole, but it does establish the correctness of certain features of the basic formalism. For example, our gauge fixing procedure is shown to be consistent, and the ψ^3 and the $\psi\overline{\omega}\omega$ vertex operators are checked. The result also demonstrates the validity of our tensor contraction routines, and the procedures whereby derivative operators from the vertices are acted on propagators. Since the same vertices and reduction procedures were used throughout the two loop work, many features of the larger calculation are checked as well. For example, the diagrams in Fig. 1(a) and 1(b) are obtained by contracting the one loop self-energy, through two pseudograviton propagators, into the ψ^3 vertex operator, as illustrated in Fig. 2. No new vertices or propagators appear, and the same procedures were used to perform the contractions and to act the derivatives. These comments apply as well to the outer ghost loop of Fig. 1(c) and to everything but the four-point vertex operator in the 4-3 diagram of Fig. 1(d). One point the current work does not check is the procedures for integrating over free interaction vertices. Of course the integrals have been checked extensively in other ways [4], but not by the one loop self-energy.

TABLE V. Act each operator on the corresponding coefficient function in Tables IV(a) and IV(b), and then sum the results to check the Ward identity. (a) Operator coefficients of $\eta_{\mu\nu}$. (b) Operator coefficients of $t_\mu t_\nu$. (c) Operator coefficients of $t_\mu x_\nu$. (d) Operator coefficients of $x_\mu t_\nu$. (e) Operator coefficients of $x_\mu x_\nu$.

No.	Operator
	(a)
1	$-2\dfrac{\partial}{\partial x^2}$
2	$-\dfrac{1}{2}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-5\dfrac{\partial}{\partial x^2}-(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u''}\right)\dfrac{\partial}{\partial x^2}-2x^2\dfrac{\partial^2}{\partial x^4}$
3	0
4	0
5	$\dfrac{1}{2}\dfrac{\partial}{\partial u''}-(u''-u')\dfrac{\partial}{\partial x^2}$
6	$-\dfrac{1}{2}\dfrac{\partial}{\partial u'}-(u''-u')\dfrac{\partial}{\partial x^2}$
7	$-5-(u''-u')\dfrac{\partial}{\partial u''}-2x^2\dfrac{\partial}{\partial x^2}$
8	$-5+(u''-u')\dfrac{\partial}{\partial u'}-2x^2\dfrac{\partial}{\partial x^2}$
9	$\dfrac{1}{4}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+\dfrac{1}{2}\dfrac{\partial}{\partial x^2}+\dfrac{1}{2}(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)\dfrac{\partial}{\partial x^2}-(u''-u')^2\dfrac{\partial^2}{\partial x^4}$
10	$-\dfrac{5}{4}\dfrac{\partial}{\partial u'}+\dfrac{1}{4}\dfrac{\partial}{\partial u''}-\dfrac{1}{4}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{7}{2}(u''-u')\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}(u''-u')^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}x^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-(u''-u')x^2\dfrac{\partial^2}{\partial x^4}$
11	$-\dfrac{1}{4}\dfrac{\partial}{\partial u'}+\dfrac{5}{4}\dfrac{\partial}{\partial u''}-\dfrac{1}{4}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{7}{2}(u''-u')\dfrac{\partial}{\partial x^2}+\dfrac{1}{2}(u''-u')^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+\dfrac{1}{2}x^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-(u''-u')x^2\dfrac{\partial^2}{\partial x^4}$
12	$-8-\dfrac{3}{2}(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)+\dfrac{1}{4}(u''-u')^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{13}{2}x^2\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)x^2\dfrac{\partial}{\partial x^2}-x^4\dfrac{\partial^2}{\partial x^4}$
13–21	0
	(b)
1	$\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+\dfrac{4}{u'}\dfrac{\partial}{\partial u''}+\dfrac{4}{u''}\dfrac{\partial}{\partial u''}+\dfrac{16}{u'u''}$
2	$\dfrac{1}{2}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+\dfrac{1}{u'}\dfrac{\partial}{\partial u''}+\dfrac{1}{u''}\dfrac{\partial}{\partial u'}+\dfrac{4}{u'u''}$
3	$-\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-10\dfrac{\partial}{\partial x^2}-\dfrac{4}{u'}\dfrac{\partial}{\partial u''}+2(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-\dfrac{1}{u''}\dfrac{\partial}{\partial u'}+\dfrac{4}{u'u''}+8\dfrac{u''}{u'}\dfrac{\partial}{\partial x^2}$
4	$-\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-10\dfrac{\partial}{\partial x^2}-\dfrac{1}{u'}\dfrac{\partial}{\partial u''}-2(u''-u')\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-\dfrac{4}{u''}\dfrac{\partial}{\partial u'}-\dfrac{4}{u'u''}+8\dfrac{u'}{u''}\dfrac{\partial}{\partial x^2}$
5	$\dfrac{7}{2}\dfrac{\partial}{\partial u'}-\dfrac{5}{2}\dfrac{\partial}{\partial u''}+\dfrac{14}{u'}+\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{5}{u''}-\dfrac{u'}{u''}\dfrac{\partial}{\partial u'}+2\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}+x^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+4\dfrac{x^2}{u'}\dfrac{\partial}{\partial x^2}$
6	$\dfrac{5}{2}\dfrac{\partial}{\partial u'}-\dfrac{7}{2}\dfrac{\partial}{\partial u''}+\dfrac{5}{u'}+\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{14}{u''}-2\dfrac{u'}{u''}\dfrac{\partial}{\partial u'}+\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}-x^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-4\dfrac{x^2}{u''}\dfrac{\partial}{\partial x^2}$
7	$\dfrac{x^2}{u''}\dfrac{\partial}{\partial u'}+4\dfrac{x^2}{u'u''}$
8	$\dfrac{x^2}{u'}\dfrac{\partial}{\partial u''}+4\dfrac{x^2}{u'u''}$
9	$-\dfrac{3}{4}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-7\dfrac{\partial}{\partial x^2}-\dfrac{1}{u'}\dfrac{\partial}{\partial u''}-(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)\dfrac{\partial}{\partial x^2}-\dfrac{1}{u''}\dfrac{\partial}{\partial u'}-\dfrac{1}{u'u''}+2\dfrac{u'}{u''}\dfrac{\partial}{\partial x^2}+2\dfrac{u''}{u'}\dfrac{\partial}{\partial x^2}-x^2\dfrac{\partial^2}{\partial x^4}$
10	$\dfrac{7}{4}\dfrac{\partial}{\partial u'}-\dfrac{5}{4}\dfrac{\partial}{\partial u''}+\dfrac{7}{2u'}+\dfrac{1}{4}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{7}{2u''}-\dfrac{1}{2}\dfrac{u'}{u''}\dfrac{\partial}{\partial u'}+\dfrac{1}{2}\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}+\dfrac{1}{2}x^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+\dfrac{x^2}{u'}\dfrac{\partial}{\partial x^2}-\dfrac{x^2}{u''}\dfrac{\partial}{\partial x^2}$

TABLE V. (*Continued*).

No.	Operator
11	$\dfrac{5}{4}\dfrac{\partial}{\partial u'}-\dfrac{7}{4}\dfrac{\partial}{\partial u''}+\dfrac{7}{2u'}+\dfrac{1}{4}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{7}{2u''}-\dfrac{1}{2}\dfrac{u'}{u''}\dfrac{\partial}{\partial u'}+\dfrac{1}{2}\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}-\dfrac{1}{2}x^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}+\dfrac{x^2}{u'}\dfrac{\partial}{\partial x^2}-\dfrac{x^2}{u''}\dfrac{\partial}{\partial x^2}$
12	$\dfrac{x^2}{u'u''}$
13	$\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+6\dfrac{\partial}{\partial x^2}+\dfrac{1}{u'}\dfrac{\partial}{\partial u''}+2(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)\dfrac{\partial}{\partial x^2}-4(u''-u')^2\dfrac{\partial^2}{\partial x^4}+\dfrac{1}{u''}\dfrac{\partial}{\partial u'}+\dfrac{1}{u'u''}-2\dfrac{u'}{u''}\dfrac{\partial}{\partial x^2}-2\dfrac{u''}{u'}\dfrac{\partial}{\partial x^2}$
14	$-\dfrac{7}{2}\dfrac{\partial}{\partial u'}+\dfrac{3}{2}\dfrac{\partial}{\partial u''}-\dfrac{7}{2u'}-\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial u''}+12u'\dfrac{\partial}{\partial x^2}-(u''-u')^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}+\dfrac{2}{u''}+\dfrac{u'}{u''}\dfrac{\partial}{\partial u'}-2\dfrac{u'^2}{u''}\dfrac{\partial}{\partial x^2}-10u''\dfrac{\partial}{\partial x^2}$ $-\dfrac{1}{2}\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}-x^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-\dfrac{x^2}{u'}\dfrac{\partial}{\partial x^2}-2(u''-u')x^2\dfrac{\partial^2}{\partial x^4}$
15	$-\dfrac{3}{2}\dfrac{\partial}{\partial u'}+\dfrac{7}{2}\dfrac{\partial}{\partial u''}-\dfrac{2}{u'}-\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+10u'\dfrac{\partial}{\partial x^2}+(u''-u')^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+\dfrac{7}{2u''}+\dfrac{1}{2}\dfrac{u'}{u''}\dfrac{\partial}{\partial u'}+12u''\dfrac{\partial}{\partial x^2}-\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}$ $+2\dfrac{u''^2}{u'}\dfrac{\partial}{\partial x^2}+x^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-2(u''-u')x^2\dfrac{\partial^2}{\partial x^4}+\dfrac{x^2}{u''}\dfrac{\partial}{\partial x^2}$
16	$-7-(u''-u')\dfrac{\partial}{\partial u''}+2\dfrac{u'}{u''}-4x^2\dfrac{\partial}{\partial x^2}-\dfrac{x^2}{u''}\dfrac{\partial}{\partial u'}-\dfrac{x^2}{u'u''}+2\dfrac{u'}{u''}x^2\dfrac{\partial}{\partial x^2}$
17	$-7+(u''-u')\dfrac{\partial}{\partial u'}+2\dfrac{u''}{u'}-4x^2\dfrac{\partial}{\partial x^2}-\dfrac{x^2}{u'}\dfrac{\partial}{\partial u''}-\dfrac{x^2}{u'u''}+2\dfrac{u''}{u'}x^2\dfrac{\partial}{\partial x^2}$
18	$-16-\dfrac{5}{2}u'\dfrac{\partial}{\partial u'}+2u'\dfrac{\partial}{\partial u''}+\dfrac{1}{4}(u''-u')^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+4\dfrac{u'}{u''}+\dfrac{1}{2}\dfrac{u'^2}{u''}\dfrac{\partial}{\partial u'}+2u''\dfrac{\partial}{\partial u'}-\dfrac{5}{2}u''\dfrac{\partial}{\partial u''}+4\dfrac{u''}{u'}+\dfrac{1}{2}\dfrac{u''^2}{u'}\dfrac{\partial}{\partial u''}$ $-\dfrac{17}{2}x^2\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)x^2\dfrac{\partial}{\partial x^2}+\dfrac{u'}{u''}x^2\dfrac{\partial}{\partial x^2}+\dfrac{u''}{u'}x^2\dfrac{\partial}{\partial x^2}-x^4\dfrac{\partial^2}{\partial x^4}$
19	$\dfrac{1}{2}\dfrac{u''-u'}{u''}x^2\dfrac{\partial}{\partial u'}-\dfrac{9}{2}\dfrac{x^2}{u''}+\dfrac{x^2}{u'}-\dfrac{x^4}{u''}\dfrac{\partial}{\partial x^2}$
20	$\dfrac{1}{2}\dfrac{u''-u'}{u'}x^2\dfrac{\partial}{\partial u''}+\dfrac{9}{2}\dfrac{x^2}{u'}-\dfrac{x^2}{u''}+\dfrac{x^4}{u'}\dfrac{\partial}{\partial x^2}$
21	$\dfrac{x^4}{u'u''}$
	(c)
1	$2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+\dfrac{8}{u'}\dfrac{\partial}{\partial x^2}$
2	$-\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}+\dfrac{2}{u'}\dfrac{\partial}{\partial x^2}$
3	0
4	$-2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-\dfrac{2}{u'}\dfrac{\partial}{\partial x^2}-4(u''-u')\dfrac{\partial^2}{\partial x^4}$
5	$-\dfrac{1}{2}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-5\dfrac{\partial}{\partial x^2}-\dfrac{2}{u'}\dfrac{\partial}{\partial u''}+(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+4\dfrac{u''}{u'}\dfrac{\partial}{\partial x^2}$
6	$-9\dfrac{\partial}{\partial x^2}+(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+\dfrac{u''}{u'}\dfrac{\partial}{\partial x^2}-2x^2\dfrac{\partial^2}{\partial x^4}$
7	$-5\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)+\dfrac{20}{u'}+(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+4\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}+2x^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+8\dfrac{x^2}{u'}\dfrac{\partial}{\partial x^2}$
8	$\dfrac{2}{u'}+2\dfrac{x^2}{u'}\dfrac{\partial}{\partial x^2}$
9	$\dfrac{1}{2}\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-(u''-u')\dfrac{\partial^2}{\partial x^4}$
10	$-\dfrac{1}{2}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{15}{2}\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}\dfrac{1}{u'}\dfrac{\partial}{\partial u''}+\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-\dfrac{3}{2}(u''-u')\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}+\dfrac{u''}{u'}\dfrac{\partial}{\partial x^2}-2x^2\dfrac{\partial^2}{\partial x^4}$

TABLE V. (*Continued*).

No.	Operator
11	$-\dfrac{3}{2}\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}\dfrac{1}{u'}\dfrac{\partial}{\partial u''}+\dfrac{u''}{u'}\dfrac{\partial}{\partial x^2}$
12	$\dfrac{1}{2}\dfrac{\partial}{\partial u'}-\dfrac{5}{2}\dfrac{\partial}{\partial u''}+\dfrac{5}{u'}+\dfrac{1}{4}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}-\dfrac{1}{2}x^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}+2\dfrac{x^2}{u'}\dfrac{\partial}{\partial x^2}$
13	0
14	$\dfrac{1}{2}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+2\dfrac{\partial}{\partial x^2}+\dfrac{1}{2}\dfrac{1}{u'}\dfrac{\partial}{\partial u''}+\dfrac{1}{2}(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)\dfrac{\partial}{\partial x^2}-2(u''-u')^2\dfrac{\partial^2}{\partial x^4}-\dfrac{u''}{u'}\dfrac{\partial}{\partial x^2}$
15	0
16	$-5\dfrac{\partial}{\partial u'}+2\dfrac{\partial}{\partial u''}-\dfrac{5}{u'}-(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-14(u''-u')\dfrac{\partial}{\partial x^2}-2(u''-u')^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}-2x^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}$
	$-2\dfrac{x^2}{u'}\dfrac{\partial}{\partial x^2}-4(u''-u')x^2\dfrac{\partial^2}{\partial x^4}$
17	0
18	$-\dfrac{1}{2}\dfrac{\partial}{\partial u'}+2\dfrac{\partial}{\partial u''}-\dfrac{1}{2}\dfrac{1}{u'}-\dfrac{1}{4}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+\dfrac{11}{2}u'\dfrac{\partial}{\partial x^2}+\dfrac{1}{2}(u''-u')^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-\dfrac{13}{2}u''\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}\dfrac{u''}{u'}\dfrac{\partial}{\partial u''}+\dfrac{u''^2}{u'}\dfrac{\partial}{\partial x^2}$
	$+\dfrac{1}{2}x^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-(u''-u')x^2\dfrac{\partial^2}{\partial x^4}$
19	$-24+3(u''-u')\dfrac{\partial}{\partial u'}+\dfrac{9}{2}u'\dfrac{\partial}{\partial u''}+\dfrac{1}{2}(u''-u')^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{11}{2}u''\dfrac{\partial}{\partial u''}+6\dfrac{u''}{u'}+\dfrac{u''^2}{u'}\dfrac{\partial}{\partial u''}-16x^2\dfrac{\partial}{\partial x^2}$
	$-(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)x^2\dfrac{\partial}{\partial x^2}+2\dfrac{u''}{u'}x^2\dfrac{\partial}{\partial x^2}-2x^4\dfrac{\partial^2}{\partial x^4}$
20	$-4+\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u'}+\dfrac{u''}{u'}-2x^2\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}\dfrac{x^2}{u'}\dfrac{\partial}{\partial u''}+\dfrac{u''}{u'}x^2\dfrac{\partial}{\partial x^2}$
21	$-x^2\dfrac{\partial}{\partial u''}+\dfrac{7}{u'}x^2+\dfrac{u''}{u'}x^2\dfrac{\partial}{\partial u''}+2\dfrac{x^4}{u'}\dfrac{\partial}{\partial x^2}$
	(d)
1	$-2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-\dfrac{8}{u''}\dfrac{\partial}{\partial x^2}$
2	$\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-\dfrac{2}{u''}\dfrac{\partial}{\partial x^2}$
3	$2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-4(u''-u')\dfrac{\partial^2}{\partial x^4}+\dfrac{2}{u''}\dfrac{\partial}{\partial x^2}$
4	0
5	$-9\dfrac{\partial}{\partial x^2}-(u''-u')\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}+2\dfrac{u'}{u''}\dfrac{\partial}{\partial x^2}-2x^2\dfrac{\partial^2}{\partial x^4}$
6	$-\dfrac{1}{2}\dfrac{1}{u'}\dfrac{\partial}{\partial u''}-5\dfrac{\partial}{\partial x^2}-(u''-u')\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-\dfrac{2}{u''}\dfrac{\partial}{\partial u'}+4\dfrac{u'}{u''}\dfrac{\partial}{\partial x^2}$
7	$-\dfrac{2}{u''}-2\dfrac{x^2}{u''}\dfrac{\partial}{\partial x^2}$
8	$-5\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)+(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{20}{u''}-4\dfrac{u'}{u''}\dfrac{\partial}{\partial u'}-2x^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-8\dfrac{x^2}{u''}\dfrac{\partial}{\partial x^2}$
9	$-\dfrac{1}{2}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-(u''-u')\dfrac{\partial^2}{\partial x^4}$
10	$-\dfrac{3}{2}\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}\dfrac{1}{u''}\dfrac{\partial}{\partial u'}+\dfrac{u'}{u''}\dfrac{\partial}{\partial x^2}$
11	$-\dfrac{1}{2}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-\dfrac{15}{2}\dfrac{\partial}{\partial x^2}+\dfrac{3}{2}(u''-u'')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-\dfrac{1}{2}\dfrac{1}{u''}\dfrac{\partial}{\partial u'}+\dfrac{u'}{u''}\dfrac{\partial}{\partial x^2}-2x^2\dfrac{\partial^2}{\partial x^4}$

TABLE V. (*Continued*).

No.	Operator
12	$\frac{5}{2}\frac{\partial}{\partial u'}-\frac{1}{2}\frac{\partial}{\partial u''}+\frac{1}{4}(u''-u')\frac{\partial}{\partial u'}\frac{\partial}{\partial u''}-\frac{5}{u''}-\frac{u'}{u''}\frac{\partial}{\partial u'}+\frac{1}{2}x^2\frac{\partial}{\partial u'}\frac{\partial}{\partial x^2}-2\frac{x^2}{u''}\frac{\partial}{\partial x^2}$
13	0
14	0
15	$\frac{1}{2}\frac{\partial}{\partial u'}\frac{\partial}{\partial u''}+2\frac{\partial}{\partial x^2}+(u''-u')\left(\frac{\partial}{\partial u''}-\frac{\partial}{\partial u'}\right)\frac{\partial}{\partial x^2}-2(u''-u')^2\frac{\partial^2}{\partial x^4}+\frac{1}{2}\frac{1}{u''}\frac{\partial}{\partial u'}-\frac{u'}{u''}\frac{\partial}{\partial x^2}$
16	0
17	$-2\frac{\partial}{\partial u'}+5\frac{\partial}{\partial u''}-(u''-u')\frac{\partial}{\partial u'}\frac{\partial}{\partial u''}-14(u''-u')\frac{\partial}{\partial x^2}+2(u''-u')^2\frac{\partial}{\partial u'}\frac{\partial}{\partial x^2}+\frac{5}{u''}+\frac{u'}{u''}\frac{\partial}{\partial u'}$ $+2x^2\frac{\partial}{\partial u''}\frac{\partial}{\partial x^2}-4(u''-u')x^2\frac{\partial^2}{\partial x^4}+2\frac{x^2}{u''}\frac{\partial}{\partial x^2}$
18	$-2\frac{\partial}{\partial u'}+\frac{1}{2}\frac{\partial}{\partial u''}-\frac{1}{4}(u''-u')\frac{\partial}{\partial u'}\frac{\partial}{\partial u''}+\frac{13}{2}u'\frac{\partial}{\partial x^2}-\frac{1}{2}(u''-u')^2\frac{\partial}{\partial u''}\frac{\partial}{\partial x^2}+\frac{1}{2}\frac{1}{u''}+\frac{1}{2}\frac{u'}{u''}\frac{\partial}{\partial u'}-\frac{u'^2}{u''}\frac{\partial}{\partial x^2}-\frac{11}{2}u''\frac{\partial}{\partial x^2}$ $-\frac{1}{2}x^2\frac{\partial}{\partial u'}\frac{\partial}{\partial x^2}-(u''-u')x^2\frac{\partial^2}{\partial x^4}$
19	$-4-\frac{1}{2}(u''-u')\frac{\partial}{\partial u''}+\frac{u'}{u''}-2x^2\frac{\partial}{\partial x^2}-\frac{1}{2}\frac{x^2}{u''}\frac{\partial}{\partial u'}+\frac{u'}{u''}x^2\frac{\partial}{\partial x^2}$
20	$-24-\frac{11}{2}u'\frac{\partial}{\partial u'}-3(u''-u')\frac{\partial}{\partial u''}+\frac{1}{2}(u''-u')^2\frac{\partial}{\partial u'}\frac{\partial}{\partial u''}+6\frac{u'}{u''}+\frac{u'^2}{u''}\frac{\partial}{\partial u'}+\frac{9}{2}u''\frac{\partial}{\partial u'}-16x^2\frac{\partial}{\partial x^2}$ $-(u''-u')\left(\frac{\partial}{\partial u''}-\frac{\partial}{\partial u'}\right)x^2\frac{\partial}{\partial x^2}+2\frac{u'}{u''}x^2\frac{\partial}{\partial x^2}-2x^4\frac{\partial^2}{\partial x^4}$
21	$x^2\frac{\partial}{\partial u'}-7\frac{x^2}{u''}-\frac{u'}{u''}x^2\frac{\partial}{\partial u'}-2\frac{x^4}{u''}\frac{\partial}{\partial x^2}$

(e)

No.	Operator
1	$-4\frac{\partial^2}{\partial x^4}$
2	$-2\frac{\partial^2}{\partial x^4}$
3	0
4	0
5	$\frac{\partial}{\partial u''}\frac{\partial}{\partial x^2}-2(u''-u')\frac{\partial^2}{\partial x^4}$
6	$-\frac{\partial}{\partial u'}\frac{\partial}{\partial x^2}-2(u''-u')\frac{\partial^2}{\partial x^4}$
7	$-14\frac{\partial}{\partial x^2}-2(u''-u')\frac{\partial}{\partial u''}\frac{\partial}{\partial x^2}-4x^2\frac{\partial^2}{\partial x^4}$
8	$-14\frac{\partial}{\partial x^2}+2(u''-u')\frac{\partial}{\partial u'}\frac{\partial}{\partial x^2}-4x^2\frac{\partial^2}{\partial x^4}$
9	0
10	$-\frac{1}{2}\frac{\partial}{\partial u'}\frac{\partial}{\partial x^2}-(u''-u')\frac{\partial^2}{\partial x^4}$
11	$\frac{1}{2}\frac{\partial}{\partial u''}\frac{\partial}{\partial x^2}-(u''-u')\frac{\partial^2}{\partial x^4}$
12	$-\frac{1}{4}\frac{\partial}{\partial u'}\frac{\partial}{\partial u''}-11\frac{\partial}{\partial x^2}-(u''-u')\left(\frac{\partial}{\partial u''}-\frac{\partial}{\partial u'}\right)\frac{\partial}{\partial x^2}-3x^2\frac{\partial^2}{\partial x^4}$
13	0
14	0
15	0

TABLE V. (*Continued*).

No.	Operator
16	0
17	0
18	$\dfrac{1}{4}\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}+\dfrac{1}{2}\dfrac{\partial}{\partial x^2}+\dfrac{1}{2}(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)\dfrac{\partial}{\partial x^2}-(u''-u')^2\,\dfrac{\partial^2}{\partial x^4}$
19	$-3\dfrac{\partial}{\partial u'}+\dfrac{1}{2}\dfrac{\partial}{\partial u''}-\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-8(u''-u')\dfrac{\partial}{\partial x^2}-(u''-u')^2\dfrac{\partial}{\partial u''}\dfrac{\partial}{\partial x^2}-x^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}-2(u''-u')x^2\dfrac{\partial^2}{\partial x^4}$
20	$-\dfrac{1}{2}\dfrac{\partial}{\partial u'}+3\dfrac{\partial}{\partial u''}-\dfrac{1}{2}(u''-u')\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-8(u''-u')\dfrac{\partial}{\partial x^2}+(u''-u')^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial x^2}+\dfrac{\partial}{\partial u''}x^2\dfrac{\partial}{\partial x^2}-2(u''-u')x^2\dfrac{\partial^2}{\partial x^4}$
21	$-42-7(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)+(u''-u')^2\dfrac{\partial}{\partial u'}\dfrac{\partial}{\partial u''}-30x^2\dfrac{\partial}{\partial x^2}-2(u''-u')\left(\dfrac{\partial}{\partial u''}-\dfrac{\partial}{\partial u'}\right)x^2\dfrac{\partial}{\partial x^2}-4x^4\dfrac{\partial^2}{\partial x^4}$

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