Higher dimensional Chern-Simons supergravity

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(Received 2 January 1996)

A Chern-Simons action for supergravity in odd-dimensional spacetimes is proposed. For all odd dimensions, the local symmetry group is a nontrivial supersymmetric extension of the Poincaré group. In $2+1$ dimensions the gauge group reduces to super-Poincaré, while for $D=5$ it is super-Poincaré with a central charge. In general, the extension is obtained by the addition of a one-form field which transforms as an antisymmetric fifth-rank tensor under Lorentz rotations. Since the Lagrangian is a Chern-Simons density for the supergroup, the supersymmetry algebra closes off shell without the need of auxiliary fields. [S0556-2821(96)00314-1]

PACS number(s): 04.50.+h, 04.65.+e, 11.10.Kk

I. INTRODUCTION

In the search for an unified theory of all interactions including gravity, higher dimensional models have become standard in theoretical physics. Two important examples are $D=10$ superstrings [1] and $D=11$ supergravity [2] which give rise, upon dimensional reduction, to interesting effective models in four dimensions. In the study of higher—and lower—dimensional models, Chern-Simons densities play an important role. Although Chern-Simons forms first appeared in the physics literature in the context of anomalies, it is now clear that they have an intrinsic value as dynamical theories in their own right. For example, pure gravity and extended supergravity in three dimensions are Chern-Simons theories for the groups SO(2,2) [3] and $\mathrm{OSp}(2|p)\otimes\mathrm{OSp}(2|q)$ [4], respectively. Similarly, in five dimensions, supergravity can be written as a Chern-Simons action for the supergroup SU $(2,2|N)$ [5]. Chern-Simons forms also provide a simple description for pure gravity in all odd-dimensional spacetimes [6]. The equations of motion of these theories possess black hole solutions [7] generalizing those found in $2+1$ dimensions [8].

The Chern-Simons Lagrangian is constructed as follows. Let G_A a basis for the Lie algebra of a given (super)group **G**. Let $A=A^A G_A$ be the connection for **G**, and $F=dA+A\wedge A=F^A G_A$ its curvature two-form. The Chern-Simons Lagrangian L is a $(2n+1)$-form whose exterior derivative (defined in $2n+2$ dimensions) satisfies

$$dL=\langle F\wedge\cdots\wedge F\rangle=g_{A_1\cdots A_{n+1}}F^{A_1}\wedge\cdots\wedge F^{A_{n+1}}, \qquad (1)$$

where $g_{A_1\cdots A_{n+1}}\equiv\langle G_{A_1},\ldots,G_{A_{n+1}}\rangle$ is completely symmetric and satisfies the invariance condition

$$\nabla g_{A_1\cdots A_{n+1}}=0. \qquad (2)$$

[In the case of a supergroup, the invariant tensor should have the corresponding (anti) symmetry properties.]

For a given n, the above condition may have no solution. Indeed, Eq. (2) imposes strong restrictions on the group. As we shall see below, for $D>3$, one needs to enlarge the bosonic sector of the theory in order to produce a supersymmetric extension of Chern-Simons gravity that contains local Poincaré invariance.

It is direct to prove from Eq. (1) that, up to a total derivative, L is invariant under the gauge transformation

$$\delta_\lambda A=\nabla\lambda, \qquad (3)$$

where λ is an arbitrary zero-form parameter and $\nabla\lambda=d\lambda+[A,\lambda]$. If δ_λ and δ_η are two trasformations, with parameters λ and η, respectively, then

$$[\delta_\lambda,\delta_\eta]A=\delta_{[\lambda,\eta]}A. \qquad (4)$$

The three-dimensional supergravity studied by Achúcarro and Townsend [4], as well as the five-dimensional theory studied by Chamseddine [5], are Chern-Simons theories in the sense described above. Their supersymmetry transformations can be written in the form (3) and, therefore, the supersymmetry algebra closes off shell without the need of auxiliary fields.

It goes without saying that, apart from the invariance under Eq. (3), the Chern-Simons action is also invariant under diffeomorphisms. In $2+1$ dimensions that symmetry is not independent from the local gauge group because, as a consequence of the equations of motion, the connection is locally flat. This means that, given two configurations that differ by a diffeomorphism, there always exists a gauge transformation that deforms one into the other. In the canonical formalism this is reflected by the absence of new *independent* constraints associated with diffeomorphisms.

In dimensions greater than three, however, the Chern-Simons equations of motion do not impose the flatness condition and therefore the diffeomorphism invariance *is* an independent symmetry giving rise to independent constraints in the canonical formalism (see [9] for more details on this point). For our purposes, here, it is enough to observe that the gauge trasformations (3) form a subgroup of the whole symmetry group.

It is the purpose of this paper to show that the above scheme can be extended to supergravity in all odd dimensions, provided one chooses the bosonic Lagrangian in an appropriate way. It turns out that for $D>3$, the Lagrangian for the bosonic sector is not Hilbert's. Rather, the correct Lagrangian is nonlinear in the curvature and yet, gives rise to first order equations for the tetrad and the spin connection.

II. GENERAL RELATIVITY AS A GAUGE THEORY

In this section we shall review some aspects of the vielbein—or gauge—formulation of general relativity. The main point of this section is to display the differences between gravity in odd and even dimensions.

A. Poincaré translations vs diffeomorphisms

General relativity in four and all even dimensions cannot be construed as a ''truly'' local gauge theory [10]. On the contrary, for odd-dimensional spacetimes, gravity can be written as a Chern-Simons action for the Poincaré group [4,3,6]. [If the cosmological constant is present, then the relevant group is the (anti–)de Sitter group.] The Poincaré group has generators P_a and J_{ab} satisfying

$$[P_a,P_b]=0,$$

$$[P_a,J_{bc}]=\eta_{ab}P_c-\eta_{ac}P_b,$$

$$[J_{ab},J_{cd}]=\eta_{ac}J_{bd}-\eta_{bc}J_{ad}+\eta_{bd}J_{ac}-\eta_{ad}J_{bc}. \quad (5)$$

We define the connection for this group:

$$A=e^aP_a+\frac{1}{2}w^{ab}J_{ab}. \quad (6)$$

Let λ be an arbitrary parameter with values in the Lie algebra ($\lambda=\lambda^aP_a+\frac{1}{2}\lambda^{ab}J_{ab}$). Under the infinitesimal gauge transformation $\delta A=\nabla\lambda$, e^a and w^{ab} transform as follows:

$$\text{translations:}\quad \delta e^a=D\lambda^a,\quad \delta w^{ab}=0, \quad (7)$$

$$\text{rotations:}\quad \delta e^a=\lambda^a_b e^b,\quad \delta w^{ab}=-D\lambda^{ab}, \quad (8)$$

where D is the covariant derivative in the connection w.

It might seem puzzling that even though the tetrad and the spin connection carry a representation of the Poincaré group, the Hilbert action constructed purely out of those fields, is not invariant under the Poincaré group in four dimensions. In fact, the action

$$I=\int \epsilon_{abcd}R^{ab}\wedge e^c\wedge e^d \quad (9)$$

is invariant under Eq. (8) but not under Eq. (7). The reason is simply that under Eq. (7) the action changes, modulo boundary terms, by

$$\delta I=2\int \epsilon_{abcd}R^{ab}\wedge T^c\lambda^d, \quad (10)$$

which is not zero for arbitrary λ.

An alternative approach, often followed in the supergravity literature, is the so-called 1.5 formalism. That is to set $T^a=0$ keeping only the tetrad transformation in Eq. (7); the variation of w^{ab} is then calculated using the chain rule. This procedure brings the diffeomorphism invariance into the scene because if $T^a=0$, the variation Eq. (7) for the tetrad is equal—up to a rotation—to a Lie derivative with a parameter $\xi^\mu=e^\mu_a\lambda^a$. In sum, the Hilbert action in 3+1 dimensions is invariant under Lorentz rotations and diffeomorphisms, but not under the local translations (7), generated by P_a.

A completely different situation is observed in 2+1 dimensions, in which case the Hilbert action is linear in the dreibein field:

$$I_{2+1}=\int \epsilon_{abc}R^{ab}\wedge e^c. \quad (11)$$

It is straightforward to check, using the Bianchi identity, that the variation of Eq. (11) under Eq. (7) gives a boundary term and hence, Eq. (7) is a symmetry of the 2+1 theory. As Witten has pointed out, this simple fact has deep consequences [3]. Indeed, in three dimensions, one can replace the diffeomorphism invariance by a local Poincaré invariance whose constraint algebra is a true Lie algebra.

One may wonder if there exists an action invariant under Eq. (7) in higher dimensions. This generalization indeed exists and is given by

$$I_{2n+1}=\int \epsilon_{a_1\cdots a_{2n+1}}R^{a_1a_2}\wedge\cdots\wedge R^{a_{2n-1}a_{2n}}\wedge e^{a_{2n+1}} \quad (12)$$

which, clearly, can only occur in odd dimensions. The fact that Eq. (12) can be written for odd dimensional manifolds only is associated to the existence of Chern-Simons forms in those dimensions.

The key property of Eq. (12) is that it is linear in the vielbein field rather than being linear in the curvature, as would be the case for the Hilbert action. This fact makes Eq. (12) invariant under the transformation Eq. (7), up to a boundary term. In spite of being nonlinear in the curvature, this action yields first order differential equations for all the fields. This is not surprising as Eq. (12) is a particular case of a Lovelock action [11]. I_{2n+1} describes a Chern-Simons theory of ISO$(D-1,1)$, obtained by contraction of SO$(D-1,2)$, and possesses solutions with conical singularities [7] analogous to those found in 2+1 dimensions without cosmological constant [12]. It should be mentioned here that the action (12) has a propagating torsion [13] and therefore, the 1.5 formalism is not applicable in this case. The main goal of this paper is to describe the supersymmetric extension of the action (12).

B. Super-Poincaré vs supergravity

Supergravity is often referred to as the square root of general relativity [14] much in the same spirit as the Dirac equation is the square root of the Klein-Gordon equation. This is justified by the fact that the commutator of two supersymmetry transformations gives a general change of coordinates plus a rotation and another supersymmetry transformation. A concrete example is the usual $N=1$ local supersymmetry transformations [15]

$$\delta e_\mu^a=\frac{1}{2}\overline{\varepsilon\Gamma}^a\psi_\mu,$$

$$\delta\psi_\mu=D_\mu\varepsilon. \tag{13}$$

Here, ψ and ε are Majorana spinors; $\overline{\varepsilon}$ is the Majorana conjugate $(\overline{\varepsilon})_\alpha=C_{\alpha_\beta}\varepsilon^\alpha$ satisfying $\overline{\varepsilon\Gamma}^a\eta=-\overline{\eta\Gamma}^a\varepsilon$, and D_μ is the covariant derivative in the spin connection. It is then straightforward to prove that the commutator of two transformations with parameters ε and η acts on the tetrad as

$$[\delta_\varepsilon,\delta_\eta]e_\mu^a=\frac{1}{2}D_\mu(\overline{\varepsilon\Gamma}^a\eta). \tag{14}$$

The transformation (13) follows from the super-Poincaré algebra when one considers the vielbein and the gravitino as the compensating fields for local translations (P_a) and supersymmetry transformations (Q), respectively. It is not surprising therefore that the right-hand side (RHS) of Eq. (14) is a local translation—acting on the tetrad—with parameter $\lambda^a=\frac{1}{2}\overline{\varepsilon\Gamma}^a\eta$.

The trouble with supergravity in 3+1 dimensions is that the action *is not* invariant under local translations. Nevertheless, transformation (14) is still a symmetry of the action provided the transformation of the spin connection preserves the torsion equation $T^a=\frac{1}{2}\overline{\psi}\Gamma^a{\wedge}\psi$ (1.5 formalism). On the surface defined by the torsion equation, the RHS of Eq. (14) can be rewritten as

$$D_\mu(\overline{\varepsilon\Gamma}^a\eta)=\mathcal{L}_\xi e_\mu^a+\xi^\nu w_{b\nu}^a e_\mu^b-\xi^\nu\overline{\psi}_\nu\Gamma^a\psi_\mu. \tag{15}$$

The first term in the RHS of Eq. (15) represents a diffeomorphism with parameter $\xi^\mu=\overline{\varepsilon\Gamma}^\mu\eta$ ($\Gamma^\mu e_\mu^a=\Gamma^a$), the second term is a rotation with parameter $\xi^\mu w_{b\mu}^a$, and the third term is a supersymmetry transformation with parameter $-\xi^\mu\psi_\mu$. Equation (15) shows that, as far as the tetrad is concerned, the algebra of supersymmetry transformations closes. But the fact that we have used the torsion equation implies that the connection is no longer an independent variable. On the contrary, its variation is given in terms of δe^a and $\delta\psi$, and it *differs* from the form dictated by group theory. As a consequence, the local supersymmetry algebra acting on the gravitino closes only on shell and auxiliary fields are required for its closure off shell.

A completely different situation is observed in the case of 2+1 gravity. Although the above discussion applies to this case [4], there is an alternative route which leads more directly to supergravity. The key feature of 2+1 supergravity is that—unlike 3+1 supergravity—the action *is* invariant under local Poincaré transformations. This means that in the 2+1 theory it is not necessary to express the RHS of Eq. (14) in terms of diffeomorphisms; the commutator of two supersymmetry transformations gives a local translation, which is a symmetry of the action as well.

In other words, in 2+1 dimensions one can consider a *truly* first order formalism in which the spin connection transforms independently of the vielbein and the gravitino, just as dictated by group theory. In particular, under supersymmetry, one can set

$$\delta w_\mu^{ab}=0 \tag{16}$$

which, together with Eq. (13), is a symmetry of the 2+1 action.

The simplicity of the 2+1 theory can be explicitly exhibited. The action reads [4,16]

$$I=\int(\epsilon_{abc}R^{ab}{\wedge}e^c-\overline{\psi}{\wedge}D\psi), \tag{17}$$

where ψ is a two-component Majorana spinor. [See Appendix for a summary of conventions.]

This action is invariant under Lorentz rotations

$$\delta\omega^{ab}=-D\lambda^{ab},\quad \delta e^a=\lambda_b^a e^a,$$

$$\delta\psi=\frac{1}{4}\lambda^{ab}\Gamma_{ab}\psi, \tag{18}$$

Poincaré translations

$$\delta w^{ab}=0,\quad \delta e^a=D\lambda^a,\quad \delta\psi=0, \tag{19}$$

and supersymmetry transformations

$$\delta w^{ab}=0,\quad \delta e^a=\frac{1}{2}\overline{\varepsilon\Gamma}^a\psi,\quad \delta\psi=D\varepsilon. \tag{20}$$

The fields e^a, w^{ab}, and ψ transform as components of a connection for the super-Poincaré group and therefore the supersymmetry algebra implied by Eqs. (18)–(20) is the super-Poincaré Lie algebra. The invariance of the action (17) under the super-Poincaré group should come as no surprise because it is the Chern-Simons action for the connection $A=e^aP_a+\frac{1}{2}w^{ab}J_{ab}+\overline{Q}\psi$, whose generators are P_a, J_{ab}, and Q^α. Indeed, the action (17) can be written as [4]

$$I=\int\langle A{\wedge}dA+\tfrac{2}{3}A{\wedge}A{\wedge}A\rangle, \tag{21}$$

where the angular brackets stand for a properly normalized trace on the algebra, with $\langle J_{ab},P_c\rangle=\epsilon_{abc}$ and $\langle Q_\alpha,Q_\beta\rangle=-iC_{\alpha\beta}$ are the only nonvanishing traces. [$C_{\alpha\beta}$ is the charge conjugation matrix.]

III. FIVE-DIMENSIONAL POINCARÉ SUPERGRAVITY

We turn now to the supersymmetric version of the action (12) in higher dimensions. To illustrate the ideas we start with the five-dimensional case which already contains the main ingredients. The general case will be indicated in the next section.

A. The action

The analogue of the action (17) in five dimensions has three pieces:

$$I_{5/\mathrm{SUSY}}=I_G+I_b+I_\psi, \tag{22}$$

where I_G is the purely gravitational term, I_b is a second bosonic term needed by supersymmetry, and I_ψ is the fermionic term. The explicit formulas are

$$I_G=\frac{1}{2}\int \epsilon_{abcdf}R^{ab}{}_\wedge R^{cd}{}_\wedge e^f,$$

$$I_b=\int R^{ab}{}_\wedge R_{ab\wedge}b,$$

$$I_\psi=-\int R^{ab}{}_\wedge(\overline{\psi}\Gamma_{ab\wedge}D\psi+D\overline{\psi}\Gamma_{ab\wedge}\psi). \tag{23}$$

Here, the gravitino field is a Dirac spinor one-form, and b is a one-form Lorentz pseudoscalar. The form of I_b is dictated by the transformation of I_G and I_ψ under supersymmetry. [See section V for an alternative way to see this through the integrability conditions of the classical equations of motion.]

Since we are interested in the geometrical aspects of this theory only, we have set all the coupling constants equal to 1. The Lorentz covariant derivative D in the spinorial representation is given by $D\psi=d\psi+\frac{1}{4}w^{ab}\Gamma_{ab\wedge}\psi$. We work with Dirac spinors in order to avoid the dimensional dependence of Majorana spinors which only exist—on a Minkwoskian signature—in dimensions 2,3,4 mod8. All the main results of this paper carry through for Majorana spinors when they exist. The Dirac conjugate is defined as $\overline{\psi}=\psi^\dagger\Gamma_0$ and in the action, ψ and $\overline{\psi}$ are varied independently. [See Appendix.]

In addition to local Lorentz rotations, the above action is invariant under the Abelian translations:

$$\delta e^a=D\lambda^a,$$

$$\delta w^{ab}=0,$$

$$\delta b=d\rho,$$

$$\delta\psi=0, \tag{24}$$

where λ^a is a 0-form Lorentz vector and ρ is a 0-form Lorentz pseudoscalar. The invariance of Eq. (22) under Eq. (24) follows directly from the Bianchi identity. The action (22) is also invariant under supersymmetry transformations:

$$\delta e^a=-i(\overline{\varepsilon}\Gamma^a\psi-\overline{\psi}\Gamma^a\varepsilon),$$

$$\delta w^{ab}=0,$$

$$\delta b=\overline{\varepsilon}\psi-\overline{\psi}\varepsilon,$$

$$\delta\psi=D\varepsilon. \tag{25}$$

The proof of the invariance of Eq. (22) under Eq. (25) is straightforward. An important test of the consistency of Eq. (25) is the fact that the commutator of two supersymmetry transformations gives local translation (24). Indeed, if δ_ε and δ_η are supersymmetry transformations with parameters ε and η, we have

$$[\delta_\varepsilon,\delta_\eta]e^a=-iD(\overline{\varepsilon}\Gamma^a\eta-\overline{\eta}\Gamma^a\varepsilon),$$

$$[\delta_\varepsilon,\delta_\eta]w^{ab}=0,$$

$$[\delta_\varepsilon,\delta_\eta]b=d(\overline{\varepsilon}\eta-\overline{\eta}\varepsilon),$$

$$[\delta_\varepsilon,\delta_\eta]\psi=0. \tag{26}$$

The symmetries of the action (22) are generated by the local super-Poincaré generators P_a, J_{ab}, Q^α, $\overline{Q}_\alpha$, plus the Abelian generator K, responsible for the nonzero transformation of b in Eq. (24). These generators form an extension of the super-Poincaré algebra whose only nonvanishing (anti) commutator is

$$\{Q^\alpha,\overline{Q}_\beta\}=-i(\Gamma^a)^\alpha_\beta P_a+\delta^\alpha_\beta K, \tag{27}$$

plus the Poincaré algebra. The commutators of K, Q^α, and $\overline{Q}_\alpha$ with the Lorentz generators can be read off from their tensor character.

Note that K commutes with all the generators in the algebra and therefore, it is as a central charge in the super-Poincaré algebra. This is, however, a peculiarity of five dimensions. For other odd dimensions, the generator K is a completely antisymmetric tensor of fifth rank, which has a nonvanishing commutator with the Lorentz generator.

B. Chern-Simons formulation

The fact that the symmetries of the action (22) close without the need of any auxiliary fields strongly suggest that Eq. (22) may be written as a Chern-Simons action for the supergroup. In this section we prove that this is indeed the case.

Consider a connection A for the superalgebra found in the last section:

$$A=e^aP_a+\frac{1}{2}w^{ab}J_{ab}+bK+\overline{\psi}_\alpha Q^\alpha-\overline{Q}_\alpha\psi^\alpha. \tag{28}$$

The super-curvature $F=dA+A_\wedge A$ is then found by direct application of Eqs. (5) and (27):

$$F=F^AG_A=\widetilde{T}^aP_a+\frac{1}{2}R^{ab}J_{ab}+\widetilde{F}K+D\overline{\psi}_\alpha Q^\alpha-\overline{Q}_\alpha D\psi^\alpha. \tag{29}$$

Here, $\widetilde{T}^a:=T^a-i\overline{\psi}\Gamma^a{}_\wedge\psi$ and $\widetilde{F}:=db+\overline{\psi}_\wedge\psi$; T^a is the torsion two-form, and R^{ab} is the two-form Lorentz curvature.

We recall that a Chern-Simons Lagrangian in five dimensions $L_{5/\mathrm{SUSY}}$ is defined by the relation

$$g_{ABC}F^A{}_\wedge F^B{}_\wedge F^C=dL_{5/\mathrm{SUSY}}, \tag{30}$$

where the trilinear form $g_{ABC}\equiv\langle G_A,G_B,G_C\rangle$ is an invariant tensor of the Lie algebra with generators G_A. Different choices of the invariant tensor g_{ABC} give different five-dimensional Lagrangians $L_{5/\mathrm{SUSY}}$.

To prove that Eq. (22) is a Chern-Simons action we need to find an invariant tensor such that $L_{5/\mathrm{SUSY}}$ is equal, up to a total derivative, to the Lagrangian in Eq. (22). This tensor indeed exists and is given by

$$\langle J_{ab}, J_{cd}, P_e \rangle = \epsilon_{abcde},$$

$$\langle J_{ab}, J_{cd}, K \rangle = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc},$$

$$\langle Q^{\alpha}, J_{ab}, \overline{Q_{\beta}} \rangle = -2(\Gamma_{ab})^{\alpha}_{\beta}. \tag{31}$$

It is straightforward to prove that, up to a total derivative and an overall factor, the 5-form $L_{5/\mathrm{SUSY}}$ associated to the above tensor is equal to the supersymmetric Lagrangian in Eq. (22).

From the above result it is now evident that the action is invariant under supersymmetry transformations up to a total derivative. All the symmetry transformations of the action can now be collected together in the form $\delta A = \nabla \lambda$ where ∇ is the covariant derivative of the supergroup and λ is a zero-form Lie algebra-valued vector in the adjoint representation. From this formula it is also evident that the algebra of supersymmetry transformations closes as dictated by group theory:

$$[\delta_{\lambda}, \delta_{\eta}]A = \delta_{[\lambda,\eta]}A. \tag{32}$$

The Chern-Simons action (22) can be obtained from the action found in [5] by an appropriate Wigner-Inonu contraction. The closure of the supersymmetry algebra, however, was not mentioned in [5]. In the next section we prove that the above scheme is not exclusive of the three- and five-dimensional theories but it can be extended to any odd-dimensional spacetime.

IV. THE GENERAL CASE

In this section we show how the results of the previous sections are generalized to any odd-dimensional manifold. In order to simplify the notation we introduce the symbol R_{abc} defined by

$$R_{abc} := \epsilon_{abca_1 \cdots a_{D-3}} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{D-4} a_{D-3}}. \tag{33}$$

Just as in the five-dimensional case, the supersymmetric action in dimension $2n+1$ has three terms:

$$I_{2n+1/\mathrm{SUSY}} = I_G + I_b + I_{\psi}, \tag{34}$$

where the bosonic ''geometric'' term I_G is given by

$$I_G = \int R_{abc} \wedge R^{ab} \wedge e^c. \tag{35}$$

I_b is a second bosonic term involving a fifth-rank one-form field b^{abcde}:

$$I_b = -\frac{1}{6}\int R_{abc} \wedge R_{de} \wedge b^{abcde}. \tag{36}$$

(Note that this term vanishes in three dimensions and in five dimensions b^{abcde} is a Lorentz pseudoscalar.) Finally, the fermionic part is

$$I_{\psi} = \frac{i}{3}\int R_{abc} \wedge (\overline{\psi}\Gamma^{abc} \wedge D\psi + D\overline{\psi}\Gamma^{abc} \wedge \psi). \tag{37}$$

Each term in the action (34) is independently invariant under local Lorentz transformations. The complete action is invariant under the Abelian translations

$$\delta e^a = D\lambda^a, \quad \delta w^{ab} = 0, \quad \delta b^{abcde} = D\rho^{abcde},$$

$$\delta\psi = 0, \tag{38}$$

and supersymmetry transformations

$$\delta e^a = -i(\overline{\varepsilon}\Gamma^a\psi - \overline{\psi}\Gamma^a\varepsilon), \quad \delta w^{ab} = 0,$$

$$\delta b^{abcde} = -i(\overline{\varepsilon}\Gamma^{abcde}\psi - \text{H.c.}), \quad \delta\psi = D\varepsilon, \tag{39}$$

where D represents the Lorentz covariant derivative.

The proof of the invariance of Eq. (34) under supersymmetry transformations is straightforward. One starts by varying the fermionic part. Up to a boundary term one easily obtains

$$\delta I_{\psi} = \frac{i}{12}\int R_{abc} \wedge R_{de} \wedge (\overline{\psi}\{\Gamma^{abc}, \Gamma^{de}\}\epsilon - \text{c.c.}). \tag{40}$$

Using the formula (A4) of Appendix, we find that Eq. (40) has a term proportional to a fifth-rank Dirac matrix Γ^{abcde} plus a term proportional to a Dirac matrix Γ^a. It is direct to see that the first term is canceled by the variation of I_b while the second term is canceled by the variation of I_G.

As in the lower dimensional cases ($D=3,5$), the commutator of two supersymmetry transformations gives a local Abelian translation (38). Thus, the supersymmetric extension of the Poincaré algebra that leaves the action invariant has generators $G_A = [P_a, J_{ab}, K_{abcde}, Q^{\alpha}, \overline{Q_{\alpha}}]$. The only nonvanishing (anti)commutator is

$$\{Q^{\alpha}, \overline{Q_{\beta}}\} = -i(\Gamma^a)^{\alpha}_{\beta}P_a - i(\Gamma^{abcde})^{\alpha}_{\beta}K_{abcde}, \tag{41}$$

plus the Poincaré algebra. The commutators of Q, $\overline{Q}$, and K with the Poincaré generators can be read from their tensorial character.

The action $I_{2n+1/\mathrm{SUSY}}$ is also a Chern-Simons action. The connection now is

$$A = e^a P_a + \frac{1}{2}w^{ab}J_{ab} + b^{abcde}K_{abcde} + \overline{\psi_{\alpha}}Q^{\alpha} - \overline{Q_{\alpha}}\psi^{\alpha}, \tag{42}$$

and the Lagrangian is defined by $\langle F \wedge \cdots \wedge F \rangle = dL_{2n+1}$ where the invariant $(n+1)$ multilinear form $\langle \cdots \rangle$ is defined by

$$\langle J_{a_1a_2} \ldots J_{a_{D-2}a_{D-1}} P_{a_D} \rangle = \epsilon_{a_1 \ldots a_D},$$

$$\langle J_{a_1a_2} \ldots J_{fg} K_{abcde} \rangle = -\frac{1}{12}\epsilon_{a_1 \ldots a_{D-3}abc}\eta_{[fg][de]},$$

$$\langle Q J_{a_1a_2} \ldots J_{a_{D-4}a_{D-3}} \overline{Q} \rangle = 2i^n \Gamma_{a_1 \ldots a_{D-3}} \tag{43}$$

(the remaining brackets are zero). This completes the construction of the $(2n+1)$-dimensional Chern-Simons action for supergravity.

V. INTEGRABILITY OF THE EQUATIONS OF MOTION

A remarkable feature of supersymmetric theories, in general, and supergravity, in particular, is the fact that the integrability conditions for the fermionic field equations are the bosonic equations. The study of the integrability conditions of the fermionic equation in our model sheds some light on the role of the bosonic field b^{abcde}.

In the notation introduced in Sec. IV, the fermionic field equations are

$$R_{abc}\Gamma^{abc}{}_{\wedge}D\psi=0, \tag{44}$$

and similarly for the Dirac conjugate spinor. Taking the covariant derivative of Eq. (44) we find the integrability condition

$$R_{abc\wedge}R^{de}\Gamma^{abc}\Gamma_{de\wedge}\psi=0. \tag{45}$$

This equation should be satisfied for any ψ. Using elementary properties of the Dirac matrices we obtain the following equations for the bosonic fields:

$$R_{abc\wedge}R^{ab}\Gamma^{c}=0, \tag{46}$$

$$R_{abc\wedge}R_{de}\Gamma^{abcde}=0. \tag{47}$$

Equation (46) is the equation of motion for the vielbein field, while Eq. (47) is the equation of motion for the b field. Had we not included b, supersymmetry would not have been achieved and the integrability conditions would not have been satisfied. [This does not rule out a Lagrangian without the b field. However, the fermionic equations for such a theory would impose additional equations on the bosonic fields.]

VI. COMMENTS AND PROSPECTS

We have shown in this paper that the successful methods used in three dimensions to construct supersymetric extensions of general relativity can be generalized to any odd-dimensional spacetime. We have restricted ourselves, however, to Poincaré supergravity. The full anti-de Sitter extension remains an open problem. (In five dimensions, a Chern-Simons action for anti–de Sitter supergravity has been known for some time [5]. That action reduces to the action considered here after a proper contraction is performed.)

There are good reasons to seek a full anti–de Sitter Chern-Simons formulation of supergravity. First, the bosonic Lagrangian in the Poincaré case does not contain the Hilbert term thus making the contact with four-dimensional theories rather obscure [6]. Second, the Poincaré theory in odd dimensions does not possess black hole solutions while the anti–de Sitter theory does [7].

In principle, a Chern-Simons anti–de Sitter supergravity can be constructed from the knowledge of the associated supergroup and an invariant tensor only (finding the invariant tensor, however, may prove to be a nontrivial task). In five dimensions, the relevant supergroup is SU(2,2|1) [17] while in the important example of eleven dimensions the supergroup is OSp(32|1) [18]. As the spacetime dimension increases, one faces a growing multiplicity of choices for the invariant tensor. To illustrate this issue, consider the problem of classifying all the invariants that can be constructed out of the Lorentz curvature in a given dimension [19]. In four dimensions we only have

$$\epsilon_{abcd}R^{ab}{}_{\wedge}R^{cd}, \quad R^{ab}{}_{\wedge}R_{ab}, \tag{48}$$

while in eight dimensions we have

$$\epsilon_{abcdefgh}R^{ab}{}_{\wedge}R^{cd}{}_{\wedge}R^{ef}{}_{\wedge}R^{gh}, \quad R^{ab}{}_{\wedge}R_{ab\wedge}R^{cd}{}_{\wedge}R_{cd},$$
$$R^{ab}{}_{\wedge}R_{bc\wedge}R^{cd}{}_{\wedge}R_{da}. \tag{49}$$

Of course, all the above scalars define Chern-Simons Lagrangians in dimension three and seven, respectively. A similar proliferation of scalars appears in supergravity. A good candidate for the right theory could be a linear combination of all possible invariants such that, under an appropriated Wigner-Inonu contraction, it reduces to the Poincaré theories studied here.

The particular case of eleven dimensions seems to be particularly suited to admit an anti–de Sitter Chern-Simons formulation. As shown in [18], the super anti–de Sitter group is OSp(32|1). A natural basis for the Lie algebra of Sp(32) is given by the Dirac matrices $\Gamma_a,\Gamma_{ab},\Gamma_{abcde}$, and this basis is easily extended to span the superalgebra of OSp(32|1). Thus, the supergroup OSp(32|1) naturally accommodates the field content of the Poincaré Chern-Simons supergravity considered here. One could expect, therefore, that a Chern-Simons Lagrangian for the supergroup OSp(32|1) in eleven dimensions should reduce to the supersymmetric action Eq. (34) upon contraction. For example, it is easy to check that the superalgebra presented in Sec. IV can be obtained from OSp(32|1) by a Wigner-Inonu contraction:

$$G_a\to\lambda^{-1}P_a,$$
$$G_{ab}\to J_{ab},$$
$$G_{abcde}\to\lambda^{-1}K_{abcde},$$
$$Q^{\alpha}\to\lambda^{-(1/2)}Q^{\alpha}. \tag{50}$$

At the level of the Lagrangian, however, the problem is more complicated. Because of the ambiguity in the choice of the invariant tensors and the large number of terms in the super anti–de Sitter Chern-Simons action, it is a nontrivial problem to find an expression such that, under contraction, it reduces to the action considered in this paper [20].

Finally, we mention that the—Poincaré—supersymmetric Chern-Simons actions found in this paper are not the only possibilities. In dimensions greater than 5, the fermionic Lagrangian accepts other Poincaré-invariant terms that give rise to other supergravities. For example, in eleven dimensions one can add to the fermionic Lagrangian the term

$$[R^{ab}{}_{\wedge}R_{ab}]^2{}_{\wedge}\bar{\psi}_{\wedge}D\psi. \tag{51}$$

This term, however, requires extra bosonic fields to respect supersymmetry. This is easily seen by studying the integrability conditions generated by Eq. (51). One finds the equation over the bosonic fields:

$$[R^{ab}{\wedge}R_{ab}]^2{\wedge}R^{cd}=0. \tag{52}$$

Thus, consistency requires an extra bosonic term in the Lagrangian of the form $[R^{cd}{\wedge}R_{cd}]^2{\wedge}R^{ab}{\wedge}c_{ab}$ which involves the one-form c_{ab}. Varying this term with respect to c_{ab} gives Eq. (52). Thus, the integrability condition is satisfied and the action is supersymmetric. A complete classification of all possible fermionic Lagrangians for a given dimension and their corresponding supersymmetry algebras is beyond the scope of this work. We would like to point out, however, that the method outlined here seems to provide a simple way to generate extensions of the super-Poincaré algebra involving extra bosonic fields.

ACKNOWLEDGMENTS

The authors are grateful to M. Contreras, C. Martínez, and, especially, C. Teitelboim for helpful discussions. This work was supported in part by Grant Nos. 1930910 and 2950047 from FONDECYT (Chile), 27-953/ZI-DICYT (University of Santiago), and Grant No. PG-023-95 of Departamento de Postgrado y Postítulo (University of Chile). M.B. was partially supported by a grant from Fundación Andes. R.T. thanks CONICYT for financial support. Institutional support to CECS from a group of Chilean private companies (COPEC, CGE, CMPC, MINERA LA ESCONDIDA, NOVAGAS, BUSINESS DESIGN ASS., and XEROX-Chile) is also acknowledged.

APPENDIX: GAMMA MATRICES

The Clifford algebra in $D=2n+1$ dimensions with Minkowskian signature can be generated by a set of $2^n\times 2^n$ matrices; the unit I and D matrices Γ^a, satisfying $\{\Gamma^a,\Gamma^b\}=2\eta^{ab}I$, where $a,b,\ldots,=1,2,\ldots,2n+1$ and $\eta^{ab}=\text{diag}(-,+,\cdots,+)$. In this signature, $\Gamma^\dagger_\mu=\Gamma_0\Gamma_\mu\Gamma_0$.

It is always possible to find a representation of Γ^a matrices in which

$$\Gamma:=\Gamma^1\Gamma^2\cdots\Gamma^D=(-i)^{n+1}I. \tag{A1}$$

We define $\Gamma^{a_1\cdots a_p}$ as the totally antisymmetric product of gamma matrices,

$$\Gamma^{a_1\cdots a_p}=\frac{1}{p!}\sum_\sigma \text{sgn}(\sigma)\Gamma^{a_{\sigma(1)}}\ldots\Gamma^{a_{\sigma(p)}}. \tag{A2}$$

Two useful formulas implicitly used in the text are

$$\Gamma_{a_1\ldots a_{D-3}}=-\frac{(-i)^{n+1}}{3!}\epsilon_{a_1\ldots a_{D-3}abc}\Gamma^{abc} \tag{A3}$$

$$\frac{1}{2}\{\Gamma^{abc},\Gamma^{de}\}=\Gamma^{abcde}-[\eta^{[ab][de]}\Gamma^c+\text{perm}(abc)], \tag{A4}$$

where $\eta^{[ab][de]}=\eta^{ad}\eta^{be}-\eta^{ae}\eta^{bd}$.

[1] For a complete treatment, see M.B. Green, J.H. Schwarz, and E. Witten, *Superstrings* (Cambridge University Press, Cambridge, England, 1987).

[2] E. Cremmer, B. Julia, and J. Scherk, Phys. Lett. **76B**, 409 (1978).

[3] E. Witten, Nucl. Phys. **B311**, 46 (1988).

[4] A. Achúcarro and P. K. Townsend, Phys. Lett. B **180**, 89 (1986).

[5] A. H. Chamseddine, Nucl. Phys. **B346**, 213 (1990).

[6] A. H. Chamseddine, Phys. Lett. B **233**, 291 (1989).

[7] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. D **49**, 975 (1994); see also, *J.J. Giambiagi Festschriff*, edited by H. Falomir, R. Gamboa, P. Leal, and F. Shaposnik (World Scientific, Singapore, 1991).

[8] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).

[9] M. Bañados, L. J. Garay, and M. Henneaux, Phys. Rev. D **53**, 593 (1996).

[10] T. Regge, Phys. Rep. **137**, 31 (1986).

[11] D. Lovelock, J. Math. Phys. **12**, 498 (1971).

[12] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).

[13] M. Bañados, L. J. Garay, and M. Henneaux, hep-th/9605159.

[14] C. Teitelboim, Phys. Rev. Lett. **38**, 1106 (1977); C. Teitelboim and R. Tabensky, Phys. Lett. **69B**, 240 (1977).

[15] For a review see P. van Nieuwenhuizen, Phys. Rep. **68**, 4 (1981).

[16] For earlier work on three-dimensional supergravity see N. Marcus and J. H. Schwarz, Nucl. Phys. **B228**, 145 (1983); S. Deser and J. H. Kay, Phys. Lett. **120B**, 97 (1983).

[17] W. Nahm, Nucl. Phys. **B135**, 149 (1978).

[18] J. W. van Holten and A. Van Proeyen, J. Phys. A **15**, 3763 (1982).

[19] A. Mardones and J. Zanelli, Class. Quantum Grav. **8**, 1545 (1991).

[20] M. Bañados, R. Troncoso, and J. Zanelli (in preparation).