Influence of spin-spin coupling on inspiraling compact binaries with $M_1=M_2$ and $S_1=S_2$

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Searching for gravitational waves with laser-interferometer detectors requires a highly accurate knowledge of the gravitational wave forms in order to construct search templates with which to cross correlate the noisy detectors' output. In this spirit, we derive here an analytic approximate formula describing the precession of an inspiraling compact binary with equal masses and equal spins, where both the spin-orbit and the spin-spin couplings are taken into account. [S0556-2821(96)00614-5]

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I. INTRODUCTION

The inspiral and coalescence of neutron star and black hole binaries is a very promising source of gravitational waves for the Laser Interferometric Gravitational Wave Observatory–(LIGO-)/VIRGO network of interferometric gravitational wave detectors [1,2] and also, in the case of massive black holes, for the space-based Laser Interferometer Space Antenna (LISA) detector [3]. A large number of people have done extensive studies both on theoretical aspects of the problem connected with the motion of these binaries and with the emission of gravitational waves from them, and on practical aspects of the implementation of various strategies to detect these waves with a given detector or network of detectors.

While for a nonspinning binary the orbital plane remains fixed during inspiral, and thus the geometry of the binary with respect to the detector is fixed, if at least one of the bodies is rotating, then the binary will undergo a precessional motion due to spin-orbit and spin-spin coupling. In [4] the general aspects of this precession have been discussed and explored both analytically (whenever that was possible) and numerically. In this short paper we derive an approximate analytical formula for the spin-induced precessional motion for a special combination of masses and spins, which, nevertheless, is an astrophysically important one, namely, the case of a binary with equal masses and spins. Although the general problem of computing the precessional motion for any combination of masses and spins (based on post$^{1.5}$-Newtonian spin-orbit coupling and post^2-Newtonian spin-spin coupling, and on leading-order radiation-reaction-induced inspiral) has already been treated numerically (see Ref. [4]), analytic expressions are more favorable for detection and parameter extraction purposes, since they can be used directly when constructing search templates and are easier to handle in data analysis investigations (see [5,6]).

In this spirit, we have derived approximate analytic formulas describing the precession of the orbital plane for the equal mass, equal spin case. We present those formulas here as a supplement to our more general numerical analysis of spin-induced precession [4], and also as a supplement to our previous analytical analysis of precession without spin-spin coupling [4]. The formulas we derive here show explicitly and analytically the spin-spin-induced ''wobbling'' of the otherwise tight inspiraling motion that the orbital angular momentum vector undergoes during its otherwise tight precession (see Figs. 11–16 of [4]). The analytic precession formulas derived in this paper can be inserted into Eqs. (18c), (19), and (29) of Ref. [4] to obtain the gravitational wave forms from the precessing, inspiraling binary.

Throughout we use units where $G=c=1$.

II. THE OPENING ANGLE OF $\hat{\mathbf{L}}$

As was mentioned in the previous section, the case of a binary with equal masses is an important one, corresponding, for example, to $1.4M_\odot$, $1.4M_\odot$ neutron-star–neutron-star (NS-NS) binaries. If, additionally, the spins happen to have almost equal magnitude, then the complicated differential equations describing the precession (up to post^2-Newtonian order) simplify significantly and allow an approximate solution, which turns out to be very accurate.

As discussed in [4] the precession equations take on the following form, after specializing to circular orbits and after averaging over one orbit:

$$\dot{\mathbf{L}}=\frac{1}{r^3}\left[\frac{4M_1+3M_2}{2M_1}\mathbf{S}_1+\frac{4M_2+3M_1}{2M_2}\mathbf{S}_2\right]\times\mathbf{L}-\frac{3}{2}\frac{1}{r^3}[(\mathbf{S}_2\cdot\hat{\mathbf{L}})\mathbf{S}_1+(\mathbf{S}_1\cdot\hat{\mathbf{L}})\mathbf{S}_2]\times\hat{\mathbf{L}}-\frac{32}{5}\frac{\mu^2}{r}\left(\frac{M}{r}\right)^{5/2}\hat{\mathbf{L}}, \tag{1a}$$

$$\dot{\mathbf{S}}_1=\frac{1}{r^3}\left[\frac{4M_1+3M_2}{2M_1}\mathbf{L}\right]\times\mathbf{S}_1+\frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_2-\frac{3}{2}(\mathbf{S}_2\cdot\hat{\mathbf{L}})\hat{\mathbf{L}}\right]\times\mathbf{S}_1, \tag{1b}$$

$$\dot{\mathbf{S}}_2=\frac{1}{r^3}\left[\frac{4M_2+3M_1}{2M_2}\mathbf{L}\right]\times\mathbf{S}_2+\frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_1-\frac{3}{2}(\mathbf{S}_1\cdot\hat{\mathbf{L}})\hat{\mathbf{L}}\right]\times\mathbf{S}_2. \tag{1c}$$

Here an overdot represents ''d/dt,'' $\mathbf{L}$, $\mathbf{S}_1$, and $\mathbf{S}_2$ are the orbital angular momentum and the two spins, respectively, $\hat{\mathbf{L}}$ is the unit vector along $\mathbf{L}$, r is the distance between the two bodies, and M_1, M_2, and μ are the masses of the two bodies and their reduced mass, respectively. The assumption of precisely equal masses ($M_1=M_2$) and spins ($|\mathbf{S}_1|=|\mathbf{S}_2|=S$) makes it possible to solve Eqs. (1) analytically even if we keep, apart from the spin-orbit terms, the spin-spin terms. Then the cosines of the angles between the two spins $\mathbf{S}_1,\mathbf{S}_2$ and the total angular momentum $\mathbf{J}$ (which, as we have shown in Ref. [4], is almost constant in direction) evolve as follows:

$$\frac{d}{dt}(\hat{\mathbf{J}}\cdot\hat{\mathbf{S}}_1)=\frac{3S}{2r^3}\left[2-\frac{(\mathbf{S}_2\cdot\mathbf{J})}{L^2}+O\left(\frac{S^2}{L^2}\right)\right]\hat{\mathbf{J}}\cdot(\hat{\mathbf{S}}_1\times\hat{\mathbf{S}}_2), \tag{2a}$$

$$\frac{d}{dt}(\hat{\mathbf{J}}\cdot\hat{\mathbf{S}}_2)=-\frac{3S}{2r^3}\left[2-\frac{(\mathbf{S}_1\cdot\mathbf{J})}{L^2}+O\left(\frac{S^2}{L^2}\right)\right]\hat{\mathbf{J}}\cdot(\hat{\mathbf{S}}_1\times\hat{\mathbf{S}}_2), \tag{2b}$$

$$\frac{d}{dt}(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)=\frac{3J}{2r^3}\left[\frac{(\mathbf{S}_1\cdot\mathbf{J}-\mathbf{S}_2\cdot\mathbf{J})}{L^2}+O\left(\frac{S^2}{L^2}\right)\right]\hat{\mathbf{J}}\cdot(\hat{\mathbf{S}}_1\times\hat{\mathbf{S}}_2). \tag{2c}$$

If we had taken into account the motion of $\hat{\mathbf{J}}$ as described by Eqs. (1), it would have contributed to the $O(S^2/L^2)$ terms. Therefore the assumption of fixed $\hat{\mathbf{J}}$ is completely justified in our analysis.

Equations (2) have been derived from Eqs. (1b) and (1c) by using the approximation $\mathbf{L}\simeq\mathbf{J}\times[1+O(S/L)]$ and then rearranging in terms of successive orders of S^k/L^k. (When $M_1\simeq M_2$ then $S\ll L$ throughout the inspiral; when $M_1\gg M_2$ this need not be so.) By introducing the notation

$$\chi\equiv\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2,\quad \xi\equiv\hat{\mathbf{J}}\cdot\frac{\hat{\mathbf{S}}_1-\hat{\mathbf{S}}_2}{2},\quad \eta\equiv\hat{\mathbf{J}}\cdot\frac{\hat{\mathbf{S}}_1+\hat{\mathbf{S}}_2}{2}, \tag{3}$$

Eqs. (2) transform to

$$\frac{d\eta}{dt}=\frac{3S}{2r^3}\left[O\left(\frac{S}{L}\right)\right]w(\eta,\xi,\chi), \tag{4a}$$

$$\frac{d\xi}{dt}=\frac{3S}{2r^3}\left[2+O\left(\frac{S}{L}\right)\right]w(\eta,\xi,\chi), \tag{4b}$$

$$\frac{d\chi}{dt}=\frac{3S}{2r^3}\left[2\xi+O\left(\frac{S}{L}\right)\right]w(\eta,\xi,\chi), \tag{4c}$$

with

$$w(\eta,\xi,\chi)=\sqrt{1-\chi^2-2(\eta^2+\xi^2)+2\chi(\eta^2-\xi^2)}. \tag{4d}$$

By ignoring all the $O(S/L)$ terms, Eqs. (4) simplify to a single differential equation for one of the η,ξ,χ functions while the other two can be expressed as functions of the first one:

$$\eta\simeq\eta(0)=\text{const}, \tag{5a}$$

$$\chi\simeq\chi(0)+\frac{1}{2}[\xi^2-\xi^2(0)], \tag{5b}$$

$$\frac{d\xi}{dt}\simeq\frac{3S}{r^3}\sqrt{A+B\xi^2+\Gamma\xi^4}, \tag{5c}$$

with

$$A\equiv[1-\eta^2(0)]^2-[\eta^2(0)-\chi(0)+\xi^2(0)/2]^2,$$

$$B\equiv-2-3\chi(0)+\eta^2(0)+3\xi^2(0)/2,$$

$$\Gamma\equiv-5/4, \tag{5d}$$

where ‘‘(0)’’ means the initial value. Equation (5c) can be solved by means of Jacobian elliptic integrals [7], leading to

$$\xi=\frac{1}{c_3}\mathrm{sd}\left(c_1\int\frac{3S}{r^3}dt+c_0\middle|c_2\right) \tag{6a}$$

where

$$c_0\equiv\mathrm{sd}^{-1}[c_3\xi(0)|c_2],\quad c_1\equiv(B^2-4A\Gamma)^{1/4},$$

$$c_2\equiv(B+c_1^2)/(2c_1^2),\quad c_3\equiv c_1/\sqrt{|A|}, \tag{6b}$$

and

$$\int\frac{3S}{r^3}dt=\frac{15a}{64}Q^2\left[\left(\frac{10\ \text{Hz}}{f(0)}\right)^{2/3}-\left(\frac{10\ \text{Hz}}{f}\right)^{2/3}\right], \tag{6c}$$

where

$$Q=\frac{18.607}{(M/M_\odot)^{1/3}}. \tag{6d}$$

Here a is the spin parameter for the two stars ($S_i=aM_i^2$).

Having analytic expressions for all the angles between the spins and total angular momentum, we can compute the opening angle λ_L of the orbital angular momentum, i.e., the angle between $\hat{\mathbf{L}}$ and $\hat{\mathbf{J}}$ (recall that $\hat{\mathbf{J}}$ is fixed to the level of accuracy used, i.e., $\hat{\mathbf{J}}$ plays the role of the $\hat{\mathbf{J}}_0$ that was used in Ref. [4] when analyzing simple precession):

$$\lambda_L=\arcsin\left(\frac{\sqrt{(\mathbf{S}_1+\mathbf{S}_2)^2-(\mathbf{S}_1\cdot\hat{\mathbf{J}}+\mathbf{S}_2\cdot\hat{\mathbf{J}})^2}}{L}\right)$$

$$=\arcsin^{-1}\left(\frac{S}{L}\sqrt{\Delta+\xi^2}\right), \tag{7a}$$

where

$$\Delta=2+2\chi(0)-4\eta^2(0)-\xi^2(0). \tag{7b}$$

In Fig. 1 we have plotted the analytic expression (7a) using the solution (6a) derived above, versus frequency, for a typical case. [The initial frequency we used is $f(0)=10$ Hz since this is approximately the frequency of the waves when they enter the frequency band of ‘‘advanced’’ LIGO detectors.] To show how good our approximate analytic solution is, we have plotted on top of it the corresponding numerical solution of the exact differential equations (1) (dashed line). In the range of frequencies where the detectors are most sensitive (10–200 Hz), the approximate analytic solution is an excellent approximation to the true numerical one. If we had dropped the spin-spin terms from the beginning, we would have missed the ξ^2 term in Eq. (7a) and hence the ‘‘wiggling’’ shape of Fig. 1. This ‘‘wiggling’’ is due to the opening and closing of the spin-spin angle.

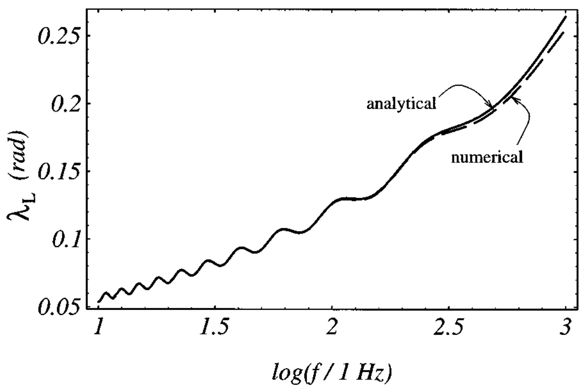

FIG. 1. Evolution of the opening angle λ_L of $\hat{\mathbf{L}}$ for two $1.4M_\odot$ neutron stars with equal spin magnitudes $0.5M_i^2$ and initial orientations $\hat{\mathbf{S}}_1=\hat{x}$, $\hat{\mathbf{S}}_2=\hat{y}$, $\hat{\mathbf{J}}=\hat{z}$. The solid line represents the analytical solution given by Eq. (7a) and the dashed one, the numerical solution. The approximation is almost perfect within the region $f\lesssim 300$ Hz where the sensitivity of detectors is high. The logarithm is to base 10.

As we said earlier, Fig. 1 corresponds to some set of parameters $\chi(0)$, $\eta(0)$, and $\xi(0)$. We have tried several cases with different initial spin and orbital angular momentum orientations, which produced either better or worse results. For the sake of completeness, in Fig. 2 we have plotted the corresponding analytic and numerical solutions for the worst case we have found.

III. THE ANGULAR POSITION OF $\hat{\mathbf{L}}$

The analysis up to this point gives no information about the angular positions of $\hat{\mathbf{L}}$, $\hat{\mathbf{S}}_1$, and $\hat{\mathbf{S}}_2$ in their precession around $\hat{\mathbf{J}}$. The angular velocity of precession can be derived from Eqs. (1). Writing these equations in a more suitable form, we get

$$\frac{d\hat{\mathbf{L}}}{dt}=-\frac{1}{L}\frac{d}{dt}(\mathbf{S}_1+\mathbf{S}_2)=\frac{1}{2r^3}[7\,\mathbf{J}-3\,\mathbf{\Sigma}]\times\hat{\mathbf{L}}, \tag{8a}$$

where

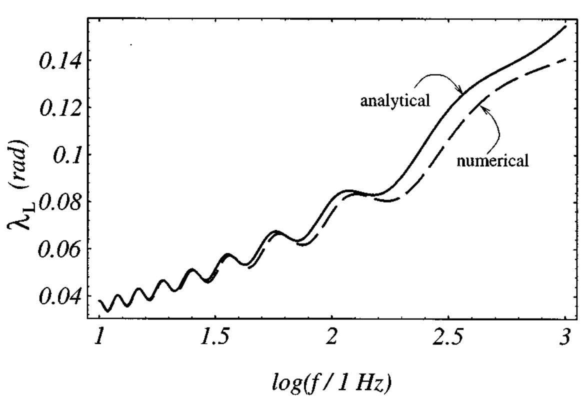

FIG. 2. Same as Fig. 1 but with initial spin and total angular momentum orientations $\hat{\mathbf{S}}_1=\hat{x}$, $\hat{\mathbf{S}}_2=\hat{z}$, $\hat{\mathbf{J}}=\hat{z}$. The logarithm is to base 10.

$$\mathbf{\Sigma}=\frac{\mathbf{S}_1(\mathbf{S}_2\cdot\mathbf{L})+\mathbf{S}_2(\mathbf{S}_1\cdot\mathbf{L})}{L^2}. \tag{8b}$$

This says that $\hat{\mathbf{L}}$ precesses with instantaneous angular velocity

$$\mathbf{\Omega}_p=\frac{1}{2r^3}[7\,\mathbf{J}-3\,\mathbf{\Sigma}]. \tag{9}$$

In Sec. IV of Ref. [4] in the analysis of *simple precession*, the $\mathbf{\Sigma}$ term was ignored, but now we have to keep it. Although it is formally of order (S^2/L^2), its effects on the precession angle will be of the same order as these produced by the (S/L) term in

$$\mathbf{J}=\hat{\mathbf{J}}L\left[1+2\,\eta(0)\frac{S}{L}+O\left(\frac{S^2}{L^2}\right)\right]. \tag{10}$$

More specifically, $\mathbf{\Sigma}$, expressed in terms of three noncoplanar vectors relevant to the problem,

$$\mathbf{\Sigma}=c_J\hat{\mathbf{J}}+c_L\hat{\mathbf{L}}+c_\perp\hat{\mathbf{J}}\times\hat{\mathbf{L}}, \tag{11}$$

contributes to the revolution of $\hat{\mathbf{L}}$ around $\hat{\mathbf{J}}$ only through c_J ($c_\perp$ produces the change of the opening angle λ_L we found in the previous section, and c_L produces no precession at all). With some effort, from Eqs. (5a), (5b), (6a), and (8b) we find that

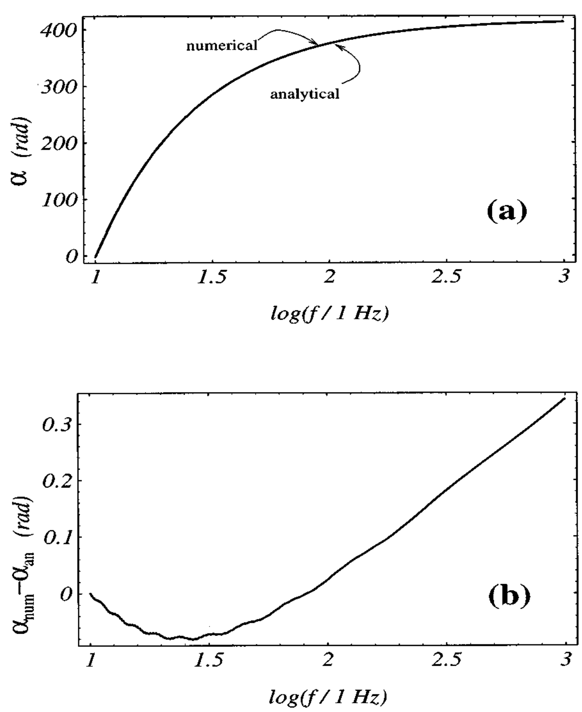

FIG. 3. (a) Evolution of the angular position α of $\hat{\mathbf{L}}$ for the same neutron stars and initial configuration as in Fig. 1. Again the solid line represents the analytical solution given by Eq. (14) and the dashed one, the numerical solution. The difference is so tiny that the two lines are hardly distinguishable. (b) The difference of the numerical and the analytical solutions has been plotted to give a feeling for how good the analytical approximation is. The logarithm is to base 10.

$$c_J = S\left[\eta(0)\frac{\Delta+5\xi^2}{\Delta+\xi^2} + O\left(\frac{S}{L}\right)\right]. \tag{12}$$

Therefore Ω_p is given by

$$\Omega_p = \frac{L}{2r^3}\left[7 + \eta(0)\frac{11\Delta-\xi^2}{\Delta+\xi^2}\left(\frac{S}{L}\right) + O\left(\frac{S^2}{L^2}\right)\right], \tag{13}$$

and the angular position of $\hat{\mathbf{L}}$, by the time integral of Ω_p

$$\alpha = \alpha(0) + \frac{35}{192}Q^3\left[\left(\frac{10\ \mathrm{Hz}}{f(0)}\right) - \left(\frac{10\ \mathrm{Hz}}{f}\right)\right] + \frac{5a}{128}Q^2\eta(0)W\left[\left(\frac{10\ \mathrm{Hz}}{f(0)}\right)^{2/3} - \left(\frac{10\ \mathrm{Hz}}{f}\right)^{2/3}\right] + O\left(\frac{S^2}{L^2}\right), \tag{14}$$

where W represents the average value of $(11\Delta-\xi^2)/(\Delta+\xi^2)$. This quantity is frequency dependent but has an oscillatory behavior which is well approximated by its average value:

$$W = -1 + \frac{12\Delta}{\xi^2_{\max}}\ln\left(\frac{\Delta+\xi^2_{\max}}{\Delta}\right), \tag{15}$$

where $\xi^2_{\max}$ can be derived by solving

$$A + B\xi^2_{\max} + \Gamma\xi^4_{\max} = 0 \tag{16}$$

[cf. Eq. (5c)]. In Fig. 3(a) we have plotted the analytic expression for α given by Eq. (14), along with the corresponding numerical solution of Eqs. (1) for the same typical case as in Fig. 1. The approximation is so good that it looks like a single curve. In Fig. 3(b) we show the difference between the numerical and analytical solutions. Other cases, with different sets of parameters used, have produced equivalently good results.

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