Universality for SU(2) Yang-Mills theory in 2+1 dimensions

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A comparison is made for SU(2) Yang-Mills theory in (2+1)D between various Hamiltonian results obtained by series expansions, linked cluster expansions, and coupled cluster methods, and the recent Euclidean Monte Carlo results of Teper. A striking demonstration of universality between the Hamiltonian and Euclidean formulations is obtained, once the difference in scales between the two formulations is taken into account. [S0556-2821(96)01815-2]

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I. INTRODUCTION

A theoretical discussion of the SU(2) Yang-Mills theory in 2+1 dimensions [(2+1)D] has been given by Feynman [1], who argued that any correlations would be of finite range, and that the flux between external charges is restricted to a ''tube'' of finite extent, leading to a strongly confining linear potential. All physical quantities should simply scale according to their physical dimensions in the continuum limit, so that the mass gap behaves as

$$Ma \sim c_1 g^2 \quad \text{as} \quad a \to 0, \tag{1.1}$$

and the string tension behaves as

$$\sigma a^2 \sim c_2 g^4 \quad \text{as} \quad a \to 0, \tag{1.2}$$

where a is the lattice spacing, and $g^2 = e^2 a$ is the dimensionless coupling.

Numerical treatments have borne out these expectations very well [2]. In the Hamiltonian formulation, most studies have employed some form of linked cluster expansion, such as strong-coupling series expansions [3,4], coupled-cluster expansions [5,6], and the so-called ''exact linked-cluster expansion'' (ELCE) [3]. The only new results to be presented here are some extended coupled-cluster expansions.

In the Euclidean formulation, Teper [7] has recently applied the full power of modern Monte Carlo techniques to this model. His results are very accurate, and now provide the benchmark with which all other approaches must be compared at weak coupling. In this report, we show a comparison between the available Hamiltonian results, and Teper's Euclidean estimates.

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II. COUPLED-CLUSTER EXPANSION METHOD

The lattice Hamiltonian is given by

$$H = \frac{g^2}{2a}\left\{ \sum_l E_l^a E_l^a - \lambda \sum_p \text{Tr}U_p \right\}, \tag{2.1}$$

where E_l^a is a component of the electric field at link l, $\lambda = 4/g^4$, and U_p denotes the product of four link operators U_l around an elementary plaquette. The commutation relation between electric field and link operators at each link may be taken as

$$[E_l^a, U_l] = \tfrac{1}{2}\tau^a U_l, \tag{2.2}$$

choosing the E_l^a as left generators of SU(2). For calculational purposes, it is most convenient to ''scale out'' a factor g^2/a, and work with the dimensionless Hamiltonian

$$H = \tfrac{1}{2}\sum_l E_l^a E_l^a - \frac{\lambda}{2}\sum \text{Tr}U_p. \tag{2.3}$$

This will be assumed henceforth, unless stated otherwise.

The coupled-cluster expansion method has been extensively used in many-body theory, and has recently been introduced to lattice gauge theory by Bishop [8], Llewellyn-Smith and Watson [6], and Guo *et al.* [5], although each uses a different truncation scheme. The basic idea of this expansion is to assume the ground state $|\Psi_0\rangle$ and first excited state (the glueball wave function) $|\Psi\rangle$ of the Hamiltonian in Eq. (2.1) can be represented by an exponential form

$$|\Psi_0\rangle = e^{R(U)}|0\rangle,$$
$$|\Psi\rangle = F(U)e^{R(U)}|0\rangle, \tag{2.4}$$

where $R(U)$ and $F(U)$ are functions of loop variables and the state $|0\rangle$ is the strong-coupling ground state, defined by

$$E_l^a|0\rangle = 0. \tag{2.5}$$

The eigenvalue equation for H can then be written as

$$\sum_{l}\{[E_l,[E_l,R]]+[E_l,R][E_l,R]\}-\frac{4}{g^4}\sum_p \mathrm{Tr}(U_p)=\frac{2a}{g^2}\epsilon_0, \tag{2.6}$$

$$\sum_{l}\{[E_l,[E_l,F]]+2[E_l,F][E_l,R]\}=\frac{2a}{g^2}(\epsilon_1-\epsilon_0),$$

where ϵ_0 (ϵ_1) is the ground state (the first excited-state) energy. $R(U)$ and $F(U)$ can be decomposed according to the order of graphs,

$$R=\sum_i R_i, \tag{2.7}$$

$$F=\sum_i F_i,$$

and the lowest order term of R and F is

$$R_1=c_1\square, \tag{2.8}$$

$$F_1=f_1\square.$$

The graphs of order i are generated by

$$\sum_{j=1}^{i-1}[E_l,R_j][E_l,R_{i-j}] \tag{2.9}$$

in Eq. (2.6). In order to make the calculation possible, some truncation scheme to truncate the eigenvalue equation must be used. The truncation scheme used by Llewellyn-Smith and Watson [6] is

$$\sum_l\left\{\left[E_l,\left[E_l,\sum_{i=1}^{n}R_i\right]\right]+\sum_{i,j=1}^{n}[E_l,R_i][E_l,R_j]\right\}-\frac{4}{g^4}\sum_p \mathrm{Tr}(U_p)=\frac{2a}{g^2}\epsilon_0, \tag{2.10}$$

$$\sum_l\left\{\left[E_l,\left[E_l,\sum_{i=1}^{n}F_i\right]\right]+2\sum_{i,j=1}^{n}[E_l,F_i][E_l,R_j]\right\}=\frac{2a}{g^2}(\epsilon_1-\epsilon_0),$$

where the new graphs generated by $[E_l,R_i][E_l,R_j]$ and $[E_l,F_i][E_l,R_j]$ are simply discarded. Guo *et al.* [5] have argued that because of this the continuum limit of this system could not be preserved, and they have proposed a better truncation scheme:

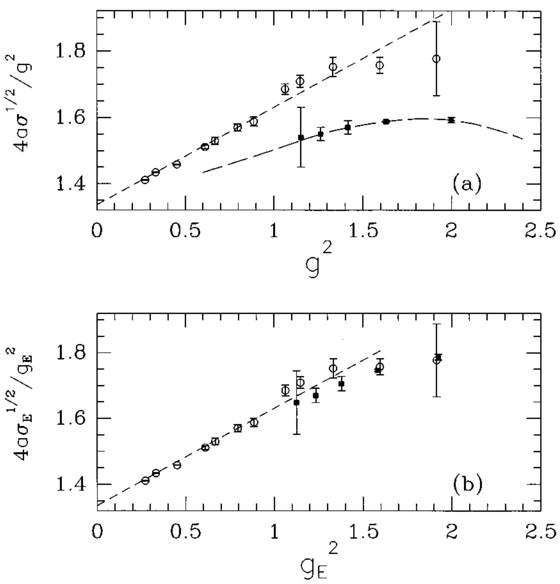

FIG. 1. (a) The string tension σ at weak coupling. Circles: Euclidean MC data [7]; squares: ELCE data [3]. (b) A comparison at weak couplings g_E between Euclidean MC [7] and ELCE [3] estimates of the string tension, including one-loop correction effects.

$$\sum_l\left\{\left[E_l,\left[E_l,\sum_{i=1}^{n}R_i\right]\right]+\sum_{i+j\leqslant n}[E_l,R_i][E_l,R_j]\right\}-\frac{4}{g^4}\sum_p \mathrm{Tr}(U_p)=\frac{2a}{g^2}\epsilon_0, \tag{2.11}$$

$$\sum_l\left\{\left[E_l,\left[E_l,\sum_{i=1}^{n}F_i\right]\right]+2\sum_{i+j\leqslant n}[E_l,F_i][E_l,R_j]\right\}=\frac{2a}{g^2}(\epsilon_1-\epsilon_0).$$

The most tedious task for the high-order approximations is to generate a list of independent loop configurations and derive the nonlinear coupled equations. So far, all these calculations in lattice gauge theory have been carried out by hand. We have tried to develop computer algorithms to overcome this problem. Borrowing some ideas from our computer algorithms used to generate a list of clusters for our linked-cluster series expansions and t expansions [10], a preliminary program was developed. Up to fourth order, a list of 70 graphs was generated, whereas Llewellyn-Smith and Watson [6] obtained only 69 graphs by hand. Some results from the truncation scheme (2.11) were presented in a previous paper [9]. Here, we make a comparison of the results of different truncation schemes with the Green's function Monte Carlo (GFMC) simulation.

III. RESULTS

A. String tension

Figure 1(a) shows some earlier Hamiltonian estimates of the string tension by Hamer and Irving [3] obtained using the

ELCE method, together with the Euclidean results of Teper [7]. A number of points may be noted.

(i) The Euclidean MC results are an order of magnitude more accurate than any Hamiltonian estimates to date, and extend to weaker couplings.

(ii) On the positive side, it can be seen that the ELCE estimates for the string tension approach a limit very similar to Teper's results [7] as $g \to 0$. This provides a nice evidence of universality between the Euclidean and Hamiltonian formulations in the continuum limit.

Let us expand on the last point a little further. When g is nonzero, the Euclidean and Hamiltonian results are not directly comparable because there is a difference in scales (i.e., coupling) between the two formulations, and also the speed of light in the Hamiltonian formulation is not equal to unity. In four dimensions, the relationship between the two scales was calculated long ago to one-loop order in weak-coupling perturbation theory by Hasenfratz and Hasenfratz [11]. The calculation has recently been repeated for the three-dimensional case by one of us [12]. The results for this model may be summarized as

$$\frac{1}{g_H^2} = \frac{1}{g_E^2} - 0.01924 + O(g_E^2), \tag{3.1}$$

where g_H, g_E are the couplings in the Hamiltonian and Euclidean models, respectively, and

$$c = 1 - 0.08365 g_E^2 + O(g_E^4) \tag{3.2}$$

for the ''speed of light,'' with the Hamiltonian normalized as in Eq. (2.1). To make a direct comparison between string tension estimates, the ''timelike'' ELCE estimates must be divided by a factor of c, and shifted from the coupling g_H to the equivalent g_E, given by Eq. (3.1). The revised estimates are then given by

$$\frac{4a\sigma_E^{1/2}}{g_E^2} = \frac{4a\sigma_H^{1/2}}{g_H^2}[1 + 0.0616 g_E^2] \tag{3.3}$$

and are compared with Teper's results in Fig. 1(b). It can be seen that the two sets of estimates now lie almost on the same curve, and the remaining discrepancy (of order 3%) may easily be attributed to higher-order (two-loop) corrections. This provides an even stronger evidence of universality between the two formulations.

B. Mass gap

There have been a number of Hamiltonian estimates of the mass gap using various linked cluster expansion techniques [5,6], and it is interesting to compare these with the Euclidean results of Teper [7].

Let us begin with estimates of the dimensionless ratio $M/\sqrt{\sigma}$. Figure 2 compares the results obtained by Hamer and Irving [3] with Teper's estimates [7]. The Hamiltonian estimates have been slightly shifted from coupling g_H^2 to the equivalent coupling g_E^2 using Eq. (3.1), and ''renormalized'' by the speed of light c according to

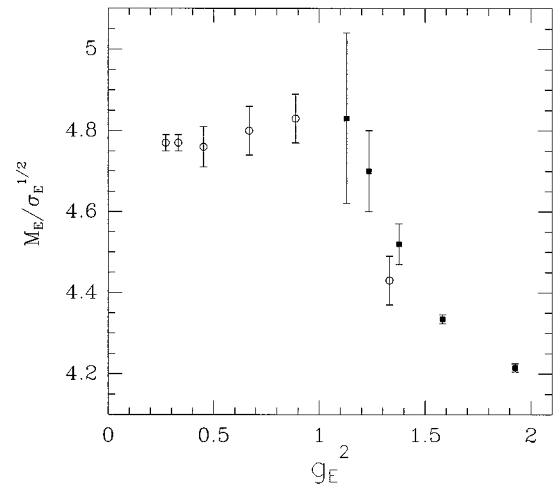

FIG. 2. The ratio $M_E/\sqrt{\sigma_E}$ plotted as a function of $\beta = 4/g^2$. Circles: Euclidean MC data [7]; squares: ELCE data [3].

$$\frac{M_E}{\sqrt{\sigma_E}} = \frac{M_H}{\sqrt{\sigma_H c}} = \frac{M_H}{\sqrt{\sigma_H}}(1 + 0.0418 g_E^2). \tag{3.4}$$

It can be seen that the two sets of data match rather well: once more, in excellent agreement with universality.

Next, Fig. 3 shows a number of different Hamiltonian estimates of the mass gap itself, or more precisely, of the quantity Ma/g^2, which should approach a finite value in the continuum limit. The strong-coupling series expansion estimates [4] rise fairly abruptly around $\beta = 4/g^2 \simeq 3$, and then begin to level off towards an asymptotic value estimated previously [4] as 2.22(5). The higher-order coupled-cluster

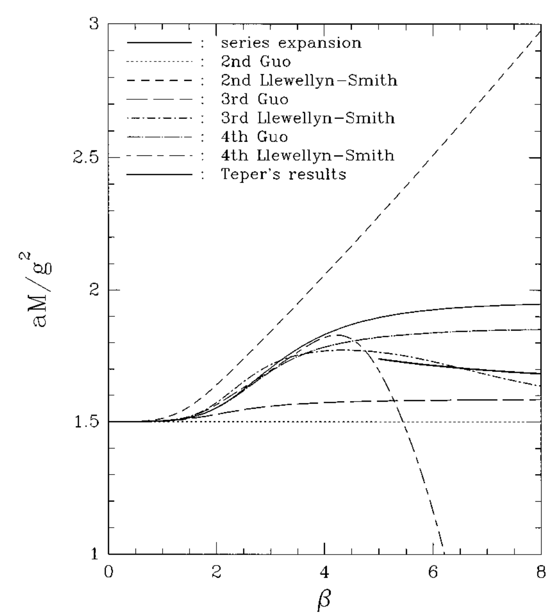

FIG. 3. Estimates of aM/g^2, where M is the mass gap, as a function of $\beta = 4/g^2$. Series expansion results from Ref. [4]; coupled-cluster estimates from the truncation scheme used by Llewellyn-Smith and Watson [6] and Guo *et al.* [5]; the heavy line is an estimate inferred from Teper [7].

estimates from the truncation schemes used by Llewellyn-Smith and Watson [6] and Guo *et al.* [5] also show a rise until $\beta \simeq 4$, but their behavior beyond that point is somewhat variable. Teper [7] does not give direct estimates of the mass gap itself; but using his results for σ and the ratio $M/\sqrt{\sigma}$, together with Eqs. (3.1) and (3.2), one may infer a behavior for the Hamiltonian mass gap at weak coupling which is shown as a heavy solid line in Fig. 3. It would appear that Ma/g^2 actually reaches a peak at around $\beta \simeq 4$, and then declines slowly towards the asymptotic value 1.59(2). The strong-coupling series extrapolations were unable to pick up this decrease at weak coupling. Some of the coupled-cluster approximants perhaps give some indication of it, but not in a very consistent or reliable manner.

IV. DISCUSSION

The most interesting result of this paper is the remarkable demonstration of universality between the Euclidean and Hamiltonian formulations which has been obtained. If account is taken of the difference in scale between the two formulations as predicted in one-loop perturbation theory [12], then the Euclidean results of Teper [7] and the Hamiltonian results of Hamer and Irving [3] for the string tension and the ratio $M/\sqrt{\sigma}$ fall almost on top of each other, not only in the continuum limit, but over a whole range of weak couplings. This provides a pleasing confirmation of this hypothesis of universality. There is little doubt that the hypothesis is correct, but it is important to check it wherever possible, since it underpins the whole program of lattice gauge theory.

In the case of the mass gap itself, it was possible to compare the results of several different linked-cluster expansion techniques [4–6]. Out to $\beta \simeq 4$, they were found to agree with each other quite well. Beyond that point, the convergence is poor. Teper's data [7] imply a gradual decline in the scaled mass gap aM/g^2 to an asymptotic limit of 1.59(2), which is not clearly indicated by any of the expansion methods. It appears, once again, that one must be cautious about placing too much trust in extrapolated linked-cluster expansions deep in the weak-coupling regime. This is not surprising, since the cluster expansions are basically strong-coupling expansions of one or another form, and they cannot be expected to converge very well in the neighborhood of the weak-coupling (continuum) limit, which is likely to be an essential singularity.

The linked-cluster expansion techniques have an important role to play, nevertheless. They are presently more accurate than every Hamiltonian Monte Carlo method for excited states. They can also be used for models containing dynamical fermions, for instance, which at present are inaccessible by Hamiltonian Monte Carlo methods.

At present, we are trying to improve the coupled-cluster expansion technique [14] by combining it with the D-function expansion used in the ''ELCE'' approach of Irving, Preece, and Hamer [13]. This avoids both the use of the Cayley Hamilton relation for the elimination of redundancies, and also the explicit handling of many SU(n) coupling coefficients, and allows one to define the truncation with respect to an orthogonal basis. The incorporation of states in the spectrum having arbitrary lattice momentum and lattice angular momentum has also been possible in this framework [14].

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