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Extraction of V_{ub} from inclusive B decays and the resummation of end point logarithms

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In this paper we discuss the theoretical difficulties in extracting V_{ub} using the data from inclusive B decays. Specifically, we address the issue of end point singularities. We perform the resummation of both the leading and next to leading end point logarithms and include the leading corrections to the hard scattering amplitude. We find that the resummation is a 20%–50% effect in the end point region where the resummation is valid. Furthermore, the resummed subleading logarithms dominate the resummed double logarithms. The consequences of this result for a model-independent extraction of the mixing angle V_{ub} are explored. [S0556-2821(96)02915-3]

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I. INTRODUCTION

Measurements in the bottom quark sector have reached the point that our knowledge of many observables is now bounded by the theoretical uncertainties [1]. Fortunately, theoretical advances in calculating both exclusive as well as inclusive rates now allow the extraction of the Cabibbo-Kobayaski-Maskawa (CKM) parameters without recourse to the models which have soiled the extraction processes to date. The present values of V_{ub} have a model dependence which introduce an uncertainty of a factor of 2 [1], which is several times larger than the experimental uncertainties. With QCD-based calculations, we can now hope to extract both V_{bc} and V_{ub} with errors on the order of tens of a percent. In this work, we concentrate on the extraction of V_{ub} from the measurement of the electron spectrum in semileptonic inclusive B meson decays.

The extraction of V_{ub} from inclusive semileptonic B decays is hindered by the fact that the background from charmed decays is overwhelming for most of the range of the lepton energy. Thus, we are forced to make a cut on the lepton energy, vetoing all events, or some large fraction thereof, with lepton energy less than the $b \rightarrow c$ end point energy. Given the proximity of the two relevant end points, this obviously hinders the statistics. However, even with a large data sample, the accuracy of the extraction will be limited by the errors induced from the approximations used in calculating the theoretical prediction in the end point region. This region of the Dalitz plot is especially nettlesome for theory, because the perturbative as well as the nonperturbative corrections become large when the lepton energy is near its end point value.

It has been shown that it is possible to calculate the decay spectrum of inclusive heavy meson decay in a systematic expansion in $\epsilon = \Lambda_{\rm QCD}/m_b$ and α_s using an operator product expansion within the confines of heavy quark effective field theory [2]. It is possible to Euclideanize the calculation of the rate for most of the region of the Dalitz plot with only minimal assumptions about local duality. However, in the end point region, the expansion in ϵ, as well as the expansion in α_s, begins to breakdown. (The end point region poses problems for local duality as well. We shall discuss this in more detail later.)

The aim of this paper is to determine the size of the errors induced from the theoretical uncertainties in the extraction of V_{ub}. A large piece of this work consists of implementing the resummation of the leading and subleading end point logarithms which cause the breakdown of the expansion in α_s, as first discussed on general grounds in [3], and the inclusion of the α_s corrections to the hard scattering amplitude. However, to determine the consistency of our calculation, we must also address the issue of the nonperturbative corrections. These issues have been previously looked at in Refs. [4] and [5]. In [4] the need for resummation was addressed on general grounds. However, the calculational methods used here are not compatible with the arguments given in [4], and thus we must recapitulate these arguments within the confines of our methods.

In the second section of this paper, we discuss the question of the need to resum the perturbative as well as nonperturbative series. The next three sections are dedicated to the resummation of the leading and next-to-leading infrared logarithms and the inclusion of the one-loop corrections to the hard scattering amplitude (that is, one-loop matching). In the fifth section we give our numerical results while the last section draws conclusions regarding what errors we can expect in the extraction process.

II. IS RESUMMATION NECESSARY?

As mentioned above, the theoretical calculation of the lepton spectrum in inclusive decays breaks down near the end point. Both the nonperturbative as well as perturbative corrections become large in this region. Here we investigate the need to perform resummations in either or both of these expansions. The one-loop decay spectrum including the leading nonperturbative corrections is given by [6]

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dx}=\theta(1-x)\left[x^2(3-2x)\left(1-\frac{2\alpha_s}{3\pi}\right)I(x)+2(3-x)x^2E_b-\frac{2}{3}x^2(9+2x)K_b-\frac{2}{3}x^2(15+2x)G_b\right]+\left[E_b-\frac{2}{3}K_b+\frac{8}{3}G_b\right]\delta(1-x)+\frac{1}{3}K_b\delta'(1-x), \tag{1}$$

where

$$I(x)=\ln^2(1-x)+\frac{31}{6}\ln(1-x)+\pi^2+\frac{5}{4},\quad x=\frac{2E_e}{m_b}, \tag{2}$$

$$\Gamma_0=|V_{ub}|^2\frac{G_F^2m_b^5}{96\pi^3}. \tag{3}$$

E_b, G_b, and K_b are hadronic matrix elements of order ϵ^2 and are given by

$$E_b=G_b+K_b,$$

$$K_b=\left\langle B(v)\left|\bar{b}_v\frac{D^2}{2m_b^2}b_v\right|B(v)\right\rangle,$$

$$G_B=\left\langle B(v)\left|\bar{b}_v g\frac{\sigma_{\alpha\beta}G^{\alpha\beta}}{4m_b^2}b_v\right|B(v)\right\rangle. \tag{4}$$

b_v is the velocity-dependent bottom quark field as defined in heavy quark effective field theory. From the above expressions we see that the breakdown of the expansions, in α_s and $\epsilon=\Lambda_{\rm QCD}/m_b$, manifests itself in the large logarithms and the derivative of δ functions, respectively.

A. Nonperturbative expansion

As one would expect for heavy meson decay, the leading order term in ϵ reproduces the parton model result. All corrections due to the fact that the b quark is in a bound state are down by ϵ^2 [2]. However, near the end point of the electron spectrum we begin to probe the nonperturbative physics. The general form of the expansion in $\epsilon=\lambda/m_b$, to leading order in α_s, is given as

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dx}=\theta(1-x)(\epsilon^0+\epsilon^2+\cdots)+\delta(1-x)(0\epsilon+\epsilon^2+\epsilon^3+\cdots) \tag{5}$$

$$+\cdots+\delta^{(n)}(1-x)(\epsilon^{n+1}+\epsilon^{n+2}+\cdots)+\cdots. \tag{6}$$

The end point singularities are there because the true end point is determined by the mesonic mass and not the partonic mass, as enforced by the θ function in the leading order term. The difference between these end points will be on the order of a few hundred MeV. To make sense of this expansion we must smear the decay amplitude with some smooth function of x. Normally, this would not pose a problem; however, given that the distance between the $b\to c$ and $b\to u$ end points is approximately 330 MeV, we are forced to integrate over a weighting function which has support in a relatively small region. On the other hand, if the weighting function is too narrow, then the expansion in ϵ will not be well behaved.

Thus we must find a smearing function that minimizes the errors due to $\Lambda_{\rm QCD}/m_b$ corrections which does not overlap with the energy region where we expect many $b\to c$ events. The question then becomes how many $b\to c$ transitions we can allow without introducing large errors due to our ignorance of the $b\to c$ end point spectrum (the theory breaks down in the $b\to c$ end point region as well though there are important differences between this case and the $b\to u$ transitions).

The issue of smearing was addressed by Falk *et al.* [4] who used Gaussian smearing functions to gain quantitative insight into the need for smearing. They found that without

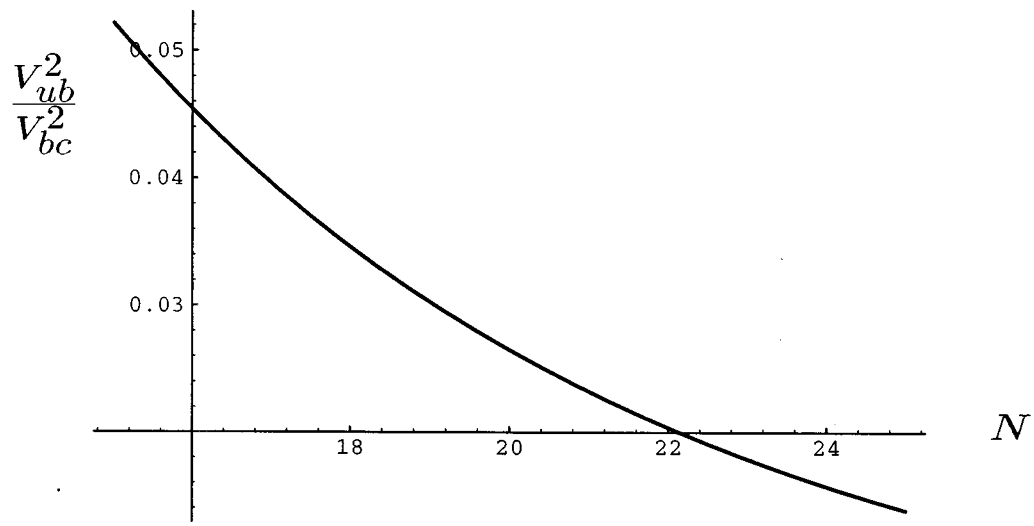

FIG. 1. The ratio of V_{ub}^2/V_{bc}^2 for which the Nth moments of the leading order spectra are equal.

any resummation, the smearing function should have a width which is greater than ϵ, but that after resumming the leading singularities, we need smear only over a region of width ϵ. Here we will smear by taking moments of the electron energy spectrum (we work with the moments of the spectrum because it greatly facilitates the resummation of the perturbative corrections). Thus, we must address the question of what range of values of N will lead to a sensible expansion which is also not overly contaminated by $b \to c$ transitions. This will obviously depend on the ratio of V_{ub} to V_{bc}. To get a handle on the numerics, let us for the moment assume that we wish that the number of $b \to u$ transitions be at least equal to the number of $b \to c$ transitions in our sample. In Fig. 1, we plot $(V_{ub}^2/V_{bc}^2)(N)$, which is the ratio of mixing angles for which the Nth moments of the leading order rates for $b \to c$ and $b \to u$ transitions will be equal, and is given by

$$\frac{V_{ub}^2}{V_{bc}^2}(N) = \frac{\int_0^{x_m} x^2[(x_m-x)^2/(1-x)^3][6-3x_m+(x_m-6)x+2x^2]x^N dx}{\int_0^1 x^2(3-2x)x^N}. \tag{7}$$

x_m is $2E_{\max}/m_b$ for the $B \to D$ transitions and takes the value $x_m \approx 0.9$. Given the bounds [7]

$$0.002 < \frac{|V_{ub}|^2}{|V_{bc}|^2} < 0.024, \tag{8}$$

we see that an understanding of the spectrum for moments around $N \simeq 20$ is necessary if we wish to keep the $b \to c$ contamination under control. (Of course we do not suggest that these moments can be measured given the finite resolution of the experiment. We will discuss this situation later in the paper.)

We now consider the issue of determining the maximum value of N for which the expansion makes sense. Let us first consider the expansion in ϵ. The moments of the leading singularities of Eq. (5) will behave as

$$M_N \approx C_n \frac{N!\,\epsilon^{n+1}}{(N-n)!}. \tag{9}$$

As a possible criterion on the size of N, we may impose that there be no growth with n. That is,

$$\frac{N!\,\epsilon^{n+1}[\ln\epsilon + \Psi(N-n+1)]}{(N-n)!} < 0. \tag{10}$$

ϵ represents the value of some matrix element in the heavy quark effective theory. It is assumed that the value of ϵ should be on the order of a few hundred MeV$/m_b$, but in theory it could vary by a factor of order 1 from term to term. To get a handle on the sizes of ϵ, we may consider the leading ϵ, which is given by

$$\epsilon_1 = \left\langle B \left| \bar{b}_v \frac{(iD)^2}{2} b_v \right| B \right\rangle \frac{1}{m_b^2}. \tag{11}$$

Quark model calculations suggest that ϵ_1^2 is on the order of 0.01 [8]. Thus, naively, it seems that to keep the expansion in ϵ under control, we must keep $N \leqslant 10$. This estimate is perhaps too crude for our purposes given that we know nothing of the growth of the coefficients C_n or of the range of possible values of ϵ_N. It does suggest that some sort of resummation may be necessary.

Neubert [8] pointed out that it is possible to resum the leading singularities, much as in the case of deep inelastic scattering, into a nonperturbative shape function

$$\tilde{f}(k_+) = \langle B | \delta(k_+ - iD_+) | B \rangle. \tag{12}$$

This function gives the probability to find the b quark within the hadron with residual light cone momentum k_+. Thus, this function is roughly determined by the kinetic energy of the b quark inside the meson. This structure function will be centered around zero and have some characteristic width δ. δ will determine the maximum size of N for which the expansion without resummation makes sense. To get a well-behaved expansion we choose N such that x^N gives order 1 support to the structure function throughout its width. The value of δ is unknown at this time, and various authors have given different estimates for its value. We can assume that this width should be on the order of $(m_B - m_b)/2$ which is around 300 MeV. We shall choose what we believe to be the conservative value of 500 MeV for δ. Since the structure function is the sum of derivatives of δ functions, we conclude that we should smear over the width of the function if we do not wish to incur large errors. Let us assume, for the sake of numerics, that x^N should not fall below the value 0.1, within 500 MeV of the end point. Then we find that N must be $\leqslant 20$. Thus, we expect that the nonperturbative effects could be quite large for the range of N that we consider here. Of course when N becomes very large, $N > 100$, it is necessary to go beyond the leading twist since the soft gluon exchange in the t channel begins to dominate, not to mention the failure of the operator product expansion (OPE) due to its asymptotic nature [9].

We see that for our purposes we should include the nonperturbative structure function in our calculation. The fact that knowledge of this nonperturbative function is needed to extract V_{ub} should not bother us too much, however, given that it is universal. That is to say, we can remove it from our final result by taking the appropriate ratio [10]. Or it can be measured on the lattice, much in the same way that the moments of the proton structure functions are now being measured. Using the Altarelli-Cabibbo-Corbo-Maiani-Martinelli (ACCMM) model [11] Blok and Mannel [5] concluded that a resummation of the nonperturbative corrections is unnecessary. If the width of the structure function is smaller than the

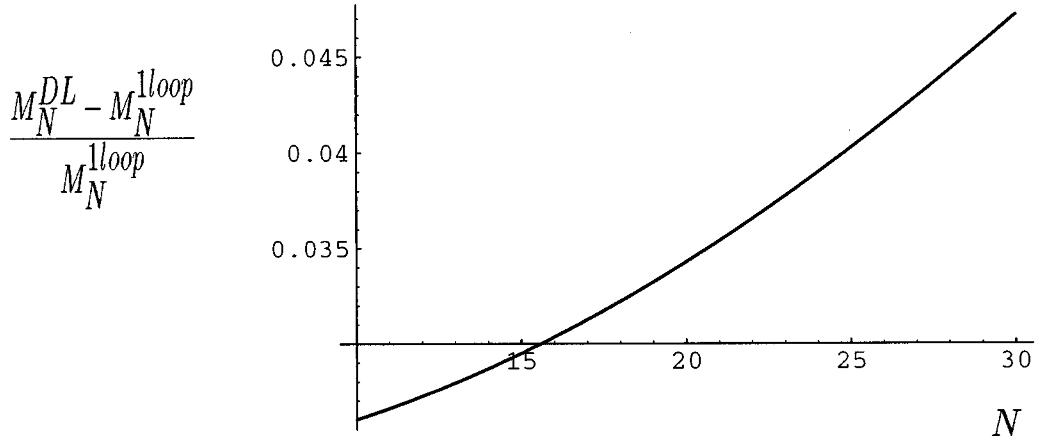

FIG. 2. The difference between the moments given by the leading α_s correction and the moments of the rate with only the double logarithms resummed.

conservative number chosen here, then this could very well be true. This would be a welcome simplification of the extraction process, since we would no longer need to rely on the extraction of nonperturbative parameters from other processes to measure the mixing angle V_{ub}.

B. Perturbative expansion

Let us now address the issue of the perturbative corrections. The corrections in α_s grow large near the electron energy end point, and, precisely at the end point, there are logarithmic infrared divergences. These divergences are due to the fact that near the end point gluon radiation is inhibited, and as a result, the usual cancellation of the infrared divergences between real and virtual gluon emission is nullified. Of course, the rate is not divergent, and we expect that a resummation procedure will have the effect of reducing the rate for the exclusive process.

Near the end point large logarithms form a series of the form

$$\begin{aligned}\frac{d\Gamma}{dx} &= C_{11}\alpha\ln^2[1-x]+C_{12}\alpha\ln[1-x]+C_{13}\alpha\\ &\quad + C_{21}\alpha^2\ln^4[1-x]+C_{22}\alpha^2\ln^3[1-x]\\ &\quad + \cdots + C_{31}\alpha^3\ln[1-x]^6+C_{32}\alpha^3\ln^5[1-x]\\ &\quad + \cdots + \cdot \ \ + \cdot \ \ + \cdots + \cdot \ \ + \cdot \ \ + \cdots,\end{aligned}$$

which in terms of a moment expansion gives[1] (for large N)

$$\begin{aligned}\int_0^1 dx x^N\frac{d\Gamma}{dx} &= \frac{1}{N}(\tilde{C}_{11}\alpha\ln^2N+\tilde{C}_{12}\alpha\ln N+\tilde{C}_{13}\alpha\\ &\quad + \tilde{C}_{21}\alpha^2\ln^4N+\tilde{C}_{22}\alpha^2\ln^3N+\cdots\\ &\quad + \tilde{C}_{31}\alpha^3\ln N^6+\tilde{C}_{32}\alpha^3\ln^5N+\cdots). \quad (13)\end{aligned}$$

[1]This form holds for $b\to s\gamma$; for the semileptonic decay, we will consider the moments of the derivative of the rate.

Given this expansion, we may ask what errors we expect to incur by truncating the expansion at order α_s. For N near 20, we see that

$$\frac{\alpha_s}{\pi}\ln^2N\simeq 0.6, \quad (14)$$

and so we might expect that truncating at leading order would not be such a good idea. We must also note that in Eq. (1) the subleading logarithm actually dominates the leading double logarithm due to the large coefficient $\frac{31}{6}$. The resummation of the double logarithms is simple and leads to the exponentiation of the double logarithms. Figure 2 shows the difference between the one-loop result and the result with only the double logarithms resummed. We see that the difference is very small, on the order of 5%. Thus, one might come to the conclusion that no resummation of the perturbative series is necessary. However, given that the coefficients of the single logarithms as well as the π^2, which are just as large as the double logarithms for the range of N we are considering here, are unknown at higher orders, we can only determine the errors induced by a truncation of the series after we have performed the resummation.

Resumming the leading double logarithms in itself does not increase the range in N over which perturbation theory is valid. Even after this resummation is performed the criteria for a convergent expansion is still $(\alpha_s/\pi)\ln^2N<1$ unless we know that the subleading logarithms exponentiate as well. However, one can show on very general grounds [12] that all the end point logarithms exponentiate as a consequence of the fact that these logarithms are really just UV logarithms in the effective field theory [13,14]. Thus, it is always possible to write down a differential equation for the rate based on its factorization scale independence. As such, the general form of the decay amplitude will be given by

$$\ln\left[\int_0^1\frac{d\Gamma}{dx}x^N\right]=C(\alpha_s)+\sum_{n=1}^{\infty}\alpha_s^n\sum_{m=1}^{2n}G_{nm}\ln^mN. \quad (15)$$

Once we have this information, the question of the region of convergence becomes the following. Do the lower order terms in question contribute numbers of order 1 in the exponent? We may continue to increase N until we find that the subleading terms in the exponent contribute on the order 1. Thus in general resumming the leading logarithms does indeed allow us to take N into the range where $(\alpha_s/\pi)\ln^2 N \simeq 1$. Here we will go further, as was done in [3], and sum the next-to-leading logarithms as well, allowing, $(\alpha_s/\pi)\ln N \leqslant 1$. This will allow us to determine the convergence of the expansion. Furthermore, we extend the analysis of [3] to include the one-loop matching corrections, thus completing the calculation at order α_s.

We wish to note that Blok and Mannel [5] analyzed the effects of the large logarithms to the end point spectrum and concluded that no resummations were necessary. These authors propose to take the lower bound on the moment integral to be the charmed quark end point x_c. Doing this allows one to stay away from larger values of N (the authors choose $N<10$). Cutting off the integral introduces errors that have the doubly logarithmic x_c dependence $\ln^2(1-x_c)$. To reduce these errors, it is necessary to go to higher values of N. These authors claim that for $N<10$ the errors induced by cutting off the integral are small, on the order of a few percent. However, we believe that these authors have underestimated their errors because they normalized their errors by the total width and not the moments themselves. Furthermore, and perhaps most importantly, the authors did not consider the possibility that the subleading logarithms could dominate the leading logarithms in the resummation, which as we shall see is indeed the case.

Finally, it should be pointed out that aside from being bounded by the size of the logarithms, N is bounded on purely logical grounds. The whole perturbative QCD framework loses meaning when the time scale for gluon emission becomes on the order of the hadronization time scale. This restriction bounds the minimum virtuality of the gluon, which we expect to be on the order of m_b/N (we will show this to be true when we perform the resummation). Thus, performing resummations can only take one so far no matter how powerful one is. However, for top quark decays it is possible to get extremely close to the end point due to the large top quark mass. In this case it is clear that the resummation of the next-to-leading logarithms will become essential. Thus, the extraction of V_{td} from inclusive top quark decays will have much smaller theoretical errors than in the b decay case. We shall discuss the issue of the breakdown of perturbation theory in greater detail after we perform the resummation.

III. FACTORIZATION

The large logarithms appearing in the perturbative expansion arise from the fact that at the edge of phase space gluon emission is suppressed. The problem of summing these large corrections has been treated previously for various applications, such as deep inelastic scattering and Drell-Yan processes [12,15], just to mention a few. The case of inclusive heavy quark decay has been treated previously in [3]. An important ingredient of the resummation procedure is the proof of factorization. As applied to the present processes, this procedure separates the particular differential rate under consideration into subprocesses with disparate scales.

In the case of inclusive semileptonic heavy quark decays, the relevant scales are m_b and $m_b(1-x)$, with $x=2E_e/m_b$ in the rest frame of the b quark. To understand how to best factorize the differential rate in the limit $x\to 1$, we need to know the momentum configurations which give leading contributions in that limit. With this in mind, let us consider the inclusive decay of the b quark into an electron and neutrino of momenta p_e and p_ν, respectively, and a hadronic jet of momenta p_h. First we note that the kinematic analysis is simplified with the following choice of variables in the rest frame of the b quark [16]

$$x=\frac{2E_e}{m_b},\quad y_0=\frac{2(E_e+E_\nu)}{m_b},\quad y=\frac{(p_e+p_\nu)^2}{m_b^2}. \tag{16}$$

The kinematic ranges for these variables are

$$0\leqslant x\leqslant 1,\quad 0\leqslant y\leqslant x,\quad (y/x+x)\leqslant y_0\leqslant (y+1). \tag{17}$$

Furthermore, define the variable

$$\eta=\left(\frac{1-y}{2-y_0}\right)\quad \text{where}\quad x\leqslant\eta\leqslant 1. \tag{18}$$

This variable plays an analogous role to the Bjorken scaling variable in deep inelastic scattering phenomena. The invariant mass of the final state hadronic jet and its energy are given by

$$p_h^2=m_b^2(1-\eta)(2-y_0),\quad p_h^0=\frac{m_b}{2}(2-y_0). \tag{19}$$

We should note that in determining the boundary values of the various variables we refer to the b-quark mass and not to that of the meson. This is justified within the perturbative framework we are working in at the moment. However, once we include the effects of the nonperturbative structure function, the phase space limits will take on their physical values.

Let us now investigate the dominant momentum configurations near $x\to 1$. First, we observe that the invariant mass of the hadronic jet + neutrino system is given by $(p_b-p_e-p_\nu)^2=m_b^2(1-x)$ which vanishes at the end point. The phase space configuration where the neutrino is soft is suppressed and hence, when the value of x approaches 1, the electron and the hadronic jet-neutrino system move back to back in the rest frame of the b quark. Furthermore, the invariant mass of the hadronic jet vanishes independently of the neutrino energy. This is readily verified using the phase space boundaries. The energy of the jet is large except near the point $x\to y\to 1$. In this region of the Dalitz plot factorization breaks down, and the techniques used here fail. However, this problematic region is irrelevant as a consequence of the fact that the rate to produce soft massless fermions is suppressed at the tree level. Thus, the following picture emerges at $x\sim 1$. The b quark decays into an electron moving back to back with the neutrino and a lightlike hadronic jet. We choose the electron to be moving in the + (light cone) direction, and the jet moves in the − (light cone) direction in the rest frame of the b quark. The constituents of the jet may interact via soft gluon radiation with each other and with the b quark, but hard gluon exchange is disallowed.

This simple picture is related by the Coleman-Norton theorem [17,18] to the type of Feynman diagrams that are

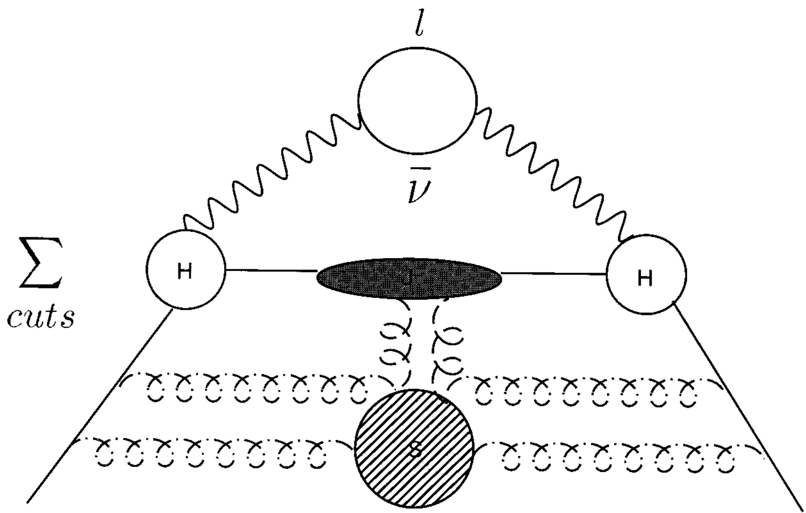

FIG. 3. Reduced diagram for B decays.

infrared sensitive. According to this theorem, if we construct a ''reduced'' diagram by contracting all off-shell lines to a point, then at the infrared singular point, such a diagram describes a physically realizable process. Thus, at $x \sim 1$ the type of diagrams that give large logarithms is precisely those described above and shown in Fig. 3 [31].

In the figure, S denotes a soft blob which interacts with the jet and the b quark via soft lines. J denotes the hadronic jet and H the hard scattering amplitude. The typical momenta flowing through the hard subprocess are $O(m_b)$. H does not contain any large end point logarithms and has a well-defined perturbative expansion in $\alpha_s(m_b)$. All the lines which constitute H are off shell and have been shrunk to a point. The soft function S contains typical soft momentum k, with $k^+ \sim k^- \sim k_\perp = O(m_b(1-x))$. Thus, by ''soft'' we mean soft compared to m_b, but still larger than Λ_{QCD}. The jet subprocess has typical momenta p such that $p^- \gg p^+$, $p_\perp^2$ with p^+, $p_\perp^2 = O(m_b(1-x))$ and $p^- = O(m_b)$. In order to delineate between momentum regimes, a factorization scale μ is introduced. The fact that the process is μ independent will be utilized to sum the large end point logarithms which are contained in the soft and jet functions. The reduced digram for the inclusive radiative decay $b \to X_s \gamma$ is exactly the same as above if we ignore the strange quark mass.

An important consequence of the factorization is the fact that the soft function S is universal. That is, it is independent of the final states as long as factorization holds. Thus the soft function in the semileptonic decay will be the same soft function as in the radiative decay. This universality will allow us to remove our ignorance of any nonperturbative physics due to bound state dynamics by taking the appropriate ratio. Thus, throughout this paper we will treat both the semileptonic as well as the radiative decays in turn.

We conclude this discussion with a few comments. First, we should point out the differences between factorization in the process considered here and in deep inelastic scattering for large values of the Bjorken scaling variable [15]. A crucial difference arises from the fact that the initial quark is massive, and hence, the semi-inclusive decays of the a heavy quark is infrared finite to all orders in perturbation theory because there are no collinear divergences arising from initial state radiation. This fact has the important consequence that the differential decay rate will be independent of μ, whereas μ independence in deep inelastic scattering is only achieved after an appropriate subtraction is made with another process, such as the Drell-Yan process, which has the same collinear divergence structure as the deep inelastic scattering process. Next we note that, in general, the separation of diagrams into soft and jet subprocesses is not unique, and some prescription must be adopted. For a discussion of this issue see [19,12]. In our case, we will determine the proper separation from the requirement of the μ independence of the decay rate from the condition that the hard scattering amplitude does not contain any large end point logarithms, and that the purely collinear divergences in the jet must satisfy an Altarelli-Parisi-like evolution equation. We will return to this point in the next section. The factorization can be made more manifest by going to the lightlike axial vector gauge with the gauge-fixing vector pointing in the jet direction. In this gauge, the soft lines decouple from the jet on a diagram by diagram basis.

In terms of the variables introduced earlier, the triply differential factorized decay amplitude may be written as

$$\frac{1}{\Gamma^0}\frac{d^3\Gamma}{dx\,dy\,dy_0} = 6m_b(x-y)(y_0-x)\int_{k_{\min}^-}^{k_{\max}^+} dk_+ f(k^+,\mu^2) \times J(p_h^-(p_h^+-k^+),\mu^2)H(m_b,\mu^2), \tag{20}$$

$$\Gamma^0 = \frac{G_F^2}{96\pi^3}|V_{ub}|^2 m_b^5. \tag{21}$$

This form will hold up to errors on the order of $O(1-x)$.

We have chosen the electron to be traveling in the $+$ direction with momenta $k = m_b(x,0,0_\perp)$, and

$$k_{\max}^+ = m_b(1-\eta), \quad k_{\min}^+ = -(M_B - m_b). \tag{22}$$

Here $f(k^+)$ is the probability for the b quark to have light cone residual momentum k^+, and thus contains not only the information in the soft function S but also the nonperturbative information regarding the nature of the bound state. If we ignore perturbative ''soft'' gluon radiation, then this function coincides with $\widetilde{f}(k_+)$ defined in the previous section. Notice that $k_+^{\rm min}$ is negative. This is important nonperturbatively and represents the leakedge past the partonic end point due to the soft gluon getting energy and momentum from the light degrees of freedom inside the B meson. Loosely speaking it is due to the Fermi motion of the b quark inside the meson.

Less formally, we may write the derivative of the decay amplitude as [3]

$$\frac{-1}{\Gamma_0}\frac{d}{dx}\left(\frac{d\Gamma}{dx}\right)=\int_1^2 dy_0 6(2-y_0)^2(y_0-1)G(x), \quad (23)$$

$$G(x)=\int_x^{M_B/m_b} dz f(z,m_b/\mu)J(m_b^2(2-y_0)\times(z-x),\mu^2)H(m_b(2-y_0)/\mu). \quad (24)$$

In this equation we have changed variables from k^+ to the residual light cone momentum fraction $z=(1-k^+/m_b)$, and absorbed a factor of m_b^2 into the jet factor.

By taking the moments of this expression with respect to x we see that we are able to treat the hard, soft, and jet functions separately. We are led to the following form for the moments of the semileptonic rate

$$M_N^{\rm sl}\equiv\frac{1}{\Gamma_0}\int_0^{M_B/m_b} x^{N-1}\frac{d}{dx}\left(\frac{d\Gamma}{dx}\right)dx$$
$$=\int_1^2 dy_0 6(2-y_0)(y_0-1)f_N J_N(m_b^2(2-y_0),\mu^2)\times H(m_b(2-y_0),\mu), \quad (25)$$

$$f_N=\int_0^{M_B/m_b} z^N dz f(z), \quad (26)$$

$$J_N(2-y_0)=\int_0^1 \lambda^N J(m_b^2(2-y_0)(1-\lambda,\mu^2))d\lambda. \quad (27)$$

In writing the last few equations we have dropped all terms of order $O(1-x)$ or equivalently taken the large N limit. The left-hand side of Eq. (25) defines the moments of the semileptonic decay electron distribution, $M_N^{\rm sl}$

The moments of the soft function f_N may be decomposed into a product of moments of a perturbatively calculable σ_N and the nonperturbative structure function S_N, which corresponds to the moments of $\tilde{f}$ discussed in the previous section. We may write

$$\tilde{f}(z)=\int_z^{M_B/m_b}\frac{dy}{y}S(y)\sigma\left(\frac{z}{y}\right), \quad (28)$$

and thus, taking moments,

$$f_N=\sigma_N S_N. \quad (29)$$

An analogous situation exists for the decay $B\to X_s\gamma$. We define $x=2E_\gamma/m_b$ in the rest frame of the b quark, and take the photon to be moving in the + direction, and, as in the semileptonic decay, at $x\sim 1$ the hadronic jet is moving in the − direction. Furthermore, the invariant mass of the hadronic jet and its energy are

$$p_h^2=m_b^2(1-x), \quad p_h^0=m_b(1-x/2). \quad (30)$$

Thus the s quark, is very energetic, and since the invariant mass of the jet vanishes as $x\sim 1$, the s quark decays into quanta which are collinear once we ignore effects on the order of m_s^2/m_b^2 Clearly the factorization picture discussed earlier for the semileptonic decay holds here as well and the reduced diagram is the same as in Fig 3.

As before we may take the moments of the differential rate

$$M_N^\gamma\equiv\frac{1}{\Gamma_\gamma}\int_0^{M_B/m_b} dx x^{N-1}\frac{d\Gamma}{dx}=S_N\sigma_N J_N, \quad (31)$$

where [20],

$$\Gamma_\gamma=\frac{\alpha G_F^2}{32\pi^4}m_b^5|V_{tb}V_{ts}^*|^2 C_7^2(m_b) \quad (32)$$

and

$$J_N=\int_0^1 dy y^{N-1}J(m_b^2(1-y),\mu^2). \quad (33)$$

σ_N and S_N are the same functions defined in Eq. (28) and C_7 is the Wilson coefficient of O_7 as defined in [20]. For the radiative decay, the $\ln(1-x)/(1-x)_+$ distribution in the amplitude will correspond to $\ln^2(N)$ in the moment, whereas, in the semileptonic decay, taking the derivative of the amplitude will generate plus distributions which will then generate $\ln N$ and $\ln^2 N$ after taking the moments. Thus, we have reduced the problem of the resummation of the large logarithms in the amplitude to resumming the logarithms in J_N and σ_N separately. This greatly simplifies the calculation as will be seen below.

IV. RESUMMATION

The resummation of the infrared logarithms is analogous to summing ultraviolet logarithms. One takes advantage of the μ independence of the amplitude. In the case of infrared logarithms, μ is the factorization scale or, equivalently, the renormalization scale within the appropriate effective field theory, which for this case would the field theory of Wilson lines [14].

We first outline the derivation of a representation of the soft function σ_N near $x=1$ following the techniques developed in Ref. [15]. We will work in the eikonal approximation where soft momenta are ignored wherever possible. At the one-loop level, the real gluon emission contribution factorizes and the quantity multiplying the tree level rate is

$$F_{\rm eik}^{\rm real}(x)=g^2C_F\int\frac{d^3k}{(2\pi)^3 2k_0}\left(\frac{2p\cdot q}{p\cdot kq\cdot k}-\frac{m_b^2}{(p\cdot k)^2}\right)\times\delta\left(1-x-\frac{2k_0}{m}\right).\tag{34}$$

The δ function enforces the phase space constraint. Similarly the one-loop virtual gluon contribution is given by

$$F_{\rm eik}^{\rm virt}=-g^2C_F\delta(1-x)\int\frac{d^2k}{(2\pi)^3 2k_0}\times\left\{\frac{2p\cdot q}{p\cdot kq\cdot k}-\frac{m_b^2}{(p\cdot k)^2}\right\},\tag{35}$$

where p and q are the b-quark and light-quark momenta respectively. In the Abelian theory exponentiation follows simply as a consequence of the factorization in the eikonal approximation. For each gluon emission one gets a factor of $F_{\rm eik}^{\rm virt}$ which is unitarized by the virtual contribution. After appropriate symmetrization the exponentiation follows. Next we use the result that even in a non-Abelian theory, for the semi-inclusive process under consideration, exponentiation of the one-loop result takes place [21,13]. By considering the Nth moment of the soft part, we obtain

$$\sigma_N=\exp\left\{g^2C_F\int\frac{d^3k}{(2\pi)^3 2k_0}\left[\left(1-\frac{2k_0}{M}\right)^{N-1}-1\right]\times\left[\frac{2p\cdot q}{p\cdot kq\cdot k}-\frac{m_b^2}{(p\cdot k)^2}\right]\right\}.\tag{36}$$

It should be noted that the ultraviolet cutoff is determined by the factorization scale μ. This cutoff is necessary despite the fact that the process under consideration is infrared finite. All momenta above this scale get shuffled into the hard scattering amplitude H. The need for a cutoff stems from the fact that we have used the eikonal approximation. This approximation is equivalent to a Wilson line formulation of the problem, and thus, as in heavy quark effective field theory, generates a new velocity-dependent anomalous dimension [14].

By an appropriate change of variables σ_N may be written as

$$\sigma_N=\exp\left\{-\int_0^{\mu/m_b}\frac{dy}{y}[1-(1-y)^{N-1}]\times\left(\int_{m_b^2y^2}^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}C_F\frac{\alpha_s}{\pi}(k_\perp^2)-C_F\frac{\alpha_s}{\pi}(m_b^2y^2)\right)\right\}.\tag{37}$$

In arriving at the above, we have made the replacement $\alpha_s\to\alpha_s(k_\perp^2)$. This change has the effect of resumming the next-to-leading logarithms coming from collinear emission of light fermion pairs [22]. However, it does not sum all the soft subleading logarithms.

Explicit calculations carried out at the two-loop level [23,15] indicate that the rest of the subleading terms in the above may be included [15]

$$\frac{C_F\alpha_s(k_\perp^2)}{\pi}\to A(\alpha_s(k_\perp^2)),\tag{38}$$

with

$$A(\alpha_s)=\frac{\alpha_s}{\pi}C_F+\left(\frac{\alpha_s}{\pi}\right)^2\frac{1}{2}C_Fk,$$

$$k=C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right)-\frac{10}{9}T_RN_f.\tag{39}$$

This resums all the leading and next-to-leading logarithms of N in the soft function.

Thus, we obtain

$$\sigma_N(m_b/\mu)=\exp\left\{-\int_0^{\mu/m_b}\frac{dy}{y}[1-(1-y)^{N-1}]\times\left(\int_{m_b^2y^2}^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))+B(\alpha_s(m_b^2y^2))\right)\right\},\tag{40}$$

with $B(\alpha_s)=-\alpha_s/\pi$. This integral is not well defined due to the existence of the Landau pole, and a prescription is needed to define the integral. Choosing a prescription leaves an ambiguity on the order of the power corrections [3,24, 25]. If we use the large N identity

$$1-x^{N-1}=\theta\left(1-x-\frac{1}{\tilde N}\right),$$

$$\tilde N=\frac{N}{N_0},\quad N_0=e^{-\gamma_E},\tag{41}$$

which is accurate to within 2% at $N=10$, to rewrite

$$\sigma_N(m_b/\mu)=\exp\left\{-\int_{m_b/\tilde N}^{\mu}\frac{dk_\perp}{k_\perp}\left[2A(\alpha_s(k_\perp))\ln\frac{k_\perp\tilde N}{m_b}-B(\alpha_s(k_\perp^2))\right]\right\},\tag{42}$$

then we have fixed a prescription which is unambiguous to the accuracy we are concerned with in this paper. From this result, we find that $\sigma_N(m_b/\mu)$ satisfies the renormalization group (RG) equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}\right)\sigma_N\left(\frac{m_b}{\mu}\right)=-\left[2A(\alpha_s(\mu^2))\ln\frac{\mu\tilde N}{m_b}+B(\alpha_s(\mu^2))\right]\sigma_N\left(\frac{m_b}{\mu}\right).\tag{43}$$

We note that this is in agreement with Ref. [3] where Wilson line techniques were utilized.

We may now use the μ dependence of the soft function, together with the fact that the total amplitude is μ independent, to determine the renormalization group equation satisfied by the jet and hard functions. We have seen in Sec. III that the Nth of the derivative of the moments of the semileptonic decay has the factorized form

$$\sigma_N(m_b\sqrt{2-y_0}/\mu)J_N^{\rm sl}(m_b/\mu)H^{\rm sl}(m_b^2(2-y_0)/\mu^2), \quad (44)$$

whereas, for the radiative decay, the moments of the decay spectrum are given by

$$\sigma_N(m_b/\mu)J_N^\gamma(m_b/\mu)H^\gamma(m_b^2/\mu^2). \quad (45)$$

We have now labeled the jet and hard functions according to their processes since these function are not universal. We will first consider the RG equation satisfied by J_N^γ and H^γ. The equations satisfied by $J^{\rm sl}$ and $H^{\rm sl}$ can then be determined by simply making the appropriate replacements. We may derive the RG equations satisfied by these functions by using the following facts: $\mu(d/d\mu)[\sigma_N(m_b/\mu)J_N(m_b/\mu)H(m_b/\mu)]=0$; the RG equation satisfied by $\sigma_N(m_b/\mu)$ is given by Eq. (43); the hard scattering amplitude by definition has no N dependence; the jet functional form which is $J_N[m_b^2/\tilde{N})\cdot(1/\mu^2)]$. This leads to the RG equations

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+f(\alpha_s)\right)J_N(m_b/\mu)$$
$$=2A(\alpha_s(\mu^2))\ln\frac{\mu^2\tilde{N}}{m_b^2}J_N(m_b/\mu), \quad (46)$$

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-f(\alpha_s)-B(\alpha_s(\mu^2))\right)H(m_b/\mu)$$
$$=-2A\left(\alpha_s(\mu^2)\ln\frac{\mu}{m}\right)H(m_b/\mu). \quad (47)$$

$f(\alpha_s)$ is an arbitrary function which can only be determined from additional input. We fix $f(\alpha_s)$ by requiring that the purely collinear divergences of the jet factor be determined by an Altarelli-Parisi-type equation as discussed in [15,12]. We note that for these purposes the jet factor is a cut light quark propagator in the axial gauge.

By requiring that we correctly reproduce the pure collinear divergences at the one-loop level it is found that

$$f(\alpha_s)=2\gamma(\alpha_s), \quad (48)$$

where $\gamma(\alpha_s)$ is the axial vector gauge anomalous dimension [15,12]:

$$\gamma(\alpha_s)=-\frac{3}{4}\frac{\alpha_s}{\pi}C_F+\cdots. \quad (49)$$

The solution of the jet RG equation may be written (to the desired accuracy) in the form

$$J_N^\gamma(m_b/\mu)=\exp\left\{\int_{1/\tilde{N}}^{\mu^2/m_b^2}\frac{dy}{y}\left[\int_{m_b^2y}^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))\right.\right.$$
$$\left.\left.-\gamma(\alpha_s(m_b^2y))\right]\right\}. \quad (50)$$

For future purposes, we rewrite this in the form

$$J_N^\gamma(m_b/\mu)=\exp\left\{\int_0^1\frac{dy}{y}[1-(1-y)^{N-1}]\right.$$
$$\times\left[\int_{m_b^2y}^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))-\gamma(\alpha_s(m_b^2y))\right]$$
$$+\int_1^{\mu^2/m_b^2}\frac{dy}{y}\left[\int_{m_b^2y}^{\mu^2}\frac{dy^2}{k_\perp^2}A(\alpha_s(k_\perp^2))\right.$$
$$\left.\left.-\gamma(\alpha_s(m_b^2y))\right]\right\}. \quad (51)$$

We may now write the explicit expressions for the resummed jet and soft factors. For the radiative decay $B\to X_s\gamma$ we rewrite the various representations obtained earlier, leaving

$$\sigma_N(m_b/\mu)=\exp\left(\int_0^1\frac{dz}{1-z}(1-z^{N-1})N(z)\right), \quad (52)$$

$$J_N(m_b/\mu)=\exp\left(\int_0^1\frac{dz}{1-z}(1-z^{N-1})I(z)\right), \quad (53)$$

$$N(z)=\int_{m_b^2(1-z)^2}^{\mu^2}\left[\frac{dk_\perp}{k_\perp^2}-A(\alpha_s(k_\perp^2))\right]-B(\alpha_s(m_b^2(1-z)^2))$$
$$-\int_1^{\mu/m_b}\left[\int_{m_b^2y^2}^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))+B(\alpha_s(m_b^2y^2))\right], \quad (54)$$

$$I(z)=\left[\int_{m_b^2(1-z)}^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))\right]-\gamma(\alpha_s(m_b^2(1-z)))$$
$$+\int_1^{\mu^2/m_b^2}\frac{dy}{y}\left[\int_{m_b^2y}^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))-\gamma(\alpha_s(m_b^2y))\right]. \quad (55)$$

Combining these two factors we see that for the N-dependent piece in the exponent, the μ^2 dependence exactly cancels. There are, however, pieces which are independent of N which are μ dependent and these will combine with similar terms in the hard scattering amplitude to give a μ-independent answer which must be true by construction. Combining all the factors we find

$$\sigma_N(m_b/\mu)J_N(m_b/\mu)=\exp\left(-\int_0^1\frac{dz}{1-z}(1-z^{N-1})K(z)\right), \quad (56)$$

$$K(z)=\int_{m_b^2(1-z)^2}^{m_b^2(1-z)}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))-\gamma(\alpha_s(m_b^2(1-z)))$$
$$-B(\alpha_s(m_b^2(1-z)^2)). \quad (57)$$

The Nth moment of the decay rate in then given by

$$M_N^\gamma=S_N\sigma_NJ_N^\gamma H^\gamma(\alpha_s(m_b^2)). \quad (58)$$

The value of the one-loop hard scattering amplitude H^γ is given in the Appendix.

For the case of the semileptonic decay the expression for the soft factor is the same as above. However, for the jet, we must rescale $J_N^{\rm sl}(m_b/\mu)\rightarrow J_N(m_b/\mu\sqrt{2-y_0})$. Thus, we get

$$J_N^{\rm sl}(m_b/\mu\sqrt{1-x_\nu})=\exp\left\{\int_0^1\frac{dy}{y}[1-(1-y)^{N-1}]L(y,x_\nu)\right\}, \tag{59}$$

$$L(y,x_\nu)=\int_{m_b^2y(1-x_\nu)}^{\mu^2}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))-\gamma(\alpha_s(m_b^2y(1-x_\nu))). \tag{60}$$

In writing the above, we have used the fact that $y_0=x+x_\nu$ and $x\sim1$ to replace the variable y_0 by x_ν, the neutrino energy fraction. After some algebra the above may be combined with the expression for the perturbative soft function, such that for the product we may write

$$\sigma_N\left(\frac{m_b}{\mu}\right)J_N^{\rm sl}\left(\frac{m_b}{\mu}\sqrt{1-x_\nu}\right)=\exp\left\{-\int_0^1\frac{dz}{1-z}(1-z^{N-1})Q(z)+P(N,x_\nu)\right\}, \tag{61}$$

$$Q(z)=\int_{m_b^2(1-z)^2}^{m_b^2(1-z)}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))-\gamma(\alpha_s(m_b^2(1-z)))-B(\alpha_s(m_b^2(1-z)^2)), \tag{62}$$

$$P(N,x_\nu)=\int_{1/\tilde N}^1\frac{dy}{y}\int_{m_b^2y(1-x_\nu)}^{m_b^2y}\frac{dk_\perp^2}{k_\perp^2}A(\alpha_s(k_\perp^2))-\int_{1-x_\nu}^1\frac{dy}{y}\left[\gamma\left(\alpha_s\left(\frac{m_b^2y}{\tilde N}\right)\right)-\gamma(\alpha_s(m_b^2y))\right]. \tag{63}$$

We note that in deriving this form we have kept only the N-dependent pieces in the exponent. Our analysis shows that certain N-independent terms, such as those proportional to $\ln(1-x_\nu)$, can also be resummed using the above-mentioned procedure. However, we have taken $\alpha_s(m_b^2)\ln(1-x_\nu)$ to be small in the relevant x_ν range and hence relegated all of these logarithms to the hard scattering amplitude. Thus, we may write for the Nth moment of the semileptonic decay rate, up to corrections $O(1/N)$,

$$M_N^{\rm sl}=S_N\int_0^1dx_\nu 6x_\nu(1-x_\nu)\sigma_NJ_N(m_b^2(1-x_\nu),\mu^2)\times H(m_b(1-x_\nu),\mu). \tag{64}$$

The complete expressions for the one-loop hard scattering amplitude in both the radiative and semileptonic processes at $x\sim1$ are given in the Appendix.

We conclude this section by giving some simplified expressions for the product σ_NJ_N which will be useful for numerical analysis. We begin by noticing that, as long as $\alpha_s(m_b^2)\ln N\leqslant1$, the resummation formulas given above have a convergent power series expansion in $\alpha_s(m_b^2)\ln N$ to the next-to-leading logarithmic accuracy. Thus, we compute these expressions to this accuracy and delegate all the nonperturbative effects phenomenologically to the structure function S_N. For a similar approach for the case of e^+e^- annihilation, see [26]. To evaluate the integrals in the exponent, we may perform the z integration using the large N identity (41). The $k_\perp$ integration is simplified by using the RG equation for the running coupling to change variables to α_s: i.e.,

$$\frac{dk_\perp^2}{k_\perp^2}=-\frac{1}{\beta_0}\frac{d\alpha_s}{\alpha_s^2}\left(1-\frac{\beta_1}{\beta_0}\alpha_s+O(\alpha_s^2)\right), \tag{65}$$

where

$$\beta_0=\frac{11C_A-2N_f}{12\pi},\quad \beta_1=\frac{17C_A^2-5C_AN_f-3C_FN_f}{24\pi^2}. \tag{66}$$

Next we use the expansion, correct to next-to-leading logarithmic accuracy,

$$\alpha_s(m_b^2/N)=\frac{\alpha_s(m_b^2)}{1-\beta_0\alpha_s(m_b^2)\ln N}\left(1-\frac{\beta_1}{\beta_0}\times\frac{\alpha_s(m_b^2)}{1-\beta_0\alpha_s(m_b^2)\ln N}\times\ln[1-\beta_0\alpha_s(m_b^2)\ln N]\right), \tag{67}$$

to obtain

$$\ln(\sigma_NJ_N)=\ln N(g_1^\gamma(\chi))+g_2^\gamma(\chi), \tag{68}$$

where

$$\chi=\beta_0\alpha_s(m_b^2)\ln N. \tag{69}$$

In the above, the functions g_1 and g_2 have the following form for the two processes discussed in this paper.

For the radiative decay

$$g_1^\gamma=-\frac{A^{(1)}}{2\pi\beta_0\chi}[(1-2\chi)\ln(1-2\chi)-2(1-\chi)\ln(1-\chi)], \tag{70}$$

and

$$g_2^\gamma=-\frac{A^{(2)}}{2\pi^2\beta_0^2}[-\ln(1-2\chi)+2\ln(1-\chi)]-\frac{A^{(1)}\beta_1}{2\pi\beta_0^3}\left(\ln(1-2\chi)-2\ln(1-\chi)+\frac{1}{2}\ln^2(1-2\chi)-\ln^2(1-\chi)\right)+\frac{\gamma^{(1)}}{\pi\beta_0}\ln(1-\chi)+\frac{B^{(1)}}{2\pi\beta_0}\ln(1-2\chi)-\frac{A^{(1)}}{\pi\beta_0}\ln N_0[\ln(1-2\chi)-\ln(1-\chi)]. \tag{71}$$

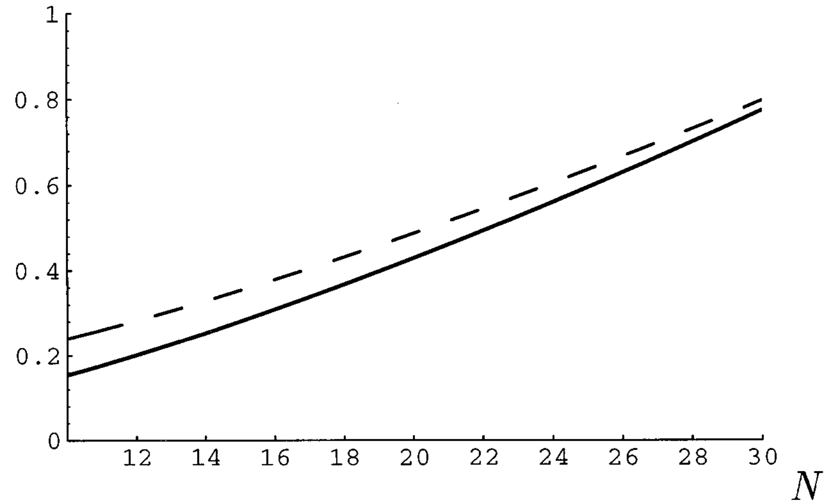

FIG. 4. The difference between the moments of the one-loop result and the resummed result with (dashed line) and without (solid line) the resummation of the π^2, normalized to the one-loop result. N varies from 10 to 30.

For the semileptonic decay

$$g_1^{\mathrm{sl}}=g_1^{\gamma} \tag{72}$$

and

$$g_2^{\mathrm{sl}}=g_2^{\gamma}+\frac{A^{(1)}}{\pi\beta_0}\ln(1-x_\nu)\ln(1-\chi). \tag{73}$$

We have kept only the N-dependent terms in these g factors which exponentiate. There are also N_0-dependent constant terms which we will shuffle into the hard scattering amplitude. These terms are given by

$$h^{\gamma}=\frac{\alpha_s}{\pi}\ln N_0(B^{(1)}+g^{(1)})-\frac{\alpha_s}{2\pi}\ln^2 N_0,$$

$$h^{\mathrm{sl}}=h^{\gamma}+A^{(1)}\frac{\alpha_s}{\pi}\ln N_0\ln(1-x_\nu). \tag{74}$$

Furthermore, Eq. (64) for the semileptonic case becomes

$$-\frac{1}{\Gamma_0}\int_0^1 x^{N-1}\frac{d}{dx}\frac{d\Gamma}{dx}=S_N\int_0^1 dx_\nu 6(1-x_\nu)\times x_\nu[H(x_\nu)+h^{\mathrm{sl}}]\exp(g_1+g_2^{\mathrm{sl}}). \tag{75}$$

In writing the above, we have used the notation

$$A(\alpha_s)=\left(\frac{\alpha_s}{\pi}\right)A^{(1)}+\left(\frac{\alpha_s}{\pi}\right)^2 A^{(2)} \tag{76}$$

and

$$B(\alpha_s)=\left(\frac{\alpha_s}{\pi}\right)B^{(1)}, \quad \gamma(\alpha_s)=\left(\frac{\alpha_s}{\pi}\right)\gamma^{(1)}. \tag{77}$$

The values for $A^{(1)}$, $A^{(2)}$, $B^{(1)}$, and $\gamma^{(1)}$ have been given previously. $H(x_\nu)$ is the one-loop correction to the hard scattering amplitude given in the Appendix.

It is interesting to note that if we expand the expressions for g_1^{sl} and g_2^{sl} in Eq. (68), we see that G_{24}, as defined in Eq. (15), vanishes. Thus, the two-loop results do not trivially exponentiate as one might have naively thought. Such behavior is a universal property of the asymptotic limit of distribution functions and is a consequence of the fact that only ''maximally non-Abelian'' graphs contribute to the exponent beyond one loop [21]. Knowing this greatly reduces the number of graphs that need to be calculated in a general resummation procedure.

From expression (75) we may determine the range of N for which our calculation is valid. The integration over y contains a branch cut at $N=m_b/\Lambda_{\mathrm{QCD}}$, signaling the breakdown of the perturbative formalism. This breakdown is coming from the fact that the time scale for gluon emission is becoming too long. An inspection of the resummation formulas for these quantities suggests that in the region $z\sim 1$ such that $k_\perp^2\leqslant\Lambda^2$, nonperturbative effects become important. Thus, we conclude that we may only trust our results in the range

$$N<\frac{m_b}{\Lambda_{\mathrm{QCD}}}. \tag{78}$$

V. ANALYSIS AND RESULTS

With the resummation now in hand, let us consider the relative sizes of all the contributions. In Fig. 4 we show the difference between the one-loop result and the resummed rate given by Eq. (75) normalized to the moments of the one-loop result:

$$-\frac{1}{\Gamma_0}\int_0^1 x^{N-1}\frac{d}{dx}\frac{d\Gamma}{dx}=1-\frac{2\alpha_s}{3\pi}\left(\pi^2+\frac{5}{4}-\frac{31}{6}\ln\widetilde{N}+\ln^2\widetilde{N}\right). \tag{79}$$

In our calculation we take $\Lambda_{\mathrm{QCD}}^{n_f=4}\approx 200$ MeV. We see that resumming the next-to-leading logarithms has a 20%–50% effect in the range of N we are considering. Furthermore, for completeness we have included the effect of resumming the

π^2. The result of this resummation is given by the dashed line. We see that the effect of resumming the π^2 is small.[2] As a check of the numerics we compared the resummed expression to the one-loop result (79) and found that for small N the two coincide to within less than a percent. The fact that the resummation of the next-to-leading logarithms is more important that the leading logarithms, is rather disheartening. It leads one to believe that perhaps the next-to-next-to-leading logarithms will be even more important. However, the fact that the effect of subleading logarithms is larger than the leading is already hinted at one loop, given that the ratio of the coefficients in front of these logarithms is 31/6. It could be *hoped* that the ratio of the coefficients of the next-to-leading and next-to-next-to-leading logarithms is not so large and the terms left over in our resummation will be on the order of 10%.

VI. DISCUSSION

Before we conclude with a discussion of the future prospects of the extraction of V_{ub} we wish to point out that there is one tacit assumption which has been made up to this point in our investigation. That is, we have assumed that local duality will hold when we are a few hundred MeV from the end point. The whole formalism of using the OPE in calculating inclusive decay rates assumes that at certain parts of the Dalitz plot, the Minkowski space calculation will give the correct result. This should be a good approximation as long as we stay away from the resonance region. The question is, how far from the end point does this region begin? If it is found that single resonances dominate, even as far as a few hundred MeV from the end point, then the extraction of V_{ub} through inclusive decays is surely doomed. The quark model seems to indicate that this may be the case [27], though other theoretical predictions say otherwise. We will have to wait to see the data before we can decide on the fate of the extraction methods discussed here.

Next we wish to reiterate that *completely* eliminating the background from the $b\to c$ transition by going to very large $N>30$ is not feasible since there is no way to reliably calculate the soft gluon emission which takes place. This is because when one goes that far out on the tail, the time scale for gluon emission is too long compared to the QCD scale to have any hope of perturbation theory making any sense. Again, this statement is independent of how many soft logarithms one is willing to resum. Another way of saying this is that when x gets to close to 1, there is no operator product expansion since the expansion parameters is $\Lambda/m_b(1-x)$. Thus, we are stuck with the fact that there will always be contamination from transitions to charmed final states. Calculating the end point of the charmed spectrum using the techniques discussed above fails as well since resonances will dominate. Thus, there does not seem to be any way to avoid having to use a model to determine the background in the extraction process. The best we can hope to do, using the results in this paper, is to go to a large enough value of N that we can reduce the model dependence as much as possible. Certainly, we can greatly reduce the model dependence from what it is in present extractions which rely solely on models.

The last point that needs to be mentioned is the fact that measuring large moments is not experimentally feasible, as x^N varies much too rapidly. For instance, if we assume that the bin size is given by $\delta = \delta E/m_b$, then the error at point x for the Nth moment will be $\delta N/x$. Therefore, the error can accumulate quite rapidly. Thus, it will be necessary to take the Mellin transform of our result. Given that our result is only trustable for $N<25$, one must be careful to calculate the contribution to the inverse transform from higher moments, if one hopes to impose the bounds on the errors discussed in this paper. Also, for smaller values of N one must be sure not to use the resummed formula as we have dropped terms that go like $1-x$.

Given these caveats, we may now address the issue as to what accuracy we can determine V_{ub} using inclusive decays. Since the subleading logarithms dominate the leading logarithms, the conservative conclusion would be that a model-independent extraction of V_{ub} is not possible. However, let us proceed under the assumption the sub-subleading logarithms will be smaller than the subleading logarithms. In this case we may say that we have been able to reduce the errors from radiative corrections down to the order of 10%. However, the QCD perturbative expansion is notoriously asymptotic, and though we may hope that we have resummed the dominant pieces of the expansion, there could still be large constants (independent of N) which could arise.

Another source of errors will come from the fact that we need to eliminate the dependence of the decay rate on the moments of the nonperturbative structure function [8] by taking the ratio of the semileptonic decay moments to the moments of the radiative decay. This will introduce the errors in the radiative decay into the semileptonic decay. One could calculate without any nonperturbative resummation, thus eliminating these errors (the results in this paper are easily modified to include this possibility), but then it is difficult to quantify the model-dependent errors introduced in the truncation. Finally, there are the errors introduced due to the model dependence from the calculation of the background. This error will be reduced as we choose larger values of N. This is the most difficult error to quantify, and we shall not discuss it here.

The authors believe that, if the end point is not dominated by single-particle resonances, and if we assume that the fact that the subleading logarithms dominate the leading logarithms is just an anomaly, then we may hope to eventually extract V_{ub} at the 30% level using the results presented here. Moreover, resumming the next-to-leading logarithms is indeed necessary. However, the more conservative view would be that the end point calculation is just intractable at this time, since it could be that the sub-subleading logarithms will dominate. To be sure that this is not the case the sub-subleading logarithms would need to be resummed. This would entail calculating A to three loops, and B and γ to two loops. Without this calculation, we cannot determine with certainty the size of the errors.

[2]Note that in resumming the π^2 we only resum part of the π^2 in the expression (2), since part of the π^2 contribution comes from integration over the neutrino energy.

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APPENDIX

In this appendix we give explicit expressions for the hard scattering amplitude at the one-loop level and to leading order in $(1-x)$. We first present the results of the computation of the QCD corrections to the doubly differential rate $d^2\Gamma/dxdy_0$ for the semileptonic b-quark decay. It is clear from Secs. II and III that this is the quantity whose moments factorize and which is relevant for the resummation. We write

$$\frac{d^2\Gamma}{dxdy_0}=6\Gamma_0(y_0-1)(2-y_0)\left(1-\frac{2\alpha_s}{3\pi}G(x,y_0)\right). \tag{A1}$$

where Γ_0 was defined earlier. The contributions of the real and the virtual gluon emission diagrams to $G(x,y_0)$ are given by

$$\begin{aligned}G_{fin}^{\text{real}}=&\ln^2(1-x)+\frac{7}{2}\ln(1-x)-2\ln(2-y_0)\ln(1-x)\\&-\ln^2(2-y_0)+\frac{3}{2}\ln(2-y_0)+\frac{\pi^2}{6}-\frac{1}{2}\ln^2\left(\frac{\lambda^2}{m_b^2}\right)\\&-\frac{5}{2}\ln\left(\frac{\lambda^2}{m_b^2}\right)+2\ln(2-y_0)\ln\left(\frac{\lambda^2}{m_b^2}\right)\end{aligned} \tag{A2}$$

and

$$\begin{aligned}G_{\text{fin}}^{\text{virt}}=&3\ln^2(2-y_0)-2\ln(2-y_0)\ln(y_0-1)\\&+2\,\mathrm{Re}\left[\mathrm{Li}_2\left(\frac{1}{2-y_0}\right)\right]+3\frac{(2-y_0)}{y_0-1}\ln(2-y_0)\\&-2\frac{\ln(2-y_0)}{y_0-1}+\frac{5}{2}+\frac{\pi^2}{6}+\frac{1}{2}\ln^2\left(\frac{\lambda^2}{m_b^2}\right)+\frac{5}{2}\ln\left(\frac{\lambda^2}{m_b^2}\right)\\&-2\ln(2-y_0)\ln\left(\frac{\lambda^2}{m_b^2}\right).\end{aligned} \tag{A3}$$

In the above, λ is the gluon mass used to regulate the infrared divergences at intermediate stages of the calculation. Combining these results and integrating over y_0 gives the electron spectrum which agrees with [28] but disagrees with [29].

From this we see that to the approximation we are working in, the hard scattering amplitude as defined in Eq. 25 is given by

$$\begin{aligned}H_{\text{sl}}=1-\frac{2\alpha_s}{3\pi}\Bigg\{&\frac{\pi^2}{3}+2\ln^2(2-y_0)-2\ln(2-y_0)\ln(y_0-1)\\&-\frac{3}{2}\ln(2-y_0)+\frac{\ln(2-y_0)}{y_0-1}+2\,\mathrm{Re}\left[\mathrm{Li}_2\left(\frac{1}{2-y_0}\right)\right]+\frac{5}{2}\Bigg\}.\end{aligned} \tag{A4}$$

For the radiative decay, we may extract the hard scattering amplitude from [30]

$$H_\gamma=1-\frac{2\alpha_s}{3\pi}\left(\frac{3}{2}-\frac{2\pi^2}{3}\right). \tag{A5}$$

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