Sidewise dispersion relations and the structure of the nucleon vertex

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(Received 21 March 1996)

We reexamine sidewise dispersion relations as a method to relate the nucleon off-shell form factor to observable quantities, namely, the meson-nucleon scattering phase shifts. It is shown how for meson-nucleon scattering a redefinition of the intermediate fields leaves the scattering amplitude invariant, but changes the behavior of the off-shell form factor as expressed through dispersion relations, thus, showing representation dependence. We also employ a coupled-channel, unitary model to test the validity of approximations concerning the influence of inelastic channels in the sidewise dispersion relation method. [S0556-2821(96)02615-X]

PACS number(s): 11.55.Fv, 13.40.Gp, 13.60.Fz, 13.75.Gx

I. INTRODUCTION

In the theoretical description of processes at intermediate energies, the structure of hadrons is often described by multiplying the pointlike vertex operators by form factors. It is a common practice to assume that these vertices, i.e., their operator structures and the associated form factors, are in all situations the same as for a free on-shell hadron. This is done, for example, in the description of electron-nucleus scattering or in two-step reactions on a free nucleon, such as Compton scattering, where one is dealing with an intermediate nucleon not on its mass shell. In these cases, however, the electromagnetic vertices can have a much richer structure: there can be more independent vertex operators and the form factors can depend on more than one scalar variable. The common treatment of such ''off-shell'' effects is to presume them small and to ignore them by using the free vertices. However, as much of the present effort in intermediate energy physics focuses on delicate effects, such as evidence of quark-gluon degrees of freedom or small components in the hadronic wave function, it is mandatory to examine these issues in detail.

One theoretical tool for the description of the off-shell vertex of the nucleon is the method of sidewise dispersion relations. The ''sidewise'' here indicates that one uses the method to get at the dependence of the form factors on the invariant mass of the nucleon rather than, e.g., the t-channel four-momentum transfer. It has been used, e.g., for the electromagnetic form factors of the nucleon [1–3], electromagnetic transition form factors [4], the nucleon axial-vector coupling constant [5], and the pion-nucleon form factor [6]. If one wants to calculate the half-off-shell πNN form factor, knowledge of its phase along the cut in the energy plane is sufficient to determine it via a sidewise dispersion integral. Below the two-pion threshold this phase is given in terms of the known pion-nucleon phase shifts. However, above this threshold assumptions have been made for the phase [1,7] which lead to quite different predictions for the half-off-shell form factor. Since neither of these prescriptions has been tested, we use a coupled-channel, unitary model to investigate the validity of the assumptions regarding the phase of the off-shell meson-nucleon form factor.

Another objective of this work is to investigate the ''representation dependence'' of off-shell effects, and specifically how this representation dependence enters the sidewise dispersion relations analysis. Off-shell vertices are described within the framework of the reduction formalism [8] using interpolating (interacting) fields for the off-shell nucleon. The choice of this interpolating field is not unique. It is well known that on-shell S-matrix elements are oblivious to the choice of interpolating field: different unitarily equivalent Lagrangians constitute different representations of the theory and physically measurable quantities, such as on-shell amplitudes, are representation independent in accord with the Coleman-Wess-Zumino theorem [9]. In fact, the transformation need not be unitary; any reversible field redefinition will leave the on-shell amplitudes unchanged [10]. However, different interpolating fields in general lead to different off-shell extrapolations [11] and, therefore, off-shell form factors cannot be uniquely determined. This was recently demonstrated [12,13] in the framework of chiral perturbation theory. It was shown how the off-shell electromagnetic form factor of the pion changes under a unitary transformation of the Lagrangian which leaves, e.g., the Compton amplitude unchanged. While the total amplitude for the on-shell pion is representation independent, and certainly observable, the individual contributions from ''pole'' and contact terms are not. In other words, representation-dependent ''off-shell effects'' in pole contributions in one representation appear as contact terms [13] in another representation.

One now faces the following puzzle: although the off-shell form factors are not unique and not measurable, it appears that through sidewise dispersion relations they can be determined from physical quantities such as meson-nucleon phase shifts. However, we will show below how representation dependence appears in the sidewise dispersion relations, making a unique determination of the half-off-shell πNN form factor impossible. This is in contrast to the use of dispersion relations for the determination of the pion-nucleon scattering amplitude at the nonphysical point $\nu = t = 0$ [14], a quantity crucial in determining the pion-nucleon sigma term [15].

The outline of this paper is as follows. In Sec. II we discuss the general features of an off-shell vertex and of the representation dependence. In Sec. III we review the sidewise dispersion relations and the different assumptions proposed in the literature about their use at energies above an inelastic threshold. These assumptions are tested in Sec. IV in a simple unitary, coupled-channel model. A summary and our conclusions are given in Sec. V. Some details of our calculations are contained in appendices.

II. THE VERTEX OF AN OFF-SHELL NUCLEON AND FIELD REDEFINITIONS

The most general pion-nucleon vertex, where the incoming nucleon of mass m has momentum p_μ, the outgoing nucleon has momentum p'_μ and the pion has momentum $q_\mu = p'_\mu - p_\mu$, can be written as [16]

$$\Gamma^5(p',p) = \left[\gamma_5 G_1 + \gamma_5 \frac{\not p - m}{m} G_2 + \frac{\not p' - m}{m} \gamma_5 G_3 + \frac{\not p' - m}{m} \gamma_5 \frac{\not p - m}{m} G_4 \right]. \tag{1}$$

By sandwiching Γ^5 between on-shell spinors one obtains $G_1(q^2,m^2,m^2)\overline{u(p')}\gamma_5 u(p)$. Clearly, the off-shell vertex has a much richer structure, in that there are more independent operators and, moreover, each of them depends on more kinematical variables than just the four-momentum transfer q^2.

Below, we will, for simplicity only, consider the ''half-off-shell'' vertex, with the incoming nucleon on shell. Defining $\omega' = \sqrt{p'^2}$ and introducing the projection operators

$$P_\pm = \frac{w' \pm \not p'}{2w'}, \tag{2}$$

we obtain, in that case,

$$\Gamma^5(p',p)u(p) = [P_+ K(q^2,w') + P_- K(q^2,-w')]\gamma_5 u(p). \tag{3}$$

Because of the incoming on-shell nucleon spinor, the terms proportional to G_2 and G_4 do not contribute. The function K is obtained as

$$K(q^2,\pm w') = G_1(q^2,w'^2,m^2) + \frac{\pm w' - m}{m} G_3(q^2,w'^2,m^2). \tag{4}$$

The general electromagnetic vertex of the nucleon is more complicated [1]. Its general form is

$$\Gamma^\mu = \sum_{j,k=0,1} (\not p')^j [A_1^{jk}\gamma^\mu + A_2^{jk}\sigma^{\mu\nu}q_\nu + A_3^{jk}q^\mu](\not p)^k, \tag{5}$$

where the 12 form factors A_i^{jk} are again functions of three scalar variables, usually taken to be q^2, p^2, and p'^2. By using the constraints provided by the Ward-Takahashi identity, the number of independent form factors can be shown to reduce to eight. Upon evaluating the vertex between two on-shell spinors, one recovers the familiar form of the electromagnetic current of a free nucleon, involving two independent contributions with their associated form factors, such as the Dirac and Pauli form factors. It is important to stress that in calculations of electromagnetic reactions involving bound nucleons or two-step reactions on a free nucleon, such as Compton scattering or meson electroproduction, one necessarily deals with the electromagnetic current or vertex of an off-shell nucleon.

In this situation, it has been quite common to make use of *ad hoc* assumptions which use as much as possible the on-shell information while maintaining current conservation. Most widely used is the prescription introduced by de Forest [17] for the off-shell electromagnetic current. It allows one to use the free current by changing its kinematical variables according to the off-shell situation. Another often used version for the nucleon vertex operator was introduced by Gross and Riska [18] and is given by

$$\Gamma_\mu(q^2) = \gamma_\mu F_1(q^2) + \frac{q_\mu \not q}{q^2}[1 - F_1(q^2)] + \sigma_{\mu\nu}q^\nu \frac{F_2(q^2)}{2m}. \tag{6}$$

It also only involves on-shell information, the free Dirac and Pauli form factors, F_1 and F_2, but has a more general Dirac structure. The second term on the right-hand side of Eq. (6) vanishes when the vertex is evaluated between on-shell spinors, but contributes when one or both nucleons are off shell. It is easily seen that this vertex satisfies the Ward-Takahashi identity when free Feynman propagators are used for the nucleon. In pion electroproduction, this prescription is equivalent to adding a contact term to the Born amplitude which is needed to restore gauge invariance [19]. The validity of this and other recipes can only be assessed on the basis of a realistic microscopic calculation and will depend on the kinematics of the process.

Several attempts have been made to calculate properties of the off-shell vertices and to estimate their effects in processes at intermediate energies. Bincer [1], for example, proposed using sidewise dispersion relations in which the electromagnetic and strong nucleon off-shell form factors were related to pion-nucleon phase shifts (see Sec. III). This approach was used by Nyman [20] and Minkowski and Fischer [21]. Studies in the context of meson loop models have been performed, e.g., in Refs. [22–25]. Typically, effects of the order of 5–15% were found for the dependence of the form factors on the variable w'.

Recently, the off-shell pion electromagnetic vertex was investigated in the framework of meson chiral perturbation theory [12,13]. The computation was performed by using two different chiral Lagrangians, related through a unitary transformation of the fields, which leaves the observables unchanged. It was shown explicitly how in the description of Compton scattering off a pion the off-shell form factors are not the same while the observable on-shell form factor and the amplitude are the same in the two representations. This general result concerning the representation dependence of the off-shell effects can be illustrated by the following simple example for pion-nucleon scattering. We consider the pseudoscalar meson-nucleon Lagrangian

$$L_P = \frac{1}{2}[(\partial\phi)^2 - \mu^2\phi^2] + \overline{\psi}(i\not\partial - m)\psi - ig\overline{\psi}\gamma_5\phi\psi, \tag{7}$$

and perform the transformation

$$\psi \to \exp(i\beta\gamma_5\phi)\psi. \tag{8}$$

Then, up to and including order β^2 terms, the new Lagrangian reads

$$\tilde{L} = L_P - 2im\beta\overline{\psi}\gamma_5\phi\psi + \beta\overline{\psi}\gamma_5(\partial\!\!\!/\phi)\psi + 2\beta(g+2m\beta)\overline{\psi}\phi^2\psi + O(\beta^3). \quad (9)$$

This transformed Lagrangian has both pseudoscalar (PS) and pseudovector (PV) πNN interaction terms, as well as a contact term. Choosing $\beta = -g/2m$ corresponds to the ''Dyson'' transformation [26] and the resulting πNN coupling becomes purely PV. For our discussion of the representation dependence we leave β free and show that physical, observable quantities are β independent [27].

The meson-nucleon vertex at the tree level for both representations is readily obtained (the dressed vertex at the one-loop level will be discussed in Appendix B). From L_P, we find for the vertex

$$\Gamma_P^5 = g\gamma_5, \quad (10)$$

corresponding to the trivial half-off-shell vertex function

$$K(q^2, \pm w') = g. \quad (11)$$

On the other hand, $\tilde{L}$ yields

$$\tilde{\Gamma}^5(p',p) = \gamma_5[g + 2m\beta + \beta(p\!\!\!/' - p\!\!\!/)], \quad (12)$$

corresponding to the half-off-shell vertex function [cf. Eq. (4)]

$$\tilde{K}(q^2, \pm w') = g + \beta(m \mp w'), \quad (13)$$

which is β dependent and clearly has a different off-shell behavior. However, the on-shell matrix element of the vertex operator is the same for both representations.

What happens if we consider a two-step process on a free nucleon, such as pion-nucleon scattering, that involves the propagation of an intermediate off-shell nucleon? Since this is an overall on-shell process, the total amplitude must be independent of the value one chooses for β. This means that the β-dependent contributions from the off-shell vertices in the pole terms, i.e., in the contributions involving two πNN vertices connected by an intermediate nucleon propagator, must be compensated by some other β-dependent contribution. To show that, we consider the on-shell pion-nucleon scattering T matrix at the tree level. Using L_P, it involves pole terms only and reads

$$\mathcal{T}_P = -ig^2\overline{u}(p')\left\{ \frac{1}{p\!\!\!/ + q\!\!\!/ - m} + \frac{1}{p\!\!\!/ - q\!\!\!/' - m} \right\} u(p), \quad (14)$$

where p and p' are the initial and final nucleon four-momenta, and q and q' are the initial and final pion four-momenta, respectively. The pole term contribution to the T matrix for the mixed PS and PV Lagrangian $\tilde{L}$ is at the tree level,

$$\mathcal{T}_{\text{pole}} = i\overline{u}(p')\Big\{ [g + \beta(2m + q\!\!\!/')]\gamma_5 \frac{1}{p\!\!\!/ + q\!\!\!/ - m}\gamma_5[g + \beta(2m + q\!\!\!/)] + [g + \beta(2m - q\!\!\!/)]\gamma_5 \frac{1}{p\!\!\!/ - q\!\!\!/' - m}\gamma_5[g + \beta(2m - q\!\!\!/')] \Big\} u(p), \quad (15)$$

where the terms in the square brackets arise from the transformed vertex, Eq. (12). Using the Dirac equation for the on-shell spinors, this may be cast in the form

$$\mathcal{T}_{\text{pole}} = -ig^2\overline{u}(p')\left\{ \frac{1}{p\!\!\!/ + q\!\!\!/ - m} + \frac{1}{p\!\!\!/ - q\!\!\!/' - m} + 4\beta(g + m\beta) \right\} u(p), \quad (16)$$

where the β-dependent term reflects the different ''off-shell'' behavior of the vertex obtained from $\tilde{L}$. However, there is now also a contribution from the contact term in $\tilde{L}$, the term proportional to $\overline{\psi}\phi^2\psi$, which yields

$$\mathcal{T}_{\text{contact}} = i\overline{u}(p')\{4\beta(g + m\beta)\}u(p). \quad (17)$$

Clearly, the β-dependent terms cancel out and the total amplitude remains unchanged:

$$\mathcal{T}_P = \mathcal{T}_{\text{pole}} + \mathcal{T}_{\text{contact}}. \quad (18)$$

This simple example illustrates not only that, as expected [9,11], total on-shell amplitudes for a given process are invariant under field redefinitions, but also the interplay between ''off-shell'' effects from vertices and contact terms. This makes it impossible to define ''off-shell'' effects in a unique, representation-independent fashion.

Our considerations above concerned only rather simple vertices at the tree level. The close connection between off-shell effects in a vertex and contact terms also exists when we consider dressed vertices, as will be shown at the one-loop level (see Appendix B). It can be made plausible with the following example that concerns the dependence of the vertex on the invariant mass p^2. Consider, for simplicity, a scalar vertex for an initially on-shell particle together with the subsequent propagation. By expanding the vertex around the on-shell point,

$$\frac{\Gamma(q^2,m^2,p^2)}{p^2 - m^2} = \frac{\Gamma(q^2,m^2,m^2)}{p^2 - m^2} + \frac{\partial\Gamma}{\partial p^2}(m^2) + \cdots, \quad (19)$$

one finds that the propagator gets canceled in the second and higher order terms. Thus, off-shell effects in the pole terms through the dependence of the vertex on the scalar variable p^2 can also be related to contact terms. Equations (15) and (16) are specific examples of this. The above seems to suggest that it is possible to find a representation for an amplitude where $K(q^2,w)$ has no off-shell dependence, i.e., no dependence on w by keeping enough terms in Eq. (19) and introducing the corresponding contact terms. However, the Taylor expansion implicit in Eq. (19) is valid only up to the first branch cut, i.e., the pion threshold. Thus, this procedure is, for example, not valid in calculations of pion electroproduction on a nucleon. In Compton scattering below the pion threshold the shifting of the dependence on the nucleon invariant mass to contact terms is possible.

The above example also showed how the transformation in Eq. (8) adds one power of the nucleon four-momentum to the asymptotic behavior of the original vertex, Eq. (10). Two powers can be added by considering a transformation involving derivatives, such as

$$\psi \rightarrow \exp[\beta\gamma_5 \partial\!\!\!/ \phi]\psi. \tag{20}$$

To leading order in β, this transformation generates the following β-dependent interaction terms

$$\begin{aligned} L^{[\beta]} = & -2m\beta\overline{\psi}\gamma_5(\partial\!\!\!/ \phi)\psi + i\beta\overline{\psi}\gamma_5(\partial\!\!\!/ \phi)(\partial\!\!\!/ \psi) \\ & - i\beta(\overline{\psi}\overleftarrow{\partial\!\!\!/})\gamma_5(\partial\!\!\!/ \phi)\psi - i\beta\overline{\psi}\gamma_5(\partial^2\phi)\psi \\ & - i\beta\overline{\psi}\gamma_5(\partial\phi)(\partial\psi) - i\beta(\overline{\psi}\overleftarrow{\partial})\gamma_5(\partial\phi)\psi. \end{aligned} \tag{21}$$

We readily obtain for the contribution of the β-dependent terms to the half-off-shell vertex

$$\Gamma_5^{\beta}(p',p)u(p) = \beta(m^2 - p'^2)\gamma_5 u(p), \tag{22}$$

which vanishes on shell, as anticipated. Higher powers in the nucleon momenta can be obtained by using transformations involving higher derivatives. Of course, one can perform transformations acting on the nucleon field that induce not just a p'^2 dependence, but also a combined p'^2 and q^2 dependence of the half-off-shell vertex Γ^5. For example, the transformation $\psi \rightarrow \exp(i\beta\gamma_5\partial^2\phi)\psi$ induces a new term $\beta(p\!\!\!/' - m)q^2\gamma_5 u(p)$.

Observations similar to those we made for the strong form factor can also be made for the electromagnetic vertex in QED by starting with the QED Lagrangian and transforming the electron field. The electromagnetic vertex obtained at the tree level from the QED Lagrangian is simply $-ie\gamma^\mu$ for on- and off-shell electrons. Applying the transformation $\psi \rightarrow \exp(\beta\partial^2 A\!\!\!/)\psi$ changes, for example, the half-off-shell vertex to $-i[e + \beta q^2(p\!\!\!/' - m)\gamma^\mu]u(p)$. The β-dependent part of this vertex vanishes on shell, as expected.

III. SIDEWISE DISPERSION RELATIONS

We now turn to the method of sidewise dispersion relations which seems to suggest that one can uniquely obtain the off-shell form factor from experimentally measurable phase shifts. There are two main issues we would like to address here. The first is where does the representation dependence discussed in the previous section enter the sidewise dispersion relation method. The second is the validity of certain approximations, related to the treatment of inelastic channels, that have been used in the literature. Dispersion relations are expressions relating the real part of a function, such as a Green's function, to a principal value integral over its imaginary part. Physically, the requirement of causality implies the analyticity properties of such functions [28] which allows one to obtain dispersion relations. Scattering amplitudes, for example, are real analytic functions of the energy E when regarded as a complex variable, i.e., $f(E) = f^*(E^*)$. In the case of form factors of a particle, usually the four-momentum transfer to the particle is used as the dispersion variable.

As shown by Bincer [1] using the reduction formalism [8], one may analytically continue both the electromagnetic and the strong nucleon form factors not only as a function of the momentum transfer, but also of the invariant mass of the off-shell nucleon. For the function $K(q^2,w)$ (we henceforth denote the dispersion variable by w) appearing in the half off-shell strong vertex, Eq. (4), he showed that it is a real analytic function of w with cuts along the real axis starting at $w = \pm(m+\mu_\pi)$ and extending to $\pm\infty$. Furthermore, $K(q^2,w)$ is purely real along the real axis in the interval $-(m+\mu_\pi) < w < (m+\mu_\pi)$. Thus, $K(q^2,w)$ satisfies dispersion relations, termed "sidewise" to emphasize that the dispersion variable is now the nucleon four-momentum, $w = \sqrt{p'^2}$. Using Cauchy's theorem, one obtains

$$\begin{aligned} \mathrm{Re}\, K(q^2,w) = \frac{1}{\pi}\mathcal{P}\int_{m+\mu_\pi}^{\infty} dw' \Bigg[& \frac{\mathrm{Im}\, K(q^2,w')}{w'-w} \\ & + \frac{\mathrm{Im}\, K(q^2,-w')}{w'+w} \Bigg], \end{aligned} \tag{23}$$

provided that $|K(q^2,w)|$ vanishes fast enough for $|w| \rightarrow \infty$. If, for example, $|K(q^2,w)|$ approaches a constant as $|w| \rightarrow \infty$, one must consider a once-subtracted dispersion relation for $K(q^2,w)$:

$$\begin{aligned} \mathrm{Re}\, K(q^2,w) = & \mathrm{Re}\, K(q^2,w_0) + \frac{(w-w_0)}{\pi}\mathcal{P}\int_{m+\mu_\pi}^{\infty} dw' \\ & \times \left[\frac{\mathrm{Im}\, K(q^2,w')}{(w'-w_0)(w'-w)} - \frac{\mathrm{Im}\, K(q^2,-w')}{(w'+w_0)(w'+w)} \right], \end{aligned} \tag{24}$$

where w_0 is a "subtraction point," most conveniently taken to be the nucleon mass, $w_0 = m$, where $K(q^2,m)$ is (experimentally) known. Evidently, if $|K(q^2,w)|$ grows like $|w|^n$ (n an integer) as $w \rightarrow \infty$, $n+1$ subtractions must be performed which introduce the same number of *a priori* unknown subtraction constants into the dispersion relation. The role of subtractions in the sidewise dispersion method is important. Since we only know $K(q^2,w)$ at the on-shell point, $w = m$, a need for more than one subtraction will spoil any possible predictive power. In cases where the vertex function is not known at the on-shell point, as, e.g., in the electromagnetic vertex of the nucleon, even one subtraction will destroy predictive power.

For our discussion below we are interested in the case where the pion is on its mass shell, i.e., in $K(m_\pi^2,w)$. It is useful to note that starting from Eq. (24), one can obtain [1]

$$\begin{aligned} |K(m_\pi^2,w)| = & |K(m_\pi^2,m)| \exp\Bigg\{ \frac{(w-m)}{\pi}\mathcal{P}\int_{m+\mu_\pi}^{\infty} dw' \\ & \times \left[\frac{\phi(w')}{(w'-m)(w'-w)} - \frac{\phi(-w')}{(w'+m)(w'+w)} \right] \Bigg\}, \end{aligned} \tag{25}$$

where $\phi(\pm w)$ is the phase of $K(m_\pi^2,w)$ along the positive (+) or negative (−) cut. Thus, $K(m_\pi^2,w)$ can be determined if these phases are known.

So far, dispersion relations just reflect analyticity properties of Greens functions and are void of any predictive power. This changes when one makes use of unitarity constraints that provide additional relations between the real and imaginary parts of the Green's function. The simplest example of the power of using analyticity and unitarity in conjucntion is the forward amplitude for the scattering of light with frequency ω from atoms. Unitarity implies that the forward amplitude for positive frequencies is related to the total cross section through the optical theorem

$$\text{Im } f(\omega)=\frac{\omega}{4\pi}\,\sigma_{\text{tot}}(\omega), \quad \omega>0, \tag{26}$$

leading, with one subtraction, to the famous Kramers-König relation

$$\text{Re } f(\omega)=\text{Re } f(0)+\frac{\omega^2}{2\pi^2}\,\mathcal{P}\int_0^\infty dw'\,\frac{\sigma_{\text{tot}}(w')}{(\omega'^2-\omega^2)}, \tag{27}$$

which allows the determination of $f(\omega)$ from the experimentally measured total cross section.

For the meson-nucleon T matrix, unitarity implies the well-known matrix equation

$$\text{Im } T=TT^\dagger, \tag{28}$$

from which the optical theorem follows. This equation assumes a simple form after projecting onto states of total angular momentum J, parity P, and isospin T. We will consider in the following sections a simple situation where there are two reaction channels, πN and ηN. The T matrix may then be written in the general form

$$T^l=\begin{pmatrix} \frac{1}{2i}\,(\rho_l e^{2i\delta_\pi^l}-1) & \frac{1}{2}\,\sqrt{1-\rho_l^2}e^{i(\delta_\pi^l+\delta_\eta^l)} \\ \frac{1}{2}\,\sqrt{1-\rho_l^2}e^{(i\delta_\pi^l+\delta_\eta^l)} & \frac{1}{2i}\,(\rho_l e^{2i\delta_\eta^l}-1)\end{pmatrix}, \tag{29}$$

where l labels the quantum numbers J, P, T, and the two-body channels are denoted by π and η. Furthermore, δ_π^l and δ_η^l are the elastic scattering phase shifts for πN and ηN scattering, respectively, given by

$$\tan 2\delta_i^l=\frac{2\text{Re } T_{ii}^l}{1-2\text{Im } T_{ii}^l}, \quad i=\pi,\eta, \tag{30}$$

and ρ_l is the corresponding inelasticity parameter. The T matrix is symmetric since time-reversal invariance has been assumed. Below the η threshold, only $T_{\pi\pi}^l$ is nonzero, and the familiar elastic form of T is obtained:

$$T_{\pi\pi}^l=\sin\,\delta_l^\pi e^{i\delta_\pi^l}. \tag{31}$$

For $w>0$, $T_{\pi\pi}$ describes πN scattering in the P_{11} partial wave ($l=1/2^+,1/2$) and for $w<0$ scattering in the S_{11} partial wave ($l=1/2^-,1/2$). The consequences of unitarity for $K(m_\pi^2,w)$ may be obtained by looking at its absorptive part which receives contributions from physical on-shell intermediate states. Unitarity provides for K, which is now a vector in the space of the different reaction channels, the constraint [1,3,6]

$$\text{Im } K=F^{-1}TFK^*. \tag{32}$$

Here, F^{-1} and F are phase space factors (see Appendix A). For the πNN form factor, this constraint can be written as

$$\begin{aligned}\text{Im } K_\pi(m_\pi^2,w)&=\theta(|w|-m-\mu_\pi)T_{\pi\pi}(w)K_\pi^*(m_\pi^2,w)\\&\quad+\theta(|w|-w_T)A(w),\end{aligned} \tag{33}$$

where w_T is the threshold energy of the first inelastic channel. The first term on the right-hand side of Eq. (33) arises from the intermediate pion-nucleon two-body state and the second term represents contributions from intermediate states with higher masses, e.g., $\pi\pi N$, ηN, $K\Lambda$, etc.

For $w<w_T$, the last term in Eq. (33) does not contribute and one sees from this equation that the phase of the form factor for the πNN vertex, $\phi_\pi=\text{Arg}(K_\pi)$, is determined by the elastic πN phase shift, defined in Eq. (30),

$$\phi_\pi=\delta_\pi^l=\frac{1}{2}\,\arctan\left(\frac{2\text{ Re } T_{\pi\pi}}{1-2\text{ Im } T_{\pi\pi}}\right). \tag{34}$$

We note that since the phase shift is a representation-independent observable quantity, both the real and imaginary parts of the representation-dependent off-shell form factor, Re K_π and Im K_π, must change under a field transformation such that the phase, $\phi_\pi\equiv\arctan(\text{Im } K_\pi/\text{Re } K_\pi)$, remains unchanged for $w<w_T$.

The use of the dispersion technique to obtain the vertex function $K(w)$ for $w\neq m$ with only experimental input faces problems in practice. In order to obtain the off-shell form factor from Eq. (25), the phase must be known up to infinite energies. Therefore, it is clear that one must make approximations about the behavior of the elastic phase shift for high energies and also about the contributions coming from the inelastic channels for $w>w_T$. Two such approximations have been proposed in the literature.

The simplest assumption is to ignore inelastic contributions, i.e., set $A=0$ in Eq. (33), which would be justifiable if the dispersion integral is dominated by the interval where the contribution from A is small compared to the elastic term. This is referred to as the ''threshold'' approximation [1]. It amounts to assuming that Eq. (34) remains valid for all energies and allows one to evaluate $K(w)$ in terms of the elastic phase shift. As shown in Ref. [6], the threshold assumption implies $\rho_l^2=1$, which is quite unrealistic as soon as one gets above the threshold for the $\pi\pi N$ channel.

To avoid this problem, Epstein [6] adopted in the dispersion analysis of the off-shell πNN form factor a suggestion by Goldberger and Treiman [7] which leads to a different *ad hoc* prescription to deal with the dispersion integral. Consider the right-hand side of Eq. (33), which, although it involves complex quantities, must nevertheless be real. This leads to the following conditions for the combined effect of the inelastic states contained in the complex quantity A:

$$\begin{aligned}\text{Re } A&=\text{Im } K_\pi-\text{Re } T_{\pi\pi}\text{ Re } K_\pi-\text{Im } T_{\pi\pi}\text{ Im } K_\pi,\\ \text{Im } A&=\text{Re } T_{\pi\pi}\text{ Im } K_\pi-\text{Im } T_{\pi\pi}\text{ Re } K_\pi.\end{aligned} \tag{35}$$

Epstein assumed that the inelastic channels will not generate a significant real part for A, i.e., $\mathrm{Re}\,A=0$. This leads to a different expression for the phase ϕ_π of the K_π form factor in terms of the elastic πN T matrix [6],

$$\phi_\pi=\arctan\left(\frac{\mathrm{Re}\ T_{\pi\pi}}{1-\mathrm{Im}\ T_{\pi\pi}}\right)$$
$$=\frac{1}{2}\arctan\left(\frac{2\ \mathrm{Re}\ T_{\pi\pi}(1-\mathrm{Im}\ T_{\pi\pi})}{(1-\mathrm{Im}\ T_{\pi\pi})^2-(\mathrm{Re}\ T_{\pi\pi})^2}\right). \quad (36)$$

In the rest of this paper, we will refer to this approximation for simplicity as the ''Goldberger-Treiman'' approximation. Of course, by setting $\mathrm{Im}\,A=0$ as well, from Eq. (35) we would then again obtain $\phi_\pi=\delta^l_\pi$, the threshold approximation. Below the inelastic threshold, the unitary constraint, Eq. (28), reads $(\mathrm{Im}\ T_{\pi\pi})^2+(\mathrm{Re}\ T_{\pi\pi})^2=\mathrm{Im}\ T_{\pi\pi}$, and simple inspection shows that Eqs. (34) and (36) agree, but above the inelastic threshold they can be quite different, especially if a resonance is present (see Sec. IV). The general problem of the choice of the phase above the inelastic threshold was also discussed in connection with the dispersive analyses of the pion elastic electromagnetic form factor [28].

Even knowing all the relevant T-matrix elements, it is not at all straightforward to solve for ϕ_π. To illustrate this, we stay with the case when there are only two channels present, the two-body states πN and ηN. Defining $K(\pm w)\equiv K(m_\pi^2,\pm w)$, we find from Eq. (32) that

$$\mathrm{Im}\ K_\pi(+w)=T^{P11}_{\pi\pi}(+w)K^*_\pi(+w)+\frac{F^-_{\eta\eta}}{F^-_{\pi\pi}}\,T^{P11}_{\pi\eta}(+w)K^*_\eta(+w),$$
$$\mathrm{Im}\ K_\eta(+w)=T^{P11}_{\eta\eta}(+w)K^*_\eta(+w)+\frac{F^-_{\pi\pi}}{F^-_{\eta\eta}}\,T^{P11}_{\pi\eta}(+w)K^*_\pi(+w), \quad (37)$$

and similar equations for $K(-w)$. In the two-channel case, the term A in Eq. (33) is

$$A(\pm w)=T_{\pi\eta}(\pm w)K^*_\eta(\pm w)\sqrt{\frac{q_\eta(E_\eta\mp m)}{q_\pi(E_\pi\mp m)}}, \quad (38)$$

where $q_{\pi(\eta)}$ is the $\pi(\eta)$ three-momentum in the c.m. frame and $E_{\pi(\eta)}=\sqrt{q^2_{\pi(\eta)}+m^2}$. As for $T_{\pi\pi}$, for $w>0$ $T_{\eta\pi}$ is given by the $J^PT=1/2^+,1/2$ partial wave while for $w<0$ it is given by the $1/2^-,1/2$ partial wave.

Equations (37) seem to provide the desired constraints for extracting the phases $\phi_\pi=\mathrm{Arg}(K_\pi)$ and $\phi_\eta=\mathrm{Arg}(K_\eta)$ from the meson-nucleon T matrix without resorting to any of the aforementioned approximations. First, one eliminates the magnitudes $|K_\pi|$ and $|K_\eta|$ from Eq. (37). Since the phase space factors cancel out, one obtains

$$T^2_{\pi\eta}e^{-i(\phi_\pi+\phi_\eta)}=[\sin\ \phi_\pi-T_{\pi\pi}e^{-i\phi_\pi}][\sin\ \phi_\eta-T_{\eta\eta}e^{-i\phi_\eta}]. \quad (39)$$

The real and imaginary parts of Eq. (39) provide two equations that should allow the determination of ϕ_π and ϕ_η from the T-matrix elements. However, this is not possible because the two resulting equations are, in fact, identical because of the unitary constraint for the T matrix. To see this, notice that $\mathrm{Im}\ K=F^{-1}TFK^*$ implies both

$$\mathrm{Im}\ K=F^{-1}(\mathrm{Re}\ T)F(\mathrm{Re}\ K)+F^{-1}(\mathrm{Im}\ T)F(\mathrm{Im}\ K) \quad (40)$$

and

$$(\mathrm{Im}\ T)F(\mathrm{Re}\ K)=(\mathrm{Re}\ T)F(\mathrm{Im}\ K). \quad (41)$$

Using Eq. (41), Eq. (40) reads

$$F^{-1}[1-\mathrm{Im}\ T]F(\mathrm{Im}\ K)=F^{-1}(\mathrm{Re}\ T)F(\mathrm{Re}\ K)$$
$$\Rightarrow[1-\mathrm{Im}\ T]\mathrm{Re}\ T^{-1}\ \mathrm{Im}\ T=\mathrm{Re}\ T. \quad (42)$$

Using the real part of the unitary condition for the T matrix, $TT^\dagger=\mathrm{Im}\ T$, yields $(\mathrm{Re}\ T)^2+(\mathrm{Im}\ T)^2=\mathrm{Im}\ T$, while the imaginary part results in the vanishing of the commutator $[\mathrm{Re}\ T,\mathrm{Im}\ T]=0$. This shows that Eq. (42) is just the T-matrix unitary constraint and the real and imaginary parts of Eq. (40) do not provide independent equations allowing the determination of the phase of K from the on-shell T matrix.

This does not necessarily imply that sidewise dispersion relations cannot be used to determine K (provided that one subtraction is enough), but more work needs to be done. Above the eta threshold, we can use Eq. (37) to determine $\mathrm{Im}\,K$ in terms of $\mathrm{Re}\,K$ and the on-shell T matrix. This may then be substituted into Eq. (24) to obtain a coupled set of Fredholm-like integral equations for $\mathrm{Re}\,K$. The problem, as shown below, is that between the pion and eta thresholds, $\mathrm{Im}\,K_\eta$ is expressed in terms of an off-shell T-matrix element; it might then be possible through dispersion relations to determine $T^{\pi\eta}$ at the needed off-shell points in terms of on-shell information. A more detailed investigation of this possibility is beyond the scope of this paper.

In the (hypothetical) case of a single channel system it seems to be possible to determine the phase ϕ and thus also the function $K(w)$ for the off-shell vertex in a model-independent fashion using the observable phase δ of the on-shell T matrix. This appears to be in contradiction with the observation in Sec. II that the off-shell form factor changes when we carry out field transformations. How can this be reconciled with the sidewise dispersion relations that express $K(w)$ in terms of observable quantities?

The answer lies in the fact that in the sidewise dispersion relation approach the number of necessary subtractions is *a priori* unknown. Indeed, different choices of the nucleon-interpolating field will, in general, lead to different asymptotic behaviors of the off-shell form factor. The examples given in Sec. II illustrate this point. From Eq. (10) we see that $K(w)=g$, i.e., is of order 1 as $w\rightarrow\infty$. On the other hand, the vertex function, Eq. (13), obtained from the transformed Lagrangian, is of order w at infinity. Thus, the ''representation dependence'' in sidewise dispersion relations shows up in the *a priori* unknown needed number of subtractions. As previously remarked, any predictive power of the sidewise dispersion relations method will be lost if two (or more) subtractions are necessary since we only know the form factor at the physical point $w=m$. Another way to improve the convergence of the dispersion integral is to consider the derivative of $K(w)$. However, none of its derivatives with respect to w at $w=m$ are known and, therefore, no information about an off-shell point can be obtained.

As dispersion relations do not depend on a particular Lagrangian, it is useful to look at the above discussion for the vertex function K in a different way and to contrast it with the dispersion relations for the pion-nucleon scattering amplitude. Consider the unitarity constraint, Eq. (32): evidently, it remains valid under the replacement $K \rightarrow f(w)K$, where $f(w)$ is a real function of w, reflecting a different off-shell behavior. If $f(w)$ is a polynomial in w and one has $f(m)=1$, then the analytical properties of K are not changed and K still satisfies a dispersion relation. However, in general, (additional) subtractions will be needed, and these subtractions have to be done at unphysical points, $w \neq m$, and, therefore, cannot be done model independently. When using the dispersion relation approach for the T matrix, we also may need subtractions to make the integrals converge. However, for this purpose we can do these subtractions at different energies where we have experimental information about the T matrix. In some cases, this makes it possible to determine the pion-nucleon T matrix at an unphysical point through dispersion relations, while the off-shell form factors can never be uniquely determined. Notice also that our discussion does not imply that dispersion relations for the electromagnetic form factor, with the momentum transfer q^2 as the dispersion variable, show any representation dependence. In this case the form factor $F(q^2)$ can be measured for a number of values of the four-momentum transfer q^2.

IV. A COUPLED-CHANNEL, UNITARY MODEL

In the previous section we have discussed two inherent difficulties of the sidewise dispersion relation approach applied to the off-shell form factors. The first one is the *a priori* unknown number of subtractions, which reflects the ''representation dependence.'' The second difficulty is related to determining ϕ, the phase of the vertex function, in terms of observable physical quantities: one is, in general, unable to properly take into account the contribution of all possible intermediate states to the absorptive (imaginary) part of the form factor. With respect to this second difficulty, two approximations had been proposed in the literature; the threshold approximation, Eq. (34), and the ''Goldberger-Treiman'' [6] approximation, Eq. (36). In this section we study these approximations by using a model with a nucleon-interpolating field that leads to a K satisfying a once-subtracted dispersion relation. Even after assuming the validity of only one subtraction, a precise determination of K through the dispersion relations remains extremely difficult (if not impossible). It is, therefore, interesting to see in the framework of a simple model under what circumstances the two approximations to the phase discussed in the previous section can be trusted to give reasonable results for $K(w)$. Indeed, $K(w)$ has already been ''extracted'' from the πN phase shifts using the Goldberger-Treiman approximation [6], but the approximation itself has not been examined. Since we will use a meson-loop model, this allows us also to examine, e.g., the behavior of the absorptive part of the strong form factor under field redefinitions, extending the studies of Sec. III beyond the tree level.

We construct a unitary T matrix based on the toy model meson-nucleon Lagrangian [29]

$$L=\frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi - \Phi \mathcal{M}^2 \Phi\right) + \overline{\psi}(i\partial\!\!\!/ - m)\psi - i\overline{\psi}\gamma_5 G\Phi\psi + \frac{1}{2}\overline{\psi}\Phi\Lambda\Phi\psi. \tag{43}$$

Apart from the conventional fermionic part describing the nucleon with mass m, we take into account two isoscalar mesons, given by the two-component field Φ, and their mass matrix $\mathcal{M}$:

$$\Phi=\begin{pmatrix}\pi\\ \eta\end{pmatrix}, \quad \mathcal{M}=\begin{pmatrix}\mu_\pi & 0\\ 0 & \mu_\eta\end{pmatrix}, \tag{44}$$

where we use the suggestive names ''pion'' and ''eta.'' We assume a pseudoscalar three-point meson-nucleon coupling G, and a scalar four-point meson-meson-nucleon coupling Λ, where

$$G=\begin{pmatrix}g_\pi\\ g_\eta\end{pmatrix}, \quad \Lambda=\begin{pmatrix}\lambda_{\pi\pi} & \lambda_{\pi\eta}\\ \lambda_{\eta\pi} & \lambda_{\eta\eta}\end{pmatrix}. \tag{45}$$

The Λ matrix has the dimension of an inverse mass, while G is dimensionless. The nonvanishing off-diagonal elements, $\lambda_{\pi\eta}$ and $\lambda_{\eta\pi}$, couple the two meson channels. For simplicity, we make the choice $\lambda_{\eta\pi}=\lambda_{\pi\eta}=\sqrt{\lambda_{\pi\pi}\lambda_{\eta\eta}}$ (see Appendix A).

While we cannot solve this model exactly, it is possible to select an infinite subset of diagrams which satisfies the necessary analyticity and unitary properties. We do that by treating the three-point ''G'' coupling to leading order only, while summing higher order contributions generated by the ''Λ'' interaction. Only two-particle intermediate states, i.e., πN and ηN, but not $\pi\pi N$ or $\eta\pi N$, are considered. Our approach does not satisfy crossing symmetry and, moreover, there are no meson loops that connect the incoming and outgoing nucleons; they would be either of second order in G, or have three-particle intermediate states. This selection of contributing diagrams does not generate a q^2 dependence for the form factor (in other words, what one usually refers to as the ''on-shell'' form factor is trivial in this model). However, it does generate a nontrivial dependence on the invariant mass p'^2 of the off-shell nucleon and satisfies two-body unitarity. We should also emphasize here that the truncation to ''two-body unitarity'' is an approximation, but it is the validity of the approximations made on top of our assumptions that we wish to test here.

The diagrams that can contribute to meson-nucleon scattering with our restrictions are shown in Fig. 1, and those contributing to the half-off-shell meson-nucleon form factor in Fig. 2. The external and internal mesons may be either pions or etas. As the ''Λ'' interaction is separable, we can express the geometric series for the T matrix in a closed form:

$$\mathcal{T}=\Lambda+\Lambda\mathcal{I}\Lambda+\Lambda\mathcal{I}\Lambda\mathcal{I}\Lambda+\cdots=(1-\Lambda\mathcal{I})^{-1}\Lambda, \tag{46}$$

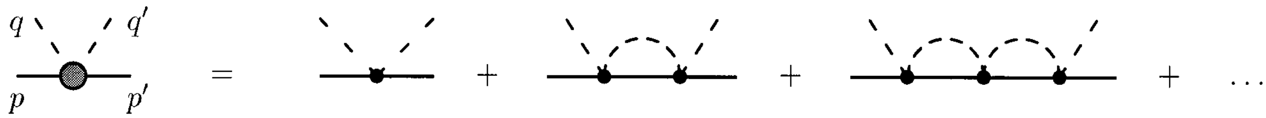

FIG. 1. Diagrams contributing to the on-shell meson-nucleon T matrix [Eq. (46) in text]. The dashed lines denote either a pion or an eta meson and (●) stands for a Λ-type coupling.

where $\mathcal{I}=\mathrm{diag}(\mathcal{I}_\pi, \mathcal{I}_\eta)$ is a diagonal matrix whose $(\pi\pi)$ and $(\eta\eta)$ entries are pion-nucleon and eta-nucleon loop integrals, respectively. The interested reader is referred to Appendix A for details of results stated without proof throughout this section. The integrals $\mathcal{I}$, and, therefore, the $\mathcal{T}$ matrix, depend only on the total center-of-mass momentum squared, $s=w^2=(p+q)^2$, and have no angular dependence. Using the projection operators defined in Eq. (2) and the standard partial wave projections, it is easy to show that P_+ projects only into the f_{0+} partial wave and P_- only into the f_{1-} partial wave, that is

$$\mathcal{T}=\mathcal{T}^{0+}P_+ + \mathcal{T}^{1-}P_- . \tag{47}$$

Taking into account the appropriate phase space factors F, it can be shown (see Appendix A) that $T=F\mathcal{T}F$, satisfies unitarity for $|w|<m+M$, where M is the cutoff needed to regularize the loop integrals $\mathcal{I}$,

$$\mathrm{Im}(T^{S11})=T^{S11}(T^{S11})^\dagger, \tag{48}$$

and analogously for T^{P11}. Thus, the T^l_{ij} may be written in the form given in Eq. (29) where $\delta^l_\pi(\delta^l_\eta)$ are the pion (eta) phase shifts (S wave for $l=0$ and P wave for $l=1$) and ρ_l are the corresponding inelasticities ($\rho_l=1$ below the eta threshold, $w_T=m+\mu_\eta$). For a numerical study of these form factors we take for m and μ_π 0.939 GeV and 0.14 GeV, respectively, and choose μ_η to be 0.42 GeV, since at $w=1.36$ GeV$=m+\mu_\eta$, the P_{11} inelasticity starts to deviate from unity. The Λ couplings are chosen to reproduce some qualitative features of the physical pion-nucleon scattering phase shifts and inelasticity, in particular, a resonance appearing above the inelastic threshold. While the actual πN scattering amplitude exhibits this feature in both the P_{11} and S_{11} channels, our model is too simple to simultaneously produce resonances in both channels. We therefore concentrate on the P_{11} channel, and show in Fig. 3 the phase shifts and inelasticity parameter obtained with $\lambda_{\pi\pi}=0.5$ MeV^{-1} and $\lambda_{\eta\eta}=0.8$ MeV^{-1}, which leads to a resonance in the P_{11} channel with a substantial inelasticity. This parametrization yields $K_\eta(m)/K_\pi(m)=-0.87$. As mentioned, the presence of a finite cutoff violates unitarity for $w>m+M$, and we, therefore, use a large cutoff, $M=10$ GeV.

The half-off-shell strong vertex in our model is generated by the series in Fig. 2, and gives

$$[\Gamma^5(p',p)]^T u(p)=\gamma_5 G^T[1-\mathcal{I}(w)\Lambda]^{-1}u(p)$$
$$\Rightarrow \Gamma^5(p',p)u(p)$$
$$=\gamma_5[1-\Lambda\mathcal{I}(w)]^{-1}Gu(p). \tag{49}$$

Here, the transpose acts on the channel space indices only, and $w=\sqrt{(p+q)^2}$. With our assumptions, the pion-nucleon strong vertex function reads

$$K_\pi(w)=\frac{[1-\lambda_{\eta\eta}\mathcal{I}_\pi(w)]g_\pi+\lambda_{\pi\eta}\mathcal{I}_\eta(w)g_\eta}{1-\lambda_{\pi\pi}\mathcal{I}_\pi(w)-\lambda_{\eta\eta}\mathcal{I}_\eta(w)}, \tag{50}$$

where we have ignored graphs of order $O(G^3)$ and higher, as well as intermediate states with more than two particles. Meson loops on the on-shell nucleon need not to be considered since their contributions can be absorbed in the definition of the on-shell vertex. The vertex function $K(w)$ is determined through the P-wave on-shell scattering amplitude, $T^{P11}(w)$, and $K(-w)$ by the S-wave on-shell amplitude, $T^{S11}(w)$. In the examples below, the values of g_π and g_η will be varied to change the ratio of the on-shell form factors, $K_\eta(m)/K_\pi(m)$.

As shown in Appendix A, the unitarity equation for K, Eq. (37), is satisfied in this model below the cutoff. As mentioned above, below the pion threshold, $|w|<m+\mu_\pi$, one has $\mathrm{Im}\,K=0$. Between the pion and eta thresholds, $m+\mu_\pi<|w|<m+\mu_\eta$, we have

$$\mathrm{Im}\;K_\pi(+w)=T^{P11}_{\pi\pi}K^*_\pi(+w),$$

$$\mathrm{Im}\;K_\eta(+w)=(F^-_{\pi\pi})^2\mathcal{T}^{P11}_{\pi\eta}K^*_\eta(+w), \tag{51}$$

where $\mathcal{T}^{P11}_{\pi\eta}$ is the $\pi\eta$ matrix element multiplying the P_- operator in Eqs. (47) and (A8), in this case evaluated at an off-shell point. Although the form of Eq. (51) is specific to our model, it is true, in general, that $\mathrm{Im}\,K_\eta$ is nonzero between the thresholds and in this region is related to off-shell quantities.

Let us now discuss the dispersion relations for the off-shell form factors in this model. While the presence of the cutoff violates unitarity, it does not affect the validity of the

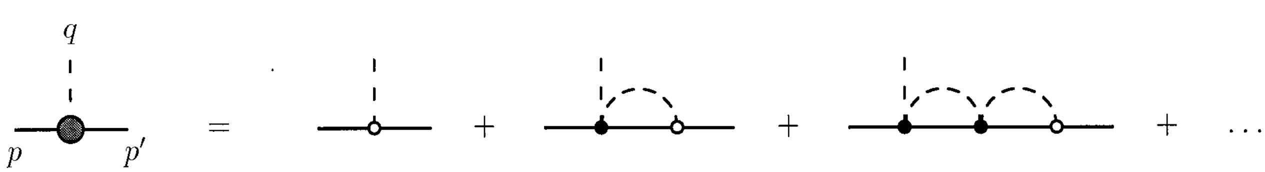

FIG. 2. Diagrams contributing to the half-off-shell ($p^2=m^2$, $p'^2\neq m^2$) meson-nucleon (strong) form factor [Eq. (49) in text]. The dashed lines may denote either a pion or an eta meson, (●) stands for a Λ-type coupling, and (○) for a G-type coupling.

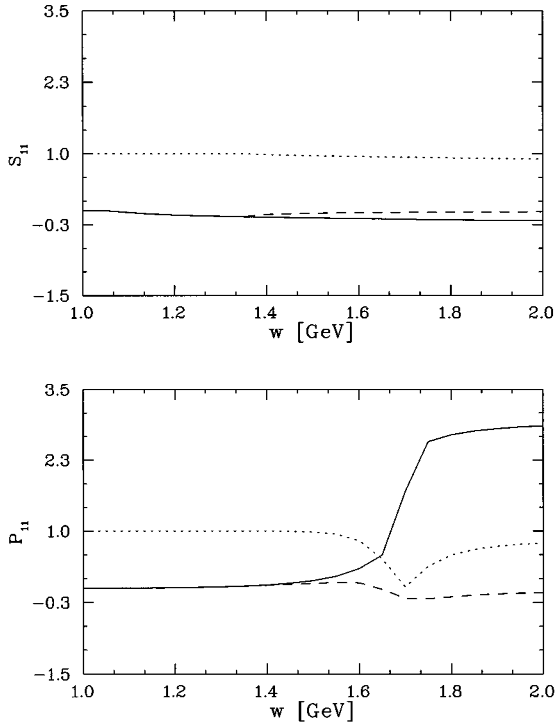

FIG. 3. Inelasticities (dotted lines), the pion-nucleon phase shift in radians (solid lines), and the phase of the pion-nucleon form factor (dashed lines) for the S_{11} and P_{11} channels in our model with parameters $\lambda_{\pi\pi}$=0.5 MeV^{-1}, $\lambda_{\eta\eta}$=0.8 MeV^{-1}, $g_\pi=-g_\eta=2$.

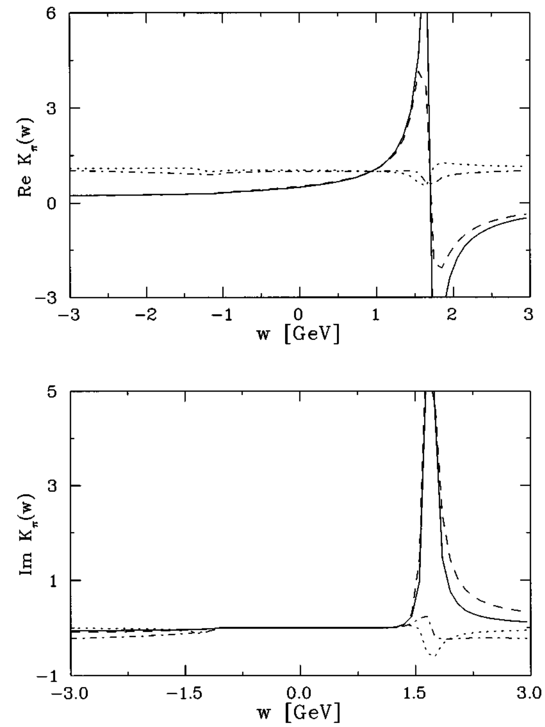

FIG. 4. Various approximations for determining the pion form factor K_π(solid=threshold, dot dashed=Goldberger Treiman), compared with the model prediction for K_π for two choices of G couplings corresponding to $K_\eta(m)/K_\pi(m)=-1$ (dots) and $K_\eta(m)/K_\pi(m)=+1$ (dashes).

dispersion relations. Because of the choice of a large cutoff, the dispersion integral has largely converged by the time the cutoff is reached. It is well known that the functions $\mathcal{I}_\pi$ and $\mathcal{I}_\eta$ satisfy once-subtracted dispersion relations [30]. K_π and K_η have the same analytical structures as $\mathcal{I}_\pi$ and $\mathcal{I}_\eta$, apart from possible additional poles on the first Riemann sheet. Furthermore, the number of needed subtractions may be different, but it is easy to establish in our model that K also satisfies a once-subtracted dispersion relation. On the other hand, the existence of poles in the complex w plane is more difficult to assess. We have simply established their absence numerically by showing that the once-subtracted dispersion relation is satisfied to six significant figures for -2 GeV $<w<2$ GeV.

In the analysis of the pion-nucleon vertex function by Epstein [6] and by Bos [3], it was found that the Goldberger-Treiman approximation leads to a much smoother off-shell behavior of K than that in the threshold approximation. This can be easily explained: if one uses the threshold approximation, the phase of the form factor is given by the scattering phase shift and we, therefore, expect the function K to show resonance behavior. When the πNN phase shift passes through $\pi/2$ the threshold approximation to $\mathrm{Re}\,K$ will change sign and $\mathrm{Im}\,K$ peaks. On the other hand, using the Goldberger-Treiman assumption, ϕ_π is constrained to be in the interval $-\pi/2<\phi_\pi<\pi/2$. This is easily seen from Eq. (36) with $\rho_l<1$, which implies $T_{\pi\pi}<1$. Therefore, $\mathrm{Re}\,K$ will not change sign since ϕ_π does not pass through $\pi/2$. Thus, we expect that the Goldberger-Treiman approximation will generate a smooth off-shell dependence, while the threshold assumption will generate a more rapid dependence on w if there is a resonance in the scattering T matrix, as happens in reality for πN scattering as well as with our P_{11} phase shifts.

Our model allows us to put these previous analyses into perspective and to confirm our qualitative expectations. In Fig. 4 we show the exact model results for K_π with g_π and g_η adjusted to give $K_\eta(m)/K_\pi(m)=-1$ and 1, all other parameters as in Fig. 3. As expected, K obtained from the threshold assumption (solid lines) displays a rapid w variation because of the resonance in the P_{11} channel (see Fig. 3), while the Goldberger-Treiman approximation (dot-dashed lines) leads to a rather smooth energy dependence of K. Whether the Goldberger-Treiman or threshold approximation is better cannot be answered in general. It depends on the details of the dynamics. At $K_\eta(m)/K_\pi(m)=-1$, the Goldberger-Treiman approximation seems to work well, while at $+1$ it is the threshold approximation that works well. We, therefore, conclude that neither approximation may be trusted a priori at any w. Figure 4 shows that there can be large discrepancies between the exact model result and the phase approximations even in the vicinity of the on-shell point.

As shown in Fig. 3, the inelasticity deviates significantly from unity for large values of w. That the qualitative features of the two approximations discussed above are not because of this large inelasticity was confirmed by considering another parametrization (results not shown). A resonance in the P_{11} channel can also be obtained with, e.g., $\lambda_{\pi\pi}$=0.5 MeV^{-1} and $\lambda_{\eta\eta}$=0.05 MeV^{-1}. Since $\lambda_{\eta\pi}=\lambda_{\pi\eta}=\sqrt{\lambda_{\pi\pi}\lambda_{\eta\eta}}$, this corresponds to a much weaker coupling between the channels and the inelasticity remains close to one. The same fea-

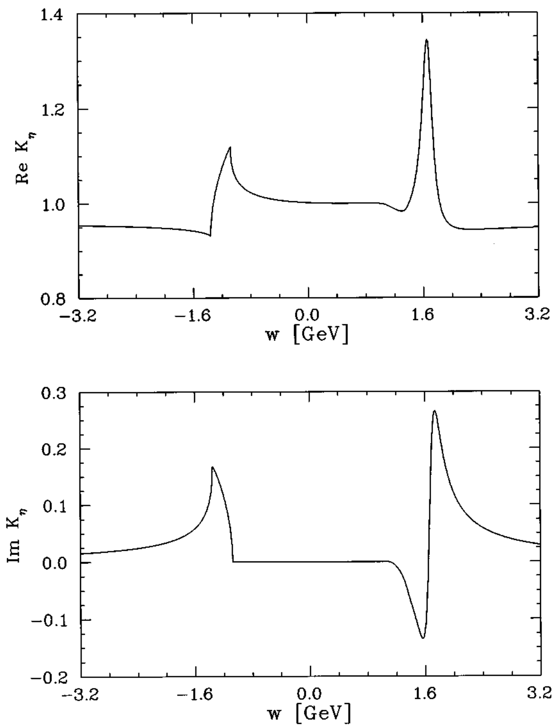

FIG. 5. The eta-nucleon form factor. Parameters as in Fig. 3.

tures as in Fig. 4 were found, thus casting doubt on the use of the threshold approximation in general. In fact, the threshold approximation requires $\rho_l = 1$, while the Goldberger-Treiman is too restrictive to allow for a resonant behavior of the form factor. Thus, neither of these approximations can be expected to be satisfactory.

The above observations are not based on some specific detail of our model, but on rather general properties such as the existence of resonances in the T matrix. The dependence on the details of the underlying reaction mechanism that we have shown with our simple model probably underestimates the real situation. In our model, the off-shell variation at positive w is mainly because of the resonance in the P_{11} channel. For example, the S_{11} resonance in πN scattering, absent in our model, would afflict the negative w sector as well.

It is also interesting to look at the model results for the eta form factor $K_\eta(w)$. The results for the same set of couplings, Λ, G, as in Fig. 3, are shown in Fig. 5. Because of the simplicity of our ''toy model,'' the results again only illustrate some general qualitative features. At negative energies, we see very pronounced effect because of the pion and eta thresholds. This effect is not visible for positive w since the P-wave phase space suppresses the cusp. It can be seen that K_η is complex even below the η threshold and displays some rapid energy dependence around the η threshold. These features arise because of the branch cuts associated with the thresholds, and, therefore, should be general features of the function K_η. The magnitude of these effects will, of course, depend on the model. Nevertheless, this casts doubt on the use of simple tree-level amplitudes with real coupling constants to extract the ηNN coupling constant, for example from photoproduction of etas [31].

V. SUMMARY AND CONCLUSIONS

Sidewise dispersion relations have been suggested in the literature as a method to obtain the electromagnetic and strong half-off-shell form factors of the nucleon. These form factors enter in calculations of, e.g., nuclear processes or of two-step reactions on a free nucleon where one includes the structure of the nucleon in terms of dressed vertices. We have focused our discussion on the strong pion-nucleon vertex, where sidewise dispersion relations relate the half-off-shell strong form factor to on-shell meson-nucleon scattering. Two aspects of this approach were examined, its representation dependence and the validity of approximations that have been used in the literature.

The strong vertex, or three-point Green's function, is not uniquely defined when one or both nucleons are not on their mass shells. They are dependent on the representation one chooses for the intermediate (off-shell) fields and, therefore, cannot be unambiguously extracted from experimental data. In order to illustrate this representation dependence and to show how it enters in the sidewise dispersion relations, we used unitarily equivalent Lagrangian models. Starting at the tree level, we showed how off-shell vertices do change under a change of representation, while the on-shell vertices are oblivious to such changes. We then showed how in on-shell amplitudes corresponding to two-step processes, this representation dependence of the vertices manifests itself through contributions of pole as well as contact terms. This means that what one would call off-shell effects resulting from a vertex in an amplitude in one representation are related to contact terms in another. We then showed how the changes of representations can change the asymptotic behavior of the off-shell form factor, thus requiring a representation-dependent number of subtractions in the dispersion relation. In other words, representation dependence enters the sidewise dispersion analysis through the number of necessary subtractions. As the form factor is only known at the on-shell point, $w = m$, only one subtraction constant is known and the sidewise dispersion analysis, thus, has no predictive power for the vertex function. We showed at the one-loop level in perturbation theory that not only the real, but also the ''absorptive'' imaginary part of the half-off-shell form factor, related to open physical channels, exhibit this representation-dependent asymptotic behavior.

Even when one chooses a particular representation (i.e., assumes one subtraction), one still faces problems when trying to obtain the corresponding off-shell vertex functions. These difficulties arise because of the contribution of inelastic channels, i.e., other than πN intermediate states. These channels contribute through the unitarity constraint that relates the half-off-shell vertex function to the meson-nucleon T matrix, or scattering amplitude. Approximations how to deal with these channels in an ad hoc fashion had been proposed in the literature, but their validity had not been examined.

In order to study these recipes, we introduced a very simple coupled-channel, unitary model for the pion-nucleon system, where the inelastic channel is represented by an ηN intermediate state. We first established that the half-off-shell form factor in this toy model satisfies a once-subtracted sidewise dispersion relation and then compared this result to the results obtained from sidewise dispersion relations using these *ad hoc* prescriptions. We found that differences among the approximations and the exact model result for the off-shell vertex functions can be sizable, particularly when w lies in the vicinity of resonances of the T matrix, where the two prescriptions we tested produce very different results. We found that which of the two prescriptions is better, i.e., is closer to the exact model result, depends on details of the dynamics assumed in the model. Therefore, neither of the two approximations is a priori preferred, and the results one obtains by using such recipes remain questionable.

We conclude that, in practice, sidewise dispersion relations cannot provide reliable and unique information about the structure of off-shell nucleons. The number of required subtractions is representation dependent and thus a priori unknown. Even if one chooses a particular representation, the inclusion of the other reaction channels cannot be dealt with without approximations. The off-shell vertex, which has a much more complicated structure than the free vertex, thus cannot be extracted from experimental data, but should instead be consistently calculated within the framework of a microscopic theory. Such a calculation will yield the dressed off-shell vertices and the concommitant contact terms. The proper interpretation of future high precision measurements of intermediate energy processes depends crucially on our ability to carry out such consistent calculations in realistic microscopic models.

ACKNOWLEDGMENTS

We would like to thank J. H. Koch for many helpful discussions and a critical reading of the manuscript. This work was supported in part by the Foundation for Fundamental Research on Matter (FOM) and the National Organization for Scientific Research (NWO). G.I.P. acknowledges support through Human Capital and Mobility Grant No. ERBCHBICT941430.

APPENDIX A

Here, we present some details of our model calculation. The integral matrix introduced in Eq. (46):

$$\mathcal{I}=\begin{pmatrix}\mathcal{I}_\pi & 0\\ 0 & \mathcal{I}_\eta\end{pmatrix} \tag{A1}$$

describes the meson-nucleon loop Feynman integrals that appear on the right-hand side (RHS) of Figs. 1 and 2. Thus,

$$\mathcal{I}_i(w)=-i\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\mu_i^2)}\frac{1}{(\not p+\not q-\not k-m)}, \tag{A2}$$

where $w=\sqrt{(p+q)^2}$. Equation (A2) is made meaningful by Pauli-Villars regularization of the meson propagators:

$$\frac{i}{k^2-\mu_i^2}\rightarrow\frac{i}{k^2-\mu_i^2}-\frac{i}{k^2-M^2}. \tag{A3}$$

For simplicity, we will use the same cutoff mass M for both the π and η propagators.

We may now write $\mathcal{I}_i=m\mathcal{I}_i^0+(\not p+\not q)\mathcal{I}_i^1$, and defining $\Delta_i=(w^2+m^2-\mu_i^2)^2-4w^2m^2$, we find, for the imaginary parts,

$$\mathrm{Im}\,\mathcal{I}_i^0=\frac{1}{16\pi}\frac{\sqrt{|\Delta_i|}}{w^2}\,\theta(\Delta_i)-[\mu_i\rightarrow M],$$

$$\mathrm{Im}\,\mathcal{I}_i^1=\frac{1}{16\pi}\frac{(w^2+m^2-\mu_i^2)}{2w^2}\frac{\sqrt{|\Delta_i|}}{w^2}\,\theta(\Delta_i)-[\mu_i\rightarrow M], \tag{A4}$$

while the real parts are given by

$$\mathrm{Re}\,\mathcal{I}_i^0=\frac{1}{16\pi^2}\int_0^1 dx\,\ln|w^2x^2+\beta_i x+m^2|-[\mu_i\rightarrow M],$$

$$\mathrm{Re}\,\mathcal{I}_i^1=\frac{1}{16\pi^2}\int_0^1 dx\,x\,\ln|w^2x^2+\beta_i x+m^2|-[\mu_i\rightarrow M], \tag{A5}$$

with $\beta_i\equiv(\mu_i^2-w^2-m^2)$. We now make the choice $\lambda_{\eta\pi}=\lambda_{\pi\eta}=\sqrt{\lambda_{\pi\pi}\lambda_{\eta\eta}}$ which considerably simplifies the formulas, rendering $\mathcal{T}$ of the form

$$\mathcal{T}_{ij}=\Lambda_{ij}\frac{1}{a+(\not p+\not q)b}, \tag{A6}$$

with

$$a\equiv1-m(\lambda_{\pi\pi}\mathcal{I}_\pi^0+\lambda_{\eta\eta}\mathcal{I}_\eta^0),$$

$$b\equiv-(\lambda_{\pi\pi}\mathcal{I}_\pi^1+\lambda_{\eta\eta}\mathcal{I}_\eta^1). \tag{A7}$$

Using the projection operators defined in Eq. (2), T may be written as

$$\mathcal{T}=\frac{\Lambda}{a+wb}P_+ +\frac{\Lambda}{a-wb}P_-. \tag{A8}$$

As our T matrix has no $x=\cos\theta$ dependence, it is easy to show using the standard partial wave projections that P_+ projects only into the f^{0+} partial wave and P_- only into the f^{1-} partial wave. The formalism for meson-nucleon scattering is well known [30] and need not be repeated here. We only mention that phase space factors must be included in the T matrix, Eq. (A8), that is, in terms of the partial waves $f_{ij}^{l\pm}$; it is the object $T_{ij}=\sqrt{|q_i||q_j|}f_{ij}^{l\pm}$ (no sum over i,j) that satisfies the simple unitarity equation (28). Including phase space factors, we, thus, obtain

$$T_{ij}^{S11}=\frac{1}{8\pi w(wb+a)}\mathcal{F}_{ij}^+, \tag{A9}$$

$$T_{ij}^{P11}=\frac{1}{8\pi w(wb-a)}\mathcal{F}_{ij}^-, \tag{A10}$$

where $\mathcal{F}^\pm=F^\pm\Lambda F^\pm$, with $F^\pm=\mathrm{diag}[\sqrt{|q_\pi|(E_\pi\pm m)},\sqrt{|q_\eta|(E_\eta\pm m)}]$, i.e.,

$$\mathcal{F}^\pm=\begin{pmatrix} f_\pi^\pm & \sqrt{f_\pi^\pm f_\eta^\pm}\\ \sqrt{f_\pi^\pm f_\eta^\pm} & f_\eta^\pm\end{pmatrix}. \tag{A11}$$

Here, $f_\pi^\pm = \lambda_{\pi\pi}|q_\pi|(E_\pi \pm m)$ (and similarly for $f_\eta^\pm$) and the $+(-)$ sign is associated with $\mathcal{T}^{S11}(\mathcal{T}^{P11})$, respectively. We now show that unitarity is satisfied. Notice that $\mathcal{F}^\pm$ is of the form $\mathcal{F}_{ij}^\pm = \sqrt{h_i^\pm h_j^\pm}$ and thus $(\mathcal{F}^\pm)^2 = \mathrm{Tr}(\mathcal{F}^\pm)\mathcal{F}^\pm$. Using $(w^2+m^2-\mu_i^2)=2wE_i$ it is easy to inspect that Eqs. (A4) and (A7) imply

$$
\begin{aligned}
-8\pi w\ \mathrm{Im}(a+wb) &= f_\pi^+ + f_\eta^+ \Rightarrow -8\pi w\ \mathrm{Im}(a+wb)\mathcal{F}^+ \\
&= \mathrm{Tr}(\mathcal{F}^+)\mathcal{F}^+ = (\mathcal{F}^+)^2 \\
&\Rightarrow -\frac{1}{8\pi w}\frac{\mathrm{Im}(a+wb)}{|a+wb|^2}\mathcal{F}^+ \\
&= \frac{1}{(8\pi w)^2}\frac{1}{|a+wb|^2}(\mathcal{F}^+)^2 \Rightarrow \mathrm{Im}(T^{S11}) \\
&= T^{S11}(T^{S11})^\dagger, \qquad \text{(A12)}
\end{aligned}
$$

and analogously for T^{P11}. Notice that, for $w \geqslant M+m$, the extra terms $[\mu_i \to M]$ contribute to the RHS of Eq. (A4). As a result, Eq. (A12) is spoiled and unitarity is violated. However, we will take M large enough so that this violation of unitarity is of no practical consequence.

Let us next consider the unitarity constraint for the strong half-off-shell vertex $\Gamma^5 u(p)$ given by Eq. (49). Writing $\mathcal{T}$ in terms of the unitary T matrix,

$$
\begin{aligned}
\mathcal{T} = 8\pi w[&(F^+)^{-1}T^{S11}(F^+)^{-1}P_+ \\
&+(F^-)^{-1}T^{P11}(F^-)^{-1}P_-], \qquad \text{(A13)}
\end{aligned}
$$

and using the definition of $\mathcal{T}$, Eq. (46), we obtain

$$
\begin{aligned}
\Gamma^5 u(p) &= \gamma_5(1-\Delta\mathcal{I})^{-1}\Lambda\Lambda^{-1}Gu(p) = \gamma_5\mathcal{T}\Lambda^{-1}Gu(p) \\
&= 8\pi w\gamma_5[(F^+)^{-1}T^{S11}(F^+)^{-1}P_+ \\
&\quad +(F^-)^{-1}T^{P11}(F^-)^{-1}P_-]\Lambda^{-1}Gu(p) \\
&= 8\pi w[(F^+)^{-1}T^{S11}(F^+)^{-1}P_- \\
&\quad +(F^-)^{-1}T^{P11}(F^-)^{-1}P_+]\Lambda^{-1}G\gamma_5 u(p), \qquad \text{(A14)}
\end{aligned}
$$

where we have used $\gamma_5 P_\pm = P_\mp \gamma_5$. Using Eq. (3), we find

$$K(+w) = 8\pi w[F^-(w)]^{-1}T^{P11}(w)[F^-(w)]^{-1}\Lambda^{-1}G,$$

$$K(-w) = 8\pi w[F^+(w)]^{-1}T^{S11}(w)[F^+(w)]^{-1}\Lambda^{-1}G. \qquad \text{(A15)}$$

Thus, $K(w)$ is related to the P-wave on-shell amplitude $T^{P11}(w)$ and $K(-w)$ to the S-wave on-shell T-matrix amplitude $T^{S11}(w)$. Taking the imaginary parts of both sides, we obtain

$$
\begin{aligned}
\mathrm{Im}\ K(+w) &= 8\pi w(F^-)^{-1}\ \mathrm{Im}(T^{P11})(F^-)^{-1}\Lambda^{-1}G \\
&= 8\pi w(F^-)^{-1}(T^{P11})^\dagger T^{P11}(F^-)^{-1}\Lambda^{-1}G \\
&= 8\pi w(F^{-1})^{-1}(T^{P11})^\dagger F^-(F^-)^{-1} \\
&\quad \times T^{P11}(F^-)^{-1}\Lambda^{-1}G \\
&= (F^-)^{-1}(T^{P11})^\dagger F^- K(+w) \\
&= (F^-)^{-1}(T^{P11})F^- K^*(+w), \qquad \text{(A16)}
\end{aligned}
$$

where, in the last step, we have taken the complex conjugate of Eq. (A16) and used the fact that T is symmetric so as to cast Eq. (A16) in the form of Eq. (32). Similar equations are found for $K(-w)$ with $F^- \to F^+$ and $T^{P11} \to T^{S11}$. Notice that we have tacitly assumed that Λ has an inverse, but we have chosen Λ such that this is not the case. However, we have explicitly checked that Eq. (37) remains valid.

APPENDIX B

Here, we study the effect of field transformations on the absorptive (imaginary) part of the strong form factor. To do this, we must go beyond the tree level. Ideally, we would like to perform the transformation, Eq. (8), to the one-channel version of our model, Eq. (43), checking that the off-shell form factor shows a different asymptotic behavior while remaining invariant on shell. To first order in β, we obtain

$$
\begin{aligned}
L' = L_f - i(g+2m\beta)\bar{\psi}\gamma_5\phi\psi + \beta\bar{\psi}\gamma_5(\partial\!\!\!/\phi)\psi + \frac{\lambda}{2}\bar{\psi}\phi^2\psi \\
+ i\beta\lambda\bar{\psi}\gamma_5\phi^3\psi, \qquad \text{(B1)}
\end{aligned}
$$

where L_f represents the free (kinetic) part of the Lagrangian. However, the equivalence theorem (representation independence of on-shell form factors) only holds if all diagrams to a given order are included. In particular, diagrams that we have omitted because they do not contribute to the imaginary part, as for example diagrams with meson loops that dress the on-shell nucleon as well as reducible diagrams with closed loops, have to be included as well. Unfortunately, that means that it is impossible to make a nonperturbative comparison since we would have to solve both theories exactly, without being able to restrict ourselves to an infinite subset of diagrams as in the previous section.

We can still, however, make a perturbative comparison. First of all, we can check to $O(\beta\lambda)$ that the on-shell form factors are the same between L and L'. That will provide an example of the representation in-dependence of on-shell form factors beyond the tree-level result of Sec. II. To show this, we need to take into account all diagrams in Fig. 6. Notice that diagram [Fig. 6(f)] is present only in the transformed Lagrangian, Eq. (B1). The comparison is most easily made by examining how the terms proportional to the ‘‘tadpole’’ integral

$$\tau \equiv -i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\mu^2} \qquad \text{(B2)}$$

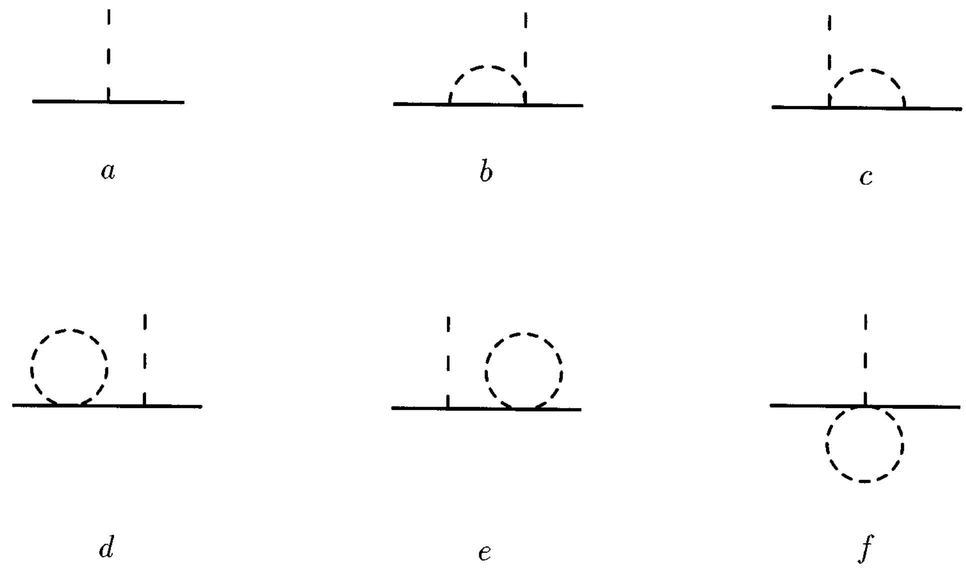

FIG. 6. Feynman diagrams for Lagrangian L' of Eq. (B1) contributing to the form factor at order $\beta\Lambda$.

compare for the on-shell matrix element [hence the $\overline{u(p')}$ spinor to the left as well] between the two models. From the transformed Lagrangian L', we obtain for the tadpole graphs

$$(d')=(d)+\frac{\beta\lambda\tau}{2}\overline{u(p')}\gamma_5(2m+\not{q})\frac{1}{\not{p}-m}u(p)$$
$$=(d)-\frac{\beta\lambda\tau}{2}\overline{u(p')}\gamma_5 u(p), \tag{B3}$$

$$(e')=(e)+\frac{\beta\lambda\tau}{2}\overline{u(p')}\frac{1}{\not{p}'-m}\gamma_5(2m+\not{q})u(p)$$
$$=(e)-\frac{\beta\lambda\tau}{2}\overline{u(p')}\gamma_5 u(p), \tag{B4}$$

$$(f')=3\beta\lambda\tau\overline{u(p')}\gamma_5 u(p). \tag{B5}$$

The β-dependent contribution from the (b') and (c') graphs can also be cast in terms of τ by using the Dirac equation for the on-shell spinors

$$(b')=i\lambda\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2}\overline{u(p')}\frac{i}{\not{p}-\not{k}-m}$$
$$\times\gamma_5(g+2m\beta-\beta\not{k})u(p)$$
$$=(b)-i\beta\lambda\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\mu^2}\overline{u(p')}\frac{1}{\not{p}-\not{k}-m}$$
$$\times(m-\not{p}+\not{k})\gamma_5 u(p)$$
$$=(b)-\beta\lambda\tau\overline{u(p')}\gamma_5 u(p), \tag{B6}$$

and similarly for (c'). From Eqs. (B3)–(B6), it is clear that the overall coefficient of the β-dependent terms $\lambda\tau\overline{u(p')}\gamma_5 u(p)$ vanishes

$$\left(-\frac{1}{2}-\frac{1}{2}+3-1-1\right)=0. \tag{B7}$$

That completes the proof of on-shell invariance. What about the imaginary part? The on-shell form factor has no imaginary part. For the half-off-shell ($p^2=m^2$) form factor, the tadpole contributions are real. The (b') contribution is also real, since, for $w=m$, $\Delta=-\mu^2(4m^2-\mu^2)<0$ [cf. Eq. (A4)]. Thus, the only diagrams that can generate an imaginary part are (c') and (c) [obtained from (c') by taking $\beta\to 0$]. We find

$$(c)'=\gamma_5\{[g+2m\beta]\mathcal{I}+\beta\mathcal{J}\}\lambda u(p), \tag{B8}$$

where $\mathcal{J}$ is a Feynman integral resulting from the ''pseudovector'' coupling and is defined analogously to $\mathcal{I}$ [Eq. (A2) and Fig. 2]:

$$\mathcal{J}(p')=-i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\mu^2}\not{k}\frac{1}{\not{p}+\not{q}-\not{k}-m}$$
$$=-i\int\frac{d^4k}{(2\pi)^4}\frac{(\not{p}'-m)}{k^2-\mu^2}\frac{(\not{p}'-\not{k}+m)}{(p'-k)^2-m^2}$$
$$+i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\mu^2}\Rightarrow\mathcal{J}(p')$$
$$=(\not{p}'-m)\mathcal{I}(p')-\tau, \tag{B9}$$

where τ is a real c number (i.e., independent of the off-shell variable w), and, therefore, does not contribute to the once-subtracted dispersion relation. Thus,

$$(c)'=\{(g-\beta[\not{p}'-m])\mathcal{I}-\beta\tau\}\gamma_5 u(p). \tag{B10}$$

We clearly see that the same off-shell operator, $(\not{p}'-m)$, multiplies both the real and imaginary parts of the integral $\mathcal{I}$. We conclude that the imaginary part of the off-shell form factor shows a higher power asymptotic behavior in w that is representation dependent, in agreement with our arguments based on unitarity.

As a last exercise that clarifies the points made in this work, consider the half-off-shell vertex function in the two-channel Lagrangians

$$L_1 = L_P + \frac{1}{2}\,\overline{\psi}\Phi\Lambda\Phi\psi, \tag{B11}$$

$$L_2 = \overline{L} + \frac{1}{2}\,\overline{\psi}\Phi\Lambda\Phi\psi. \tag{B12}$$

Here, L_P is given by Eq. (7) and $\widetilde{L}$ by Eq. (9). Keeping $O(\beta)$ terms only (where β is now a two-component vector such as G), L_1 is our original Lagrangian, Eq. (43), whereas the second is a different ''model,'' not unitarily equivalent to L_1, but, nevertheless, generating the same T matrix (in the sense of Fig. 1). To $O(G,\beta)$, the off-shell form factor generated by L_2 is [cf. Eq. (49)]

$$\Gamma_2^5 u(p) = \gamma_5\, \frac{1}{1-\Lambda\mathcal{I}} \{G + \beta(p\!\!\!/' + m) - \beta\Lambda\tau\} u(p). \tag{B13}$$

Projecting as in Eq. (2), we see that, since the imaginary part comes solely from the $1-\Lambda\mathcal{I}$ term (related to the T matrix), the same line of arguments leading to Eq. (A15) shows that both K_a, $a\in\{1,2\}$, satisfy

$$\mathrm{Im}(K_a) = F^{-1}TFK_a^*, \tag{B14}$$

with the same T matrix. As with $K_1(w)$, we renormalize $K_2(w)$ such that it is equal to the (physical) $G_{\pi NN}$ coupling at $w=m$, i.e., the two form factors $K_1(w)$ and $K_2(w)$ are equal at the on-shell point $w=m$. However, they have a different asymptotic behavior in the off-shell variable and, therefore, require a different number of subtractions in the dispersion relation. Thus, this example shows that knowledge of the T matrix cannot uniquely determine the off-shell form factor.

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