Relativistic two-photon and two-gluon decay rates of heavy quarkonia

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The decay rates of $c\bar{c}$ and $b\bar{b}$ through two-photon or two-gluon annihilations are obtained by using totally relativistic decay amplitudes and a sophisticated quantum-chromodynamic potential model for heavy quarkonia. Our results for the photonic and gluonic widths of the ${}^1S_0$, ${}^3P_0$, and the ${}^3P_2$ states are in excellent agreement with the available experimental data. The procedures and mathematical techniques used by us for the treatment of the fermion-antifermion bound states are also applicable to other decay processes. [S0556-2821(96)03615-6]

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I. INTRODUCTION

The decay rates of heavy quarkonia through two-photon or two-gluon annihilations were first obtained in the nonrelativistic approximation [1,2], and found to be inadequate in the light of the experimental data [3]. Improvements to the earlier results, therefore, have been explored by various authors by including the relativistic corrections [4–7]. The quarkonium decay is usually treated by a suitable adaptation of the matrix element for the annihilation of a free quark-antiquark pair, and this treatment is either supported by appealing to the Bethe-Salpeter approach with instantaneous approximation [5] or simply regarded as an artifice [7].

In view of the ambiguity involved in the treatment of the quarkonium decays, we shall obtain the decay rates by following two different approaches based on different assumptions regarding the role of the potential energy in a bound state. In both approaches, fully relativistic matrix elements for the quark-antiquark annihilation will be used. Moreover, unlike the earlier authors, we shall use a realistic quarkonium potential model [8] which has proved highly successful in the investigation of the $c\bar{c}$ and $b\bar{b}$ spectra with spin splittings. Our results for the decays of the ${}^1S_0$, ${}^3P_0$, and ${}^3P_2$ states of $c\bar{c}$ and $b\bar{b}$ will be compared with the earlier results of other authors as well as with the available experimental data. Additional experimental data on charmonium decays will be forthcoming from the work in progress at CLEO and LEP.

The general procedure and its applications to the S and P states of the fermion-antifermion bound states are described in Secs. II and III, while the quarkonium photonic and gluonic widths according to two different approaches are obtained in Secs. IV and V, which is followed by a discussion of our results in Sec. VI.

II. FERMION-ANTIFERMION BOUND-STATE DECAYS

Let us first consider the annihilation of a pair of an electron and a positron of four-momenta p and q into two photons of four-momenta k and k'. The second-order contribution of the scattering operator for this process is

$$S=V^{-2}(2\pi)^4\delta(p+q-k-k')Fa_{\mathbf{e}}^*(\mathbf{k})a_{\mathbf{e}'*}(\mathbf{k}')b_s(\mathbf{q})a_r(\mathbf{p}), \tag{2.1}$$

where

$$F=\frac{ie^2}{2}\frac{1}{(k_0k_0')^{1/2}}\overline{v_s}(\mathbf{q})\bigg[\mathbf{e}'\cdot\boldsymbol{\gamma}\frac{i(p-k)\cdot\gamma-m}{(p-k)^2+m^2}\mathbf{e}\cdot\boldsymbol{\gamma} + \mathbf{e}\cdot\boldsymbol{\gamma}\frac{i(p-k')\cdot\gamma-m}{(p-k')^2+m^2}\mathbf{e}'\cdot\boldsymbol{\gamma}\bigg]u_r(\mathbf{p}). \tag{2.2}$$

It is to be noted that

$$\overline{v_s}(\mathbf{q})(iq\cdot\gamma-m)=0,\quad (ip\cdot\gamma+m)u_r(\mathbf{p})=0,$$
$$p^2=q^2=-m^2,\quad k^2=k'^2=0, \tag{2.3}$$

so that F can be simplified as

$$F=\frac{ie^2}{2}\frac{1}{(k_0k_0')^{1/2}}\overline{v_s}(\mathbf{q})\bigg[\mathbf{e}'\cdot\boldsymbol{\gamma}\frac{2i\mathbf{p}\cdot\mathbf{e}-i(k\cdot\gamma)(\mathbf{e}\cdot\boldsymbol{\gamma})}{-2\mathbf{k}\cdot\mathbf{p}} + \mathbf{e}\cdot\boldsymbol{\gamma}\frac{2i\mathbf{p}\cdot\mathbf{e}'-i(k'\cdot\gamma)(\mathbf{e}'\cdot\boldsymbol{\gamma})}{-2\mathbf{k}'\cdot\mathbf{p}}\bigg]u_r(\mathbf{p}). \tag{2.4}$$

It is possible to convert the matrix element from the Dirac form to Pauli form without making any approximation by the substitutions [9]

$$u_r(\mathbf{p})=\left(\frac{m+p_0}{2p_0}\right)^{1/2}\begin{pmatrix}1\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{m+p_0}\end{pmatrix}\alpha_r,$$
$$v_s(\mathbf{q})=\left(\frac{m+q_0}{2q_0}\right)^{1/2}\begin{pmatrix}\dfrac{\boldsymbol{\sigma}\cdot\mathbf{q}}{m+q_0}\\ 1\end{pmatrix}\beta_s, \tag{2.5}$$

and

$$\gamma_i=\begin{pmatrix}0 & -i\sigma_i\\ i\sigma_i & 0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.6}$$

together with the charge-conjugation relation

$$\beta_s^* = \alpha_s^T i\sigma_2 . \tag{2.7}$$

Then, after reducing the products of the σ matrices, it is found that

$$F = \alpha_s^T i\sigma_2 O \alpha_r \tag{2.8}$$

with

$$\begin{aligned} O = \frac{ie^2}{2k_0^2[(\hat{\mathbf{k}}\cdot\mathbf{p})^2 - p_0^2]} \Big[& imk_0(\hat{\mathbf{k}}\cdot\mathbf{e}\times\mathbf{e}') - (\boldsymbol{\sigma}\cdot\hat{\mathbf{k}})(\mathbf{e}\cdot\mathbf{e}') \\ & \times(\hat{\mathbf{k}}\cdot\mathbf{p})k_0 - [(\boldsymbol{\sigma}\cdot\mathbf{e})(\mathbf{p}\cdot\mathbf{e}') + (\boldsymbol{\sigma}\cdot\mathbf{e}')(\mathbf{p}\cdot\mathbf{e})]p_0 \\ & - [(\boldsymbol{\sigma}\cdot\mathbf{e})(\mathbf{p}\cdot\mathbf{e}') - (\boldsymbol{\sigma}\cdot\mathbf{e}')(\mathbf{p}\cdot\mathbf{e})]\frac{(\hat{\mathbf{k}}\cdot\mathbf{p})(k_0 - p_0)}{p_0} \\ & + (\boldsymbol{\sigma}\cdot\mathbf{p})\frac{k_0(\mathbf{e}\cdot\mathbf{e}')(\hat{\mathbf{k}}\cdot\mathbf{p})^2 + 2p_0(\mathbf{e}\cdot\mathbf{p})(\mathbf{e}'\cdot\mathbf{p})}{p_0(m+p_0)} \Big], \end{aligned} \tag{2.9}$$

where we have used the center-of-mass relations

$$\mathbf{q} = -\mathbf{p}, \quad q_0 = p_0, \quad \mathbf{k}' = -\mathbf{k}, \quad k_0' = k_0, \tag{2.10}$$

as well as

$$\mathbf{k} = k_0\hat{\mathbf{k}}, \quad \hat{\mathbf{k}}\cdot\mathbf{e} = \hat{\mathbf{k}}\cdot\mathbf{e}' = 0, \tag{2.11}$$

but avoided the use of the energy conservation relation

$$p_0 = k_0 . \tag{2.12}$$

Now, let Ψ denote a positronium wave function, which can be Fourier decomposed as

$$\Psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{p}\Psi(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{2.13}$$

Following the usual approach, we assume that the decay amplitude for the $\mathbf{p}$ component of the positronium wave function can be obtained from the free electron-positron annihilation amplitude (2.8) by ignoring the relation (2.12), and treating p_0 as a variable given by

$$p_0 = (m^2 + \mathbf{p}^2)^{1/2}, \quad 0 < |\mathbf{p}| < \infty, \tag{2.14}$$

while

$$k_0 = \frac{1}{2} m_{e\bar{e}}. \tag{2.15}$$

This assumption implies that the effect of the bound-state potential energy is simply to nullify the energy conservation relation for the free-state annihilation amplitude.

Consequently, the amplitude for positronium decay into two photons is

$$\bar{F} = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{p} F\Psi(\mathbf{p}), \tag{2.16}$$

and the resulting decay rate is given by

$$\Gamma(e\bar{e}\to\gamma\gamma) = \frac{1}{(2\pi)^2}\int d\Omega_{\mathbf{k}}\frac{k_0^2}{2}|\bar{F}|^2. \tag{2.17}$$

Furthermore, the decay rates for quarkonia can be obtained by using the quarkonium wave functions, setting $k_0 = (1/2)M_{Q\bar{Q}}$, and making the usual multiplicative replacements in Eq. (2.17). For the decay rate $\Gamma(Q\bar{Q}\to\gamma\gamma)$, the replacement is

$$\alpha^2 \to N e_Q^4 \alpha^2, \quad N = 3, \tag{2.18}$$

while $\Gamma(Q\bar{Q}\to gg)$ can be obtained from $\Gamma(Q\bar{Q}\to\gamma\gamma)$ by the replacement

$$e_Q^4\alpha^2 \to \frac{2}{9}\alpha_s^2 . \tag{2.19}$$

III. *S* AND *P* STATE DECAY RATES

We shall apply the treatment of Sec. II to obtain the decay rates for those *S* and *P* states of positronium which can decay into two photons, and for this purpose we shall use the wave functions in the matrix representation, given in Appendix A.

A. $^1S_0$ decay

For the $^1S_0$ state (A10), the decay amplitude, given by Eqs. (2.16) and (2.8), takes the form

$$\bar{F} = \frac{i}{(2\pi)^{3/2}}\int d\mathbf{p}\frac{1}{(8\pi)^{1/2}}\mathrm{Tr}[O]\phi(p). \tag{3.1}$$

Since terms linear in $\boldsymbol{\sigma}$ in O do not contribute to the trace, $\bar{F}$ reduces to

$$\bar{F} = -\frac{ie^2}{8\pi^2}\int d\mathbf{p}\frac{m\hat{\mathbf{k}}\cdot\mathbf{e}\times\mathbf{e}'}{k_0[(\hat{\mathbf{k}}\cdot\mathbf{p})^2 - p_0^2]}\phi(p),$$

and, after angular integrations,

$$\bar{F} = -\frac{ie^2\hat{\mathbf{k}}\cdot\mathbf{e}\times\mathbf{e}'}{4\pi k_0}I_1, \tag{3.2}$$

where

$$I_1 = \int_0^\infty dp\frac{mp}{p_0}\ln\left|\frac{p_0 - p}{p_0 + p}\right|\phi(p). \tag{3.3}$$

With the substitution of Eq. (3.2) in Eq. (2.17), we have

$$\Gamma(^1S_0\to\gamma\gamma) = \frac{\alpha^2}{(2\pi)^2}\int d\Omega_{\mathbf{k}}\frac{1}{2}|\hat{\mathbf{k}}\cdot\mathbf{e}\times\mathbf{e}'|^2|I_1|^2, \tag{3.4}$$

and, upon summation over the final polarization states,

$$\sum_{\mathrm{pol}}|\hat{\mathbf{k}}\cdot\mathbf{e}\times\mathbf{e}'|^2 = 2, \tag{3.5}$$

while, in view of the indistinguishability of the two photons,

$$\int d\Omega_{\mathbf{k}}=2\pi. \tag{3.6}$$

Thus, the decay rate is given by

$$\Gamma(^1S_0\to\gamma\gamma)=\frac{\alpha^2}{2\pi}|I_1|^2. \tag{3.7}$$

This agrees with the result in Ref. [7], where the authors have obtained the decay rate for the $^1S_0$ state but not for the $^1S_0$ and $^3P_0$ states.

B. $^3P_2$ decay

For the $^3P_0$ state (A12), the decay amplitude (2.16) becomes

$$\overline{F}=\frac{i}{(2\pi)^{3/2}}\int d\mathbf{p}\,\frac{1}{(8\pi)^{1/2}}\mathrm{Tr}[\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}O]\phi(p), \tag{3.8}$$

where only terms linear in $\boldsymbol{\sigma}$ in O contribute to the trace. After trace evaluation and simplification, we find

$$\overline{F}=-\frac{ie^2}{8\pi^2}\int d\mathbf{p}\,\frac{m}{pp_0k_0^2}\times\frac{k_0(\mathbf{e}\cdot\mathbf{e}')(\hat{\mathbf{k}}\cdot\mathbf{p})^2+2p_0(\mathbf{p}\cdot\mathbf{e})(\mathbf{p}\cdot\mathbf{e}')}{(\hat{\mathbf{k}}\cdot\mathbf{p})^2-p_0^2}, \tag{3.9}$$

and, upon angular integrations with the help of Eq. (B1),

$$\overline{F}=-\frac{ie^2}{4\pi k_0^2}(\mathbf{e}\cdot\mathbf{e}')I_2, \tag{3.10}$$

where

$$I_2=\frac{1}{2\pi}\int_0^\infty dp\,\frac{mp}{p_0}[k_0A_1+(k_0+2p_0)A_2]\phi(p), \tag{3.11}$$

and A_1 and A_2 are given by Eq. (B4).

Substituting Eq. (3.10) in Eq. (2.17), and summing over the final polarization states, we arrive at the decay rate

$$\Gamma(^3P_0\to\gamma\gamma)=\frac{\alpha^2}{2\pi k_0^2}|I_2|^2. \tag{3.12}$$

C. $^3P_2$ decay

For the $^3P_2$ state (A14), the decay amplitude (2.16) is

$$\overline{F}=\frac{i}{(2\pi)^{3/2}}\int d\mathbf{p}\left(\frac{3}{8\pi}\right)^{1/2}\mathrm{Tr}[(\sigma_i\xi_{ij}^M\hat{p}_j)O]\phi(p), \tag{3.13}$$

where again only terms linear in $\boldsymbol{\sigma}$ in O contribute to the trace.

After trace evaluation, angular integrations with the help of Eqs. (B1), (B2), and (B3), and applications of the relations

$$\delta_{ij}\xi_{ij}^M=0,\quad (e_ie_j'-e_i'e_j)\xi_{ij}^M=0, \tag{3.14}$$

it is found that

$$\overline{F}=\frac{ie^2\sqrt{3}}{8\pi^2k_0^2}\xi_{ij}^M[(e_je_j'+e_i'e_j)I_3+(\mathbf{e}\cdot\mathbf{e}')\hat{k}_i\hat{k}_jI_4], \tag{3.15}$$

where

$$I_3=\int_0^\infty dp\left[-pA_2+\frac{2p}{p_0(m+p_0)}B_3\right]\phi(p), \tag{3.16}$$

$$I_4=\int_0^\infty dp\left[-\frac{pk_0}{p_0}(A_1+A_2)+\frac{k_0p}{p_0^2(m+p_0)}\times(B_1+5B_2+2B_3)+\frac{2p}{p_0(m+p_0)}B_2\right]\phi(p), \tag{3.17}$$

and A_i and B_i are given by Eqs. (B4) and (B5).

Furthermore, upon averaging over the initial states with different values of M by using Eq. (A8), and summing over the final polarization states, we obtain

$$\frac{1}{5}\sum_M\sum_{\mathbf{e},\mathbf{e}'}\overline{F}^*\overline{F}=\frac{e^4}{80\pi^4k_0^4}(6|I_3|^2+|I_4-I_3|^2), \tag{3.18}$$

which, when substituted in Eq. (2.17), gives the decay rate

$$\Gamma(^3P_2\to\gamma\gamma)=\frac{\alpha^2}{20\pi^3k_0^2}(6|I_3|^2+|I_4-I_3|^2). \tag{3.19}$$

IV. QUARKONIUM PHOTONIC AND GLUONIC WIDTHS

The quarkonium photonic and gluonic widths, which are obtainable by making the replacements (2.18) and (2.19) in the results of Sec. III, are given by

$$\begin{aligned}
\Gamma(^1S_0\to\gamma\gamma)&=\frac{3\alpha^2e_Q^4}{2\pi}|I_1|^2,\\
\Gamma(^1S_0\to gg)&=\frac{\alpha_s^2}{3\pi}|I_1|^2,\\
\Gamma(^3P_0\to\gamma\gamma)&=\frac{3\alpha^2e_Q^4}{2\pi k_0^2}|I_2|^2,\\
\Gamma(^3P_0\to gg)&=\frac{\alpha_s^2}{3\pi k_0^2}|I_2|^2,\\
\Gamma(^3P_2\to\gamma\gamma)&=\frac{3\alpha^2e_Q^4}{20\pi^3k_0^2}(6|I_3|^2+|I_4-I_3|^2),\\
\Gamma(^3P_2\to gg)&=\frac{\alpha_s^2}{30\pi^3k_0^2}(6|I_3|^2+|I_4-I_3|^2),
\end{aligned} \tag{4.1}$$

where

$$k_0=\frac{1}{2}M_{Q\overline{Q}}. \tag{4.2}$$

We have computed these widths by using the wave functions and parameters obtained from our quantum-

TABLE I. Photonic and gluonic widths of $c\bar{c}$ and $b\bar{b}$. The first two sets of theoretical results correspond to the relativistic treatments of Sec. IV and Sec. V. We also give the nonrelativistic results from Ref. [3]. The experimental results for χ_{c0} and χ_{c2} are from Ref. [10], and those for η_c are from Ref. [11].

Decay	Theory	Alternative theory	Nonrelativistic theory	Expt.
$\eta_c \to \gamma\gamma$	10.94 keV	10.81 keV		$6.7^{+2.4}_{-1.7} \pm 2.3$ keV
$\to gg$	23.03 MeV	22.76 MeV	9.01 MeV	$23.9^{+12.6}_{-7.1}$ MeV
$\chi_{c0} \to \gamma\gamma$	6.38 keV	8.13 keV		4.0 ± 2.8 keV
$\to gg$	13.44 MeV	17.10 MeV	1.63 MeV	$13.5 \pm 3.3 \pm 4.2$ MeV
$\chi_{c2} \to \gamma\gamma$	0.57 keV	1.14 keV		$0.321 \pm 0.078 \pm 0.054$ keV
$\to gg$	1.20 MeV	2.39 MeV	0.37 MeV	2.00 ± 0.18 MeV
$\eta_b \to \gamma\gamma$	0.46 keV	0.48 keV		
$\to gg$	12.46 MeV	13.02 MeV		
$\chi_{b0} \to \gamma\gamma$	0.080 keV	0.085 keV		
$\to gg$	2.15 MeV	2.29 MeV		
$\chi_{b2} \to \gamma\gamma$	0.008 keV	0.012 keV		
$\to gg$	0.22 MeV	0.33 MeV		

chromodynamic potential model for heavy quarkonia [8]. An essential feature of our model is the inclusion of the one-loop radiative corrections in the quantum-chromodynamic potential, which is known to be responsible for the remarkable agreement between the theoretical and experimental results for the spin splittings in the $c\bar{c}$ and $b\bar{b}$ spectra. Another advantage of our model is that it is based on a nonsingular form of the quarkonium potential, and thus avoids the use of an illegitimate perturbative treatment.

In addition to the wave functions, the parameters used for the computation of the widths are

$$\alpha_s(c\bar{c}) = 0.316, \quad m_c = 2.088 \text{ GeV},$$
$$M(\eta_c) = 2.979 \text{ GeV},$$
$$M(\chi_{c0}) = 3.415 \text{ GeV},$$
$$M(\chi_{c2}) = 3.556 \text{ GeV}, \tag{4.3}$$

and

$$\alpha_s(b\bar{b}) = 0.283, \quad m_b = 5.496 \text{ GeV},$$
$$M(\eta_b) = 9.417 \text{ GeV},$$
$$M(\chi_{b0}) = 9.860 \text{ GeV},$$
$$M(\chi_{b2}) = 9.913 \text{ GeV}, \tag{4.4}$$

where we have included our theoretical value for the mass of the unobserved energy level η_b. Our results for $c\bar{c}$ and $b\bar{b}$, together with the available experimental data [10,11], are given in Table I.

V. ALTERNATIVE TREATMENT OF BOUND-STATE DECAYS

Finally, we shall explore an alternative treatment of the bound-state decays by making an assumption regarding the role of the potential energy which differs from that in Sec. II.

Let us again consider the Fourier decomposition

$$\Psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p}\Psi(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} \tag{5.1}$$

of a bound-state $\Psi(\mathbf{r})$, and look upon it as a superposition of plane waves of different momenta but the same energy. Such a viewpoint is appropriate for a bound state because a wave packet consisting of waves of the same energy does not spread in time. It also allows us to treat the decay of the $\mathbf{p}$ component of a bound state of $c\bar{c}$ or $b\bar{b}$ as the annihilation of a pair of free quark and antiquark of momenta $\mathbf{p}$ and $-\mathbf{p}$, whose energy and effective mass are

$$p_0 = \frac{1}{2} M_{Q\bar{Q}}, \tag{5.2}$$

and

$$m = (p_0^2 - \mathbf{p}^2)^{1/2} = (\tfrac{1}{4} M_{Q\bar{Q}}^2 - \mathbf{p}^2)^{1/2}. \tag{5.3}$$

This approach implies that the quark and antiquark in a quarkonium can be looked upon as free particles of constant energy and variable momentum and mass.

The above treatment also leads to the quarkonium decay rates given by Eq. (4.1) in Sec. IV, but with an important difference. In Sec. IV, m is a constant, while p_0 is a variable, given by Eq. (2.14). On the other hand, in the alternative treatment p_0 is a constant, while m is a variable, and they are given by Eqs. (5.2) and (5.3). The computed photonic and gluonic widths resulting from the alternative treatment are also given in Table I.

VI. DISCUSSION

We have obtained the two-photon and the two-gluon relativistic decay rates of $c\bar{c}$ and $b\bar{b}$ by using two different approaches, which are based on apparently reasonable but very different assumptions. It is interesting to find that both approaches give quite similar results. As shown in Table I, our results for the ${}^3P_0$ and ${}^3P_2$ states are in agreement with the

Particle Data Group [10], while our results for the $^1S_0$ state agree with the recent findings of the E760 Collaboration [11].

In Table I, we have also included the nonrelativistic results obtained in Ref. [3] with the use of the Cornell potential. The nonrelativistic decay rates are much smaller than the experimental values, and this disagreement has not been resolved by the authors in Refs. [4–7], who found that the relativistic corrections amount to a reduction in the nonrelativistic decay rates.

Our treatment differs from those of the earlier authors in several respects: (1) We have used totally relativistic decay amplitudes instead of making nonrelativistic approximations or retaining only the leading relativistic corrections; (2) we have used a sophisticated quarkonium potential instead of simpler potentials such as the harmonic oscillator or the Cornell potential; (3) we have used a nonperturbative treatment for the spin-dependent interaction terms in the quarkonium potential instead of obtaining their contributions to the energy levels through first-order perturbation.

It is interesting that our values of the heavy quark masses, given by Eqs. (4.3) and (4.4), are somewhat higher than those generally found in the literature. This is a consequence of the fact that nonperturbative treatment of the spin-dependent interaction terms in the quarkonium potential yields constituent quark masses which are higher than those resulting from the commonly used perturbative treatment [12]. The nonperturbative treatment also has a pronounced effect on the quarkonium wave functions.

The remarkable agreement between our theoretical results and the experimental data represents a distinct success of our relativistic treatment of the bound-state decays as well as of our quantum-chromodynamic quarkonium model. The procedures and mathematical techniques used in this paper can also be applied to other decay processes.

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APPENDIX A: FERMION-ANTIFERMION WAVE FUNCTIONS

The fermion-antifermion wave functions in the matrix representation are given by

$$\Psi(^1S_0)=\sqrt{\frac{1}{8\pi}}\sigma_2\phi(r), \tag{A1}$$

$$\Psi(^3S_1)=\sqrt{\frac{1}{8\pi}}\sigma_i\xi_i^M\sigma_2\phi(r), \tag{A2}$$

$$\Psi(^3P_0)=\sqrt{\frac{1}{8\pi}}\sigma_i\hat{x}_i\sigma_2\phi(r), \tag{A3}$$

$$\Psi(^3P_1)=\sqrt{\frac{3}{16\pi}}\epsilon_{ijk}\sigma_i\xi_j^M\hat{x}_k\sigma_2\phi(r), \tag{A4}$$

$$\Psi(^3P_2)=\sqrt{\frac{3}{8\pi}}\sigma_i\xi_{ij}^M\hat{x}_j\sigma_2\phi(r), \tag{A5}$$

$$\Psi(^1P_1)=\sqrt{\frac{3}{8\pi}}\xi_i^M\hat{x}_i\sigma_2\phi(r), \tag{A6}$$

where ξ_i^M is a unit vector, and ξ_{ij}^M is a symmetric and traceless unit tensor, such that

$$\sum_{M=-1}^{1}\xi_i^{M\star}\xi_j^M=\delta_{ij}, \tag{A7}$$

and

$$\sum_{M=-2}^{2}\xi_{ij}^{M*}\xi_{mn}^M=\frac{1}{2}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})-\frac{1}{3}\delta_{ij}\delta_{mn}. \tag{A8}$$

It is also to be noted that since we treat the spin-dependent interaction terms nonperturbatively, the radial wave function $\phi(r)$ has a different form for each of the above states.

It is straightforward to show that these wave functions have the desired quantum numbers by applying the operators L_i^2, S_i^2, and $J_i^2=L_i^2+S_i^2+2L_iS_i$ to them, and keeping in mind that in the matrix representation of the wave functions

$$S_i\Psi=\frac{1}{2}(\sigma_i\Psi+\Psi\sigma_i^T). \tag{A9}$$

In the momentum space, the corresponding wave functions are

$$\Psi(^1S_0)=\sqrt{\frac{1}{8\pi}}\sigma_2\phi(p), \tag{A10}$$

$$\Psi(^3S_0)=\sqrt{\frac{1}{8\pi}}\sigma_i\xi_i^M\sigma_2\phi(p), \tag{A11}$$

$$\Psi(^3P_0)=\sqrt{\frac{1}{8\pi}}\sigma_i\hat{p}_i\sigma_2\phi(p), \tag{A12}$$

$$\Psi(^3P_1)=\sqrt{\frac{3}{16\pi}}\epsilon_{ijk}\sigma_i\xi_j^M\hat{p}_k\sigma_2\phi(p), \tag{A13}$$

$$\Psi(^3P_2)=\sqrt{\frac{3}{8\pi}}\sigma_i\xi_{ij}^M\hat{p}_j\sigma_2\phi(p), \tag{A14}$$

$$\Psi(^1P_1)=\sqrt{\frac{3}{8\pi}}\xi_i^M\hat{p}_i\sigma_2\phi(p), \tag{A15}$$

where we have abbreviated $|\mathbf{p}|$ as p.

APPENDIX B: ANGULAR INTEGRATIONS OF DECAY AMPLITUDES

Angular integrations of complicated integrals appearing in the decay amplitudes can be performed by setting

$$\int d\Omega_{\mathbf{p}}\frac{p_ip_j}{(\hat{\mathbf{k}}\cdot\mathbf{p})^2-p_0^2}=A_1\hat{k}_i\hat{k}_j+A_2\delta_{ij}, \tag{B1}$$

$$\int d\Omega_{\mathbf{p}} \frac{p_i p_j p_k}{(\hat{\mathbf{k}}\cdot\mathbf{p})^2 - p_0^2} = 0, \tag{B2}$$

$$\int d\Omega_{\mathbf{p}} \frac{p_i p_j p_k p_l}{(\hat{\mathbf{k}}\cdot\mathbf{p})^2 - p_0^2} = B_1 \hat{k}_i \hat{k}_j \hat{k}_k \hat{k}_l + B_2(\delta_{ij}\hat{k}_k\hat{k}_l + \delta_{jk}\hat{k}_i\hat{k}_l + \delta_{ik}\hat{k}_j\hat{k}_l + \delta_{il}\hat{k}_j\hat{k}_k + \delta_{jl}\hat{k}_i\hat{k}_k + \delta_{kl}\hat{k}_i\hat{k}_j) + B_3(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{B3}$$

Then, A_i and B_i are found to be

$$A_1 = \frac{\pi}{p_0 p}\left[6p_0 p + (3p_0^2 - p^2)\ln\left|\frac{p_0 - p}{p_0 + p}\right|\right],$$

$$A_2 = -\frac{\pi}{p_0 p}\left[2p_0 p + (p_0^2 - p^2)\ln\left|\frac{p_0 - p}{p_0 + p}\right|\right]; \tag{B4}$$

$$B_1 = \frac{\pi}{12 p_0 p}\left[210 p_0^3 p - 110 p_0 p^3 + (105 p_0^4 - 90 p_0^2 p^2 + 9 p^4)\ln\left|\frac{p_0 - p}{p_0 + p}\right|\right],$$

$$B_2 = -\frac{\pi}{12 p_0 p}\left[30 p_0^3 p - 26 p_0 p^3 + (15 p_0^4 - 18 p_0^2 p^2 + 3 p^4)\ln\left|\frac{p_0 - p}{p_0 + p}\right|\right],$$

$$B_3 = \frac{\pi}{12 p_0 p}\left[6 p_0^3 p - 10 p_0 p^3 + 3(p_0^2 - p^2)^2 \ln\left|\frac{p_0 - p}{p_0 + p}\right|\right]. \tag{B5}$$

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[12] We have checked, for instance, that if the hyperfine interaction terms in the quarkonium potential are treated by means of first-order perturbation, the charmed and bottom quark masses required to fit the experimental data for the hyperfine splittings in $c\bar{c}$ and $b\bar{b}$ are $m_c = 1.58$ GeV and $m_b = 5.1$ GeV, which are lower than the values given by Eqs. (4.3) and (4.4) in Sec. IV.