Topological defects inside domain walls

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We investigate the presence of topological defects inside domain walls in a specific system of coupled real scalar fields. This system belongs to a general class of systems of coupled real scalar fields, and presents some interesting properties in 1+1 dimensions. The potential that identifies the system is defined with two parameters, and we show that this is enough to implement the idea concerning the presence of defects inside defects. [S0556-2821(96)03414-5]

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The idea we want to investigate in this paper was born from the investigations already presented in some recent works [1–3]. It is motivated by the recent investigation introduced in [4], in which the possibility of domain walls having internal structure is nicely explored.

This idea of topological defects to present internal structure was originally introduced in [5], and has also been considered by other authors [6–8]. A feature of these works is that the potential that specifies the system in general contains several parameters. In this case, one can play with these parameters in order to implement the idea of introducing internal structure to topological defects.

In the course of our investigations [1–3], we have realized that one can present systems that have everything one needs to implement the above-mentioned idea. This appears to be interesting since our system is constrained to obey some conditions, and this makes the system to be defined by a reduced number of free parameters. This reduced set of parameter may then guide us toward some clearer understanding of the physical properties the system can perhaps comprise.

Since our former investigations rely on systems of coupled real scalar fields, in this paper we shall follow the investigation done in [4] to comment on topological defects inside domain walls. Before doing this, however, let us first briefly review the idea presented in [1–3]. In this case we investigate systems of coupled real scalar fields in bidimensional spacetime.

A general Lagrangian density describing a relativistic system of two coupled real scalar fields in bidimensional spacetime is given by

$$\mathcal{L}=\tfrac{1}{2}\,\partial_\alpha\phi\partial^\alpha\phi+\tfrac{1}{2}\,\partial_\alpha\chi\partial^\alpha\chi-U(\phi,\chi), \tag{1}$$

where $U=U(\phi,\chi)$ is the potential, which specifies the particular system one is interested in. Our notation is usual: We are using natural units, in which $\hbar=c=1$, and the metric tensor $g^{\alpha\beta}$ is diagonal, with $g^{00}=-g^{11}=1$.

In the standard way of searching for soliton solutions one considers static field configurations, and so $\phi=\phi(x)$ and $\chi=\chi(x)$. In this case, the equations of motion become

$$\frac{d^2\phi}{dx^2}=\frac{\partial U}{\partial\phi} \tag{2}$$

and

$$\frac{d^2\chi}{dx^2}=\frac{\partial U}{\partial\chi}. \tag{3}$$

In the above system, the potential $U(\phi,\chi)$ in general is a nonlinear function of the fields, and so the equations of motion (2) and (3) constitute a system of nonlinear coupled second-order differential equations. To circumvent the problem of working with second-order differential equations, we constrain the potential in some specific way [1–3], as we are going to show, and this will certainly restrict our investigation. This is the price one has to pay, although we yet get a large class of systems which can be investigated in a simpler way.

To do this, we consider the potential in the form

$$U(\phi,\chi)=\tfrac{1}{2}\,V^2(\phi,\chi)+\tfrac{1}{2}\,W^2(\phi,\chi), \tag{4}$$

in which the functions $V(\phi,\chi)$ and $W(\phi,\chi)$ are in principle arbitrary but continuous twice differentiable functions of the fields ϕ and χ. In this case, the equations of motion describing static field configurations become

$$\frac{d^2\phi}{dx^2}=V\frac{\partial V}{\partial\phi}+W\frac{\partial W}{\partial\phi} \tag{5}$$

and

$$\frac{d^2\chi}{dx^2}=V\frac{\partial V}{\partial\chi}+W\frac{\partial W}{\partial\chi}, \tag{6}$$

and this seems to give no good answer to the above referred problem; but this is not so, as we are now going to show.

Here we follow the procedure introduced in [1–3]—see also Ref. [9]. In this case one investigates the energy corresponding to static field configurations. For the system given by Eqs. (1) and (4) the energy can be written as

$$E=\frac{1}{2}\int_{-\infty}^{\infty}dx\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\chi}{dx}\right)^2+V^2+W^2\right\}. \tag{7}$$

This expression for the energy can be rewritten in the form $E=E'+E''$, where E' is given by

$$E'=\frac{1}{2}\int_{-\infty}^{\infty}dx\left\{\left(\frac{d\phi}{dx}-V\right)^2+\left(\frac{d\chi}{dx}-W\right)^2\right\} \tag{8}$$

and E'' reads

$$E''=\int_{-\infty}^{\infty}dx\left\{V\frac{d\phi}{dx}+W\frac{d\chi}{dx}\right\}. \tag{9}$$

We now introduce a general function $H=H(\phi,\chi)$ to write a new quantity E_M is the form

$$E_M=\int_{-\infty}^{\infty}dx\frac{dH}{dx}=H[\phi(\infty),\chi(\infty)]-H[\phi(-\infty),\chi(-\infty)]. \tag{10}$$

We use the chain rule to get

$$\frac{dH}{dx}=\frac{\partial H}{\partial\phi}\frac{d\phi}{dx}+\frac{\partial H}{\partial\chi}\frac{d\chi}{dx}. \tag{11}$$

Then, if we introduce the conditions

$$\frac{\partial H}{\partial\phi}=V,\quad \frac{\partial H}{\partial\chi}=W, \tag{12}$$

we obtain $E''=E_M$ in an obvious way. Moreover, from Eqs. (8), (9), and (12) we recognize that E_M is the minimum value for the energy, which is achieved when we impose the conditions

$$\frac{d\phi}{dx}=V \tag{13}$$

and

$$\frac{d\chi}{dx}=W, \tag{14}$$

since in this case the contribution E' given in Eq. (8) is zero.

The above equations (13) and (14) are first-order equations, and we can use them to verify that

$$\frac{d^2\phi}{dx^2}=V\frac{\partial V}{\partial\phi}+W\frac{\partial V}{\partial\chi} \tag{15}$$

and

$$\frac{d^2\chi}{dx^2}=V\frac{\partial W}{\partial\phi}+W\frac{\partial W}{\partial\chi}. \tag{16}$$

We now compare Eq. (5) to Eq. (15), and Eq. (6) to Eq. (16), to easily see that we can make the first-order equations (13) and (14) solve the second-order equations of motion (5) and (6), when we impose the condition

$$\frac{\partial V}{\partial\chi}=\frac{\partial W}{\partial\phi}. \tag{17}$$

Here we note that the above condition (17) is just the condition for the existence of a continuous twice differentiable function $H=H(\phi,\chi)$ satisfying Eq. (12). Therefore, for the general system (1), when the potential has the specific form (4), the second-order differential equations of motion (5) and (6) are solved by the first-order differential equations (13) and (14), if one imposes condition (17).

On the other hand, in this case the energy is bounded from below, and gets to its minimum value given by Eq. (10), where the function $H(\phi,\chi)$ can be obtained from conditions (12). This same function $H(\phi,\chi)$ can be used to define topological sectors. Here we follow [10] and introduce the topological current

$$J_T^{\alpha}=\epsilon^{\alpha\beta}\partial_\beta H(\phi,\chi), \tag{18}$$

which is trivially conserved, thanks to the asymmetry of the Levi-Civita tensor: $\epsilon^{01}=-\epsilon^{10}=1$ and $\epsilon^{00}=\epsilon^{11}=0$. The corresponding topological charge is given by

$$Q_T=\int_{-\infty}^{\infty}dxJ_T^0=H[\phi(\infty),\chi(\infty)]-H[\phi(-\infty),\chi(-\infty)]. \tag{19}$$

In this case the topological charge is equal to the energy of the static field configurations. The vacuum sector, which is identified by time- and space-independent field configurations, has zero topological charge. We use the topological charge, which is conserved, to introduce topological sectors: Nonvanishing different topological charges define different topological sectors.

Despite the above general result, we have shown explicitly in [2,3] that the soliton solutions we can find in this general class of systems are all stable, from the point of view of classical or linear stability.

Let us now examine a specific system. Here we consider V and W as

$$V(\phi,\chi)=\lambda(\phi^2-a^2)+\mu\chi^2 \tag{20}$$

and

$$W(\phi,\chi)=2\mu\phi\chi. \tag{21}$$

In this case the potential can be written in the form

$$U(\phi,\chi)=\tfrac{1}{2}\lambda^2(\phi^2-a^2)^2+\lambda\mu(\phi^2-a^2)\chi^2+\tfrac{1}{2}\mu^2\chi^4+2\mu^2\phi^2\chi^2, \tag{22}$$

and the first-order differential equations are given by

$$\frac{d\phi}{dx}=\lambda(\phi^2-a^2)+\mu\chi^2 \tag{23}$$

and

$$\frac{d\chi}{dx}=2\mu\phi\chi. \tag{24}$$

This system was already investigated in [1], and some soliton solutions were presented. Here, the function H which obeys conditions (12) has the form

$$H(\phi,\chi)=\lambda(\tfrac{1}{3}\phi^2-a^2)\phi+\mu\chi^2\phi. \tag{25}$$

At this stage we recall that to investigate internal structure of defects, we must play with the spatial part of spacetime. Then, we must change from the (1+1)-dimensional spacetime we have being working with to a (3+1)-dimensional spacetime. Thus, in the rest of this paper we shall be considering spacetime with $3+1$ dimensions. Furthermore, since we are working with real scalar fields, we follow [4] to introduce internal structure to domain walls.

As a first point we note that, because of the constraints we need to make the potential to belong to the class of systems we have already introduced, the above potential (22) contains only two free parameters, namely, λ and μ. And this is in contrast to Ref. [4], in which several parameters are introduced. In spite of this fact, the above system has everything one needs to implement the idea related to internal structure to domain walls.

To see this explicitly, let us consider the parameters λ and μ real and positive, with $\mu \geqslant \lambda$. In this case we see that

$$U(\phi,0)=\tfrac{1}{2}\lambda^2(\phi^2-a^2)^2, \tag{26}$$

and so a mass $m(a,0)$ for the ϕ field can be introduced; it is

$$m^2(a,0)=4\lambda^2a^2. \tag{27}$$

Moreover, we can write

$$U(0,\chi)=\tfrac{1}{2}\mu^2\left(\chi^2-\frac{\lambda}{\mu}a^2\right)^2, \tag{28}$$

and so a mass $m(0,a\sqrt{\lambda/\mu})$ for the χ field can be introduced; it is given by

$$m^2(0,a\sqrt{\lambda/\mu})=4\lambda\mu a^2. \tag{29}$$

We can also write

$$U(a,\chi)=2\mu^2a^2\chi^2+\tfrac{1}{2}\mu^2\chi^4, \tag{30}$$

and so another mass $m(a,\chi)$ for the χ field can be introduced; it has the form

$$m^2(a,\chi)=4\mu^2a^2. \tag{31}$$

We then see that for $\mu>\lambda$ the vacuum states are given by $(\phi^2=a^2,\chi=0)$. However, for $\mu=\lambda$ the vacuum states are $(\phi^2=a^2,\chi=0)$ and $(\phi=0,\chi^2=a^2)$. In the first case the system present a discrete Z_2 symmetry, and in the second case the discrete symmetry is Z_4.

From the above results we see that, if one uses the ϕ field to generate a domain wall, as one can see from Eq. (26), then inside this domain wall the field ϕ can be approximated to zero. In this case the χ field can generate a domain ribbon, as one can see from Eq. (28). Here we recall that domain wall and domain ribbon are time-independent defects obtained via the standard time-independent kink in $1+1$ dimensions, embedded in a $3+1$ and $2+1$ spacetime, respectively.

Everything works in the same way the investigation given in [4] has already advanced. The only point which calls our attention concerns evaporation of the domain ribbon. Here we refer to the fact that in [4] the author adjusts some of the available parameters to introduce the assumption that domain ribbons evaporate into elementary χ mesons, which are ejected to the surrounding domain wall vacuum. This assumption simplifies the calculation since it circumvents back reactions inside the domain wall.

In the above system, however, evaporation of domain ribbons into elementary χ mesons occurs *only inside* the domain wall, for $\mu>\lambda$, or *both inside and outside* the domain wall, for $\mu=\lambda$. In the first case, for $\mu>\lambda$ back reactions are unavoidably present and the results presented in [4] must be recalculated. In the second case, for $\mu=\lambda$ back reactions are present, but we believe that their effects are negligible, because of the smallness of the probability of finding χ mesons inside the wall, and so the result is essentially the one presented in [4]. We think of this as an interesting point, since the system we have just introduced can be seen as the bosonic portion of a supersymmetric theory [11], and one knows that supersymmetry in general is a important mechanism to control the underlying physics the system can comprise.

The supersymmetric theory whose real bosonic sector gives the system we have introduced can be constructed with two chiral superfields, as was already done by Morris [12], in a calculation that follows the lines of Ref. [13]. In this case the general function $H(\phi,\chi)$ we have introduced becomes the superpotential of the corresponding supersymmetric theory, and so supersymmetry introduces no further restrictions on the parameters λ and μ. Within this context, it follows that the probability of finding χ mesons inside and outside the domain wall, which depends on the values of λ and μ, is not controlled by supersymmetry.

As we have shown, models belonging to the above general class of systems may be best suitable to develop the idea related to internal structure of topological defects. We have introduced a simple model, containing only two coupling constants, which is still sufficiently complicated for internal structure to be present inside domain walls.

A natural extension of the system we have just investigated is to consider the ϕ field as a complex field. In this case we can gauge the corresponding continuum symmetry to reach the string territory. Furthermore, we can choose to work in a (2+1)-dimensional spacetime, and so we can also introduce a Chern-Simons term to the kinetic part of the gauge action to further enlarge the scope of the problem. Regarding this specific point, we note that the Maxwell-Chern-Simons system considered in [14]—see also Ref. [15]—can be seen as a natural extension of the method we have introduced in [1–3] to the case of planar systems that require a complex and a real scalar fields to be described.

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[1] D. Bazeia, M. J. dos Santos, and R. F. Ribeiro, Phys. Lett. A **208**, 84 (1995).
[2] D. Bazeia and M. M. Santos, Phys. Lett. A (to be published).
[3] D. Bazeia, R. F. Ribeiro, and M. M. Santos, Phys. Rev. E (to be published).
[4] J. R. Morris, Phys. Rev. D **51**, 697 (1995).
[5] E. Witten, Nucl. Phys. **B249**, 557 (1985).
[6] G. Lazarides and Q. Shafi, Phys. Lett. B **159**, 261 (1985).
[7] R. MacKenzie, Nucl. Phys. **B303**, 149 (1988).
[8] J. R. Morris, Phys. Rev. D **49**, 1105 (1994).
[9] E. B. Bogomol'nyi, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)].
[10] M. A. Lohe, Phys. Rev. D **20**, 3120 (1979).
[11] Z. Hlousek and D. Spector, Nucl. Phys. **B370**, 143 (1992).
[12] J. R. Morris (private communication).
[13] J. R. Morris, Phys. Rev. D **52**, 1096 (1995).
[14] C. Lee, K. Lee, and H. Min, Phys. Lett. B **252**, 79 (1990).
[15] D. Bazeia, Phys. Rev. D **43**, 4074 (1991).