

## Initial studies of bound states in light-front QCD

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We present the first numerical QCD bound state calculation based on a renormalization-group-improved light-front Hamiltonian formalism. The QCD Hamiltonian is determined to second order in the coupling, and it includes two-body confining interactions. We make a momentum expansion, obtaining an equal-time-like Schrödinger equation. This is solved for quark-antiquark constituent states, and we obtain a set of self-consistent parameters by fitting  $B$  meson spectra. [S0556-2821(96)04214-2]

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### I. INTRODUCTION

Recently, a new approach to renormalization and solving for QCD bound states, inspired by the constituent quark model, has been advocated [1–4]. Certainly the most successful approach to date is provided by lattice QCD. There are a number of reasons, however, why it remains important to develop other non-nonperturbative tools for the direct solution of QCD bound states. It is difficult (but not impossible) to gain intuitions about the quark or gluon structure of hadrons from Euclidean lattice calculations. Excited states must be extracted from statistical noise. Experimental precision for many hadron properties far exceeds that of any foreseeable lattice calculation. States involving more than one hadron are not conveniently studied using finite lattices. The lattice is the only tool for studying non-nonperturbative QCD with controlled errors, and we need other tools.

The new approach, in many aspects complimentary to lattice QCD, is tailored to build a bridge between QCD and a constituent quark model (CQM). It has been argued that it is convenient to use a light-front formulation of the theory, because on the light front it is possible to make the vacuum trivial simply by implementing a small longitudinal cutoff. As a result all partons in a hadronic state are connected to the hadron, instead of being disconnected excitations in a complicated medium. The price to pay is a considerably more complicated renormalization problem.

In this paper we briefly describe this new approach, concentrating on aspects necessary to appreciate the simple calculation we present. The calculations of the heavy meson spectra, for systems containing one heavy and one light quark, is intended to elucidate the approach and provide qualitative tests of the leading terms revealed in the effective Hamiltonian by the renormalization group. Only the first step in this calculation is taken here, and later work will focus on the spin-dependent structure. We refer the reader to the literature for many details [1–4].

The new approach consists of two steps. The first step is renormalization. The second step is a bound state calculation.

The aim of the first step is to find an effective renormalized Hamiltonian at hadronic energy scales starting with a Hamiltonian which is consistent with perturbative QCD at high energy scales. A natural starting point is the canonical light-front QCD Hamiltonian, although it cannot be complete

— there must be other operators, which cannot be determined from the perturbative behavior of the theory. Głazek and Wilson [5] designed a similarity transformation to lower the cutoff scale, which is tailored to make the Hamiltonian look more like a CQM Hamiltonian (see, for example, [6] and references in [1]). The cutoff violates manifest gauge invariance and Lorentz covariance, and thus these symmetries are no longer a guide to what operators are allowed in the Hamiltonian.

Similarity transformations can be designed to bring the Hamiltonian toward a band-diagonal form by eliminating matrix elements between states which differ drastically in light-front energy. Effects of couplings that are removed have to be put directly into the Hamiltonian as new effective interactions. One important consequence is that two-body potentials are generated. In fact, the similarity transformation generates a logarithmic confining potential already at order  $g^2$  [2,7]. This part of the calculation is done perturbatively. If the similarity transformation can be done analytically, it is easy to use a powerful method called coupling coherence [8,9] to determine all new terms. We will illustrate this procedure, which is straightforward to second order, in the following sections.

The second step is the bound state calculation itself. The effective Hamiltonian is divided into  $H_0$ , a part which is solved nonperturbatively, and  $V$ , the difference between the original Hamiltonian and  $H_0$ . The effects of  $V$  are to be computed using bound state perturbation theory. The criteria for choosing  $H_0$  are that it approximates the physics relevant for hadronic bound states as closely as possible (we take a hint from the CQM and include constituent masses and two-body potentials) and yet it must be manageable. We emphasize that the approach is tailored within limits to take advantage of the successful phenomenology, but it does not stop there. We can systematically improve the calculations, both by computing corrections to the Hamiltonian and higher terms in bound state perturbation theory. For example, any terms added by hand (e.g., constituent masses) can be added in such a way that at the physical value of the coupling they reduce to terms in the effective QCD Hamiltonian (for details see [1,4]).

In this paper we present one of the simplest possible calculations of QCD bound states based on the new approach. In the first step we find the effective Hamiltonian to order  $g^2$  using a similarity transformation and coupling coherence,

and in the second step we solve for the lowest lying  $q\bar{q}$  color singlet states with arbitrary but nonzero masses in the nonrelativistic limit. These approximations are severe, but we will see that the qualitative results are good. We will not explicitly show operators that have zero expectation value in the  $q\bar{q}$  color singlet.

We wish to derive an effective Hamiltonian that acts at the hadronic scale by lowering the similarity cutoff perturbatively as low as possible, hopefully down to the scale at which the bound state is well approximated by its two particle component,  $q\bar{q}$ . This may not be possible for all systems. The coupling may become too large for perturbation theory to be reliable before the higher Fock components are eliminated. We know, however, that it is possible in QED, and we believe that the success of the CQM suggests the same for QCD. The most favorable systems as far as the  $q\bar{q}$  and nonrelativistic approximations are concerned are heavy quarkonia. However, heavy mesons containing a light quark provide a better tool for testing our approach qualitatively, because they have fine structure that allows us to separately test spin and orbital angular momentum-dependent operators in the Hamiltonian, in the regime which is dominated by the confining interaction. This means that the scaling of momenta and energies is determined by the confining interaction. As the Coulomb part of the potential becomes more important, the simple scaling analysis breaks down, until the Coulomb potential dominates over the confining potential. Then the momenta and energies scale as in QED. We expect heavy quarkonia to be in the mixed regime, so the simple scaling analysis is probably not going to be reliable. Study of the spin and orbital angular momentum-dependent operators, which are generated by the similarity transformation to second order in the coupling, will be done in later work.

In heavy-light systems there is only one heavy quark, but we do not yet know whether the light degrees of freedom can be approximated by just one constituent in our approach. Further, as we show below, heavy-light mesons are qualitatively different from heavy quarkonia, but there are similarities with other mesonic systems: strange mesons and isospin 1 light mesons. So some of what we learn from heavy mesons may be generalized to light mesons. We will choose  $B$  mesons to check whether we can fit spectra with reasonable parameters.

After bringing the cutoff down to the hadronic scale, we do bound state calculations. We have to decide which terms will be treated as the dominant interactions and put in the unperturbed Hamiltonian, and which ones will be treated as perturbations.

In order to gain qualitative insight it is useful to study the nonrelativistic limit and to rewrite the bound state equation in position space. It turns out that in the nonrelativistic limit light-front dynamics naturally reduce to equal-time dynamics, which implies that angular momenta become kinematically defined. It is possible to transform light-front coordinates to equal-time coordinates by a specific change of variables [10] without taking the nonrelativistic limit. However, the point we want to make here is that in the nonrelativistic limit equal-time dynamics arise naturally. This will become clearer below.

Nonrelativistic reduction can be justified at best only for

the lowest lying states. We may need to do a series of bound state calculations with small coupling, and then extrapolate to the physical value of the strong coupling if this is large [1]. However, we are not yet at the stage where we can carry out the strategy with confidence that we have complete control over the bound state perturbation theory. The similarity transformation generates effective operators. The primary motivation of this work is to initiate the study of these operators. Further, the issue of chiral symmetry breaking terms, that is, what operators have to be added to the Hamiltonian by hand to restore the effects of zero modes removed by a small longitudinal cutoff, is not resolved yet. Nonrelativistic reduction provides a framework in which these and other questions can be studied.

The main questions answered in this calculation are the following.

(1) Two-body potentials generated by the similarity transformation include the Coulomb potential and a logarithmic confining potential already at order  $g^2$ . Does this effective Hamiltonian contain enough structure to provide a starting point for studying bound states in QCD? For the spin-independent part of the effective Hamiltonian, which we study here, the answer is yes.

(2) The similarity transformation generates new effective interactions. Is there a simple way to see how these new operators affect spectra? Yes.

(3) In order to answer the previous questions, one has to make some simplifications of the original Hamiltonian, including but not restricted to nonrelativistic reduction (for details see Sec. III). We do not focus on quantitative analysis, but we want to ask whether the calculation is at all reasonable; that is, can we fit any data with a set of reasonable parameters? Yes.

The paper is organized as follows. In the second section we briefly review the general strategy, sketching the similarity transformation and coupling coherence. The third section gives our calculation —  $q\bar{q}$  to order  $g^2$ . First, we find the effective Hamiltonian using the similarity transformation and coupling coherence. Then we split the Hamiltonian into a part which is treated nonperturbatively in the bound state calculation and a part which is treated perturbatively. In this paper, we solve the leading order problem, and show that it reduces to a simple Schrödinger equation in the nonrelativistic limit. We show a simple way to qualitatively analyze the physics behind this Schrödinger equation. Then we present the numerical results. The last section contains our summary and conclusions.

## II. TWO STEPS TO SOLVING QCD— GENERAL STRATEGY

In this section we briefly review the general strategy, first outlined in Ref. [1]. We start with the canonical light-front QCD Hamiltonian in light-cone gauge,  $A_a^+ = 0$ . We will not explicitly show terms that are not important for the specific calculations presented in the next section. For a detailed discussion of the light-front Hamiltonian see [1,11] and references therein. Ignoring purely gluonic terms that do not affect the effective  $q\bar{q}$  Hamiltonian until fourth order in  $g$ ,

$$H = H_{\text{free}} + V_1 + V_2, \quad (1)$$

where  $H_{\text{free}}$  is the free light-front Hamiltonian:

$$H_{\text{free}} = \sum_f \int \frac{d^3p}{(2\pi)^3 2p^+} \frac{p^{\perp 2} + m_f^2}{p^+} (b_f^\dagger b_f + d_f^\dagger d_f), \quad (2)$$

where there is a sum over flavors,  $m_f$  is a quark mass,  $b_f, d_f$  are quark and antiquark annihilation operators, and

$$V_1 = g \int dx^- d^2x_\perp \bar{\psi} \not{A} \psi \quad (3)$$

contains the standard order  $g$  quark-gluon coupling. Here  $\psi$  and  $A^\mu \equiv \sum_a A_a^\mu T^a$  are free light-front fields:

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- = \frac{1}{\partial^+} (-i \vec{\alpha}^\perp \cdot \vec{\partial}_\perp + \beta m) \psi_+ \end{pmatrix}$$

and

$$A^\mu = \left( A^+ = 0, \quad A^- = \frac{2}{\partial^+} \vec{\partial}_\perp \cdot \vec{A}^\perp, \quad \vec{A}^\perp \right).$$

The constrained fields,  $\psi_-$  and  $A^-$ , are replaced by functions of the physical degrees of freedom resulting in new terms in the canonical Hamiltonian, among which

$$V_2 = -2g^2 \int dx^- d^2x_\perp (\psi_+^\dagger T^a \psi_+) \left( \frac{1}{\partial^+} \right)^2 (\psi_+^\dagger T^a \psi_+) \quad (4)$$

is the instantaneous gluon exchange between two fermions.

We regulate the Hamiltonian with cutoffs on the change in free energy at each interaction vertex and a cutoff on longitudinal momentum fractions that is taken to zero at the end of the calculation. Then we use a similarity transformation. The similarity transformations form a renormalization group. Repeated transformations generate a sequence of Hamiltonians with decreasing cutoff. A Hamiltonian in the sequence is related to the previous one by

$$H_{\Lambda_{n-1}} = U^{-1}(\Lambda_{n-1}) H_{\Lambda_n} U(\Lambda_{n-1}). \quad (5)$$

$U$  is a unitary matrix and can be written as  $U = e^{iR}$ , where  $R$  is Hermitian and has an expansion in powers of the non-diagonal part of the Hamiltonian. We have chosen  $n$  so that it decreases as the cutoff decreases.

The similarity transformation is designed to bring the cutoff Hamiltonian to a band-diagonal form, while avoiding problems of small energy denominators in perturbative expansions [5]. In particular, the transformed Hamiltonian is required to be band diagonal relative to the new scale. This means that the matrix elements between states that differ in energy by more than the new cutoff must be zero for the simple step function cutoffs we employ here. This requirement has implications for the matrix elements of  $R$ . Given  $R$ , one can find the transformed Hamiltonian.

The initial cutoff destroys manifest symmetries and one of the criteria for the renormalized Hamiltonian is that it restores the symmetries, albeit not necessarily in manifest form. If the similarity transformation can be done analytically, it is possible to use coupling coherence [8,9] to completely fix the renormalized Hamiltonian without explicit ref-

erence to symmetries. The basic idea of coupling coherence is that in the Hamiltonian restricted by symmetries, the strengths of all operators are not independent but depend only on a finite number of independent canonical parameters; so that under a transformation, the Hamiltonian reproduces itself in form exactly, apart from the change of the explicit cutoff and the running of those few independent couplings. All dependence on the cutoff is absorbed into the independent running couplings. Once one obtains a Hamiltonian that reproduces itself as the cutoff is lowered, any initial cutoff can be sent to infinity.

In order to use coupling coherence we need to study how the Hamiltonian changes when the cutoff changes. Let  $H_{\Lambda_n} = H_{\text{free}} + v$ , where  $H_{\text{free}}$  is a free Hamiltonian and  $v \equiv V_1 + V_2$  is cut off so that

$$\langle \phi_i | v | \phi_j \rangle = 0, \quad (6)$$

if  $|E_{0i} - E_{0j}| > \Lambda_n^2 / \mathcal{P}^+$ , where  $H_{\text{free}} | \phi_i \rangle = E_{0i} | \phi_i \rangle$ . The similarity cutoff,  $\Lambda_n^2 / \mathcal{P}^+$  with the dimension of light-front energy, consists of  $\Lambda_n^2$ , which carries a dimension of transverse mass squared, and an arbitrary longitudinal momentum reference scale  $\mathcal{P}^+$ . If this cutoff is lowered to  $\Lambda_{n-1}^2 / \mathcal{P}^+$  by the similarity transformation, the new Hamiltonian matrix elements to  $O(v^2)$  are [7]

$$\begin{aligned} H_{\Lambda_{n-1}ab} &= \langle \phi_a | H_{\text{free}} + v | \phi_b \rangle \\ &- \sum_k v_{ak} v_{kb} \left[ \frac{\theta\left(|\Delta_{ak}| - \frac{\Lambda_{n-1}^2}{\mathcal{P}^+}\right) \theta(|\Delta_{ak}| - |\Delta_{bk}|)}{E_{0k} - E_{0a}} \right. \\ &\quad \left. + \frac{\theta\left(|\Delta_{bk}| - \frac{\Lambda_{n-1}^2}{\mathcal{P}^+}\right) \theta(|\Delta_{bk}| - |\Delta_{ak}|)}{E_{0k} - E_{0b}} \right], \quad (7) \end{aligned}$$

where  $\Delta_{ij} = E_{0i} - E_{0j}$  and  $|E_{0a} - E_{0b}| < \Lambda_{n-1}^2 / \mathcal{P}^+$ , and there are implicit cutoffs in this expression because the matrix elements of  $v$  have already been cut off so that  $v_{ij} = 0$  if  $|E_{0i} - E_{0j}| > \Lambda_n^2 / \mathcal{P}^+$ .

To this order, a coupling coherent Hamiltonian reproduces itself, with the only change being  $\Lambda_n \rightarrow \Lambda_{n-1}$ . At the third order one begins to see the quark-gluon coupling run. If the Hamiltonian reproduces itself, the index  $n$  becomes irrelevant. The solution is found by noting that we need the partial sum above to be added to an interaction in  $v$  that is expressed as a sum, so that the transformation merely changes the limits on the sum in a simple fashion (for details see [2]). There are two possibilities. The first is

$$\begin{aligned} H_{ab} &= \langle \phi_a | H_{\text{free}} + v | \phi_b \rangle \\ &- \sum_k v_{ak} v_{kb} \left[ \frac{\theta\left(|\Delta_{ak}| - \frac{\Lambda_{n-1}^2}{\mathcal{P}^+}\right) \theta(|\Delta_{ak}| - |\Delta_{bk}|)}{E_{0k} - E_{0a}} \right. \\ &\quad \left. + \frac{\theta\left(|\Delta_{bk}| - \frac{\Lambda_{n-1}^2}{\mathcal{P}^+}\right) \theta(|\Delta_{bk}| - |\Delta_{ak}|)}{E_{0k} - E_{0b}} \right], \quad (8) \end{aligned}$$

and the second is

$$\begin{aligned}
H_{ab} = & \langle \phi_a | H_{\text{free}} + v | \phi_b \rangle \\
& + \sum_k v_{ak} v_{kb} \left[ \frac{\theta\left(\frac{\Lambda_{n-1}^2}{\mathcal{P}^+} - |\Delta_{ak}|\right) \theta(|\Delta_{ak}| - |\Delta_{bk}|)}{E_{0k} - E_{0a}} \right. \\
& \left. + \frac{\theta\left(\frac{\Lambda_{n-1}^2}{\mathcal{P}^+} - |\Delta_{bk}|\right) \theta(|\Delta_{bk}| - |\Delta_{ak}|)}{E_{0k} - E_{0b}} \right]. \quad (9)
\end{aligned}$$

Note that  $v$  in these expressions is the same as that above only to first order. The coupling coherent interaction in  $H$  is written as a power series in  $v$  which reproduces itself under the transformation, except the cutoff changes. In higher orders the canonical couplings also run. To decide which of these equations to use one must in principle go to higher orders, but in practice it is usually obvious which choice is correct. For example, the first solution provides effective two-body interactions from one-gluon exchange, while the second provides the relevant part of the quark self-energy. If chosen otherwise, the new terms would make the effective Hamiltonian divergent.

At the end of this first step, the Hamiltonian is renormalized and the scale relative to which it is band diagonal is a hadronic scale.<sup>1</sup> The effective Hamiltonian contains complicated potentials, which result from eliminating the coupling between high and low energy states. It still contains emission and absorption interactions, but these no longer mix states of high and low energies.

In the second step, the Hamiltonian which is now given as an expansion in  $g$  is regrouped from the point of view of what is important in the bound state. Based on the success of the CQM, we believe that emission and absorption processes, which would mix different Fock states, are suppressed with respect to interactions that do not change particle number for low-lying states and sufficiently low cutoff. Thus the particle number changing interactions will be treated perturbatively using bound state perturbation theory. This is clearly justified if the gluons are massive [1], but even for massless gluons one can argue that the interacting gluon effectively acquires mass related to the confining scale [2]. It is therefore plausible to assume that adding an extra constituent is suppressed even in this case.

Next, we use constituent masses. It is possible to add the constituent mass at zeroth order and subtract it at a higher order in bound state perturbation theory, as outlined in [1]. This issue starts to be important when the effective Hamiltonian is calculated to higher orders and one tries to see that the approximations in the leading order bound state calculation lead to convergent results.

In the calculation presented here we also use a nonrelativistic reduction and we choose a rotationally symmetric  $H_0$ . We want to minimize higher order corrections, but we do not yet know what restores the rotational symmetry (e.g., adding extra gluons or terms of higher order in  $g$ ). The answer to this question may change our choice of  $H_0$  in future

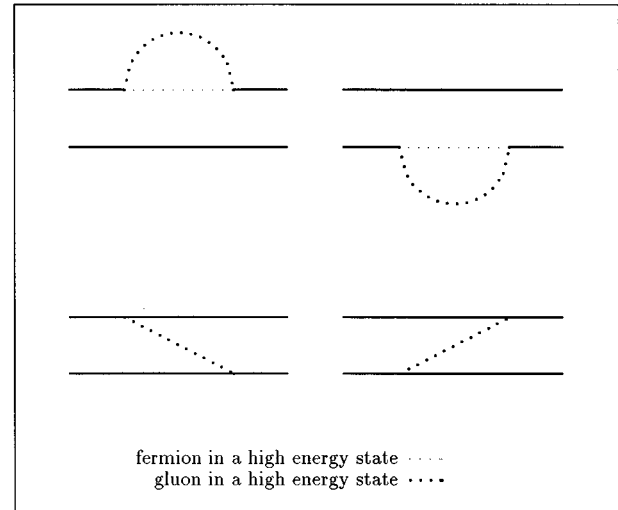


FIG. 1. Diagrams representing similarity transformation to order  $g^2$  for  $q\bar{q}$ . The top line represents new effective one-body operators, the bottom line are two-body operators. We use untypical lines for gluons and fermions to emphasize that these represent new operators, not Feynman diagrams.

calculations. At least for now, we choose the confining potential such that it does not yield any corrections in first order bound state perturbation theory. The corrections which come from the rotationally noninvariant terms are not suppressed by powers of the coupling, despite the fact that they enter at the second order of bound state perturbation theory. This is the same order at which one has to include emission and absorption processes, unless those are suppressed nonperturbatively (e.g., if the gluons are massive). Therefore, it is not clear whether it is meaningful to consider the corrections due to rotationally noninvariant terms without including  $q\bar{q}g$ . Nevertheless, we evaluated these corrections for the ground state and they are of order a few percent in a region where the confining potential dominates over the Coulomb potential.

### III. THE SIMPLE QCD CALCULATION — $q\bar{q}$ TO ORDER $g^2$

In this section we discuss mesons, i.e., color singlet QCD bound states whose valence constituents are a quark and an antiquark. The masses of the constituents are arbitrary but nonzero. We expect that the qualitative aspects of the study are relevant to all  $q\bar{q}$  systems with a possible exception of light isospin zero mesons.<sup>2</sup> We fit  $B$  mesons.

To order  $g^2$ , the similarity transformation is represented by a few diagrams, as shown in Fig. 1, so it is possible to find the effective Hamiltonian analytically. Let us work out in detail one of the operators, and then list the results for the remaining ones.

Let us consider matrix elements involving one gluon exchange between a quark and an antiquark (see the bottom two diagrams in Fig. 1). If the Hamiltonian is band diagonal

<sup>1</sup>A specific value of the “hadronic scale” is to be specified by fitting spectra.

<sup>2</sup>For light mesons with  $I=0$  we expect that operators of  $O(g^4)$  will play an important role.

relative to a scale  $\Lambda_n^2/\mathcal{P}^+$ , then only those matrix elements in which the energy difference is less than this value are non-zero:

$$\begin{aligned}
& -g_{\Lambda_n}^2 \bar{u}(p_2, \sigma_2) \gamma^\mu u(p_1, \sigma_1) \bar{v}(k_2, \lambda_2) \gamma^\nu v(k_1, \lambda_1) \langle T_a T_b \rangle \\
& \times \left[ \frac{\theta(q^+)}{q^+} D_{\mu\nu}(q) \left( \frac{\theta\left(\frac{\Lambda_n^2}{\mathcal{P}^+} - |D_1|\right) \theta(|D_1| - |D_2|)}{D_1} \right. \right. \\
& \left. \left. + \frac{\theta\left(\frac{\Lambda_n^2}{\mathcal{P}^+} - |D_2|\right) \theta(|D_2| - |D_1|)}{D_2} \right) \right. \\
& \left. + \frac{\theta(-q^+)}{-q^+} D_{\mu\nu}(-q) \left( \frac{\theta\left(\frac{\Lambda_n^2}{\mathcal{P}^+} - |-D_1|\right) \theta(|D_1| - |D_2|)}{-D_1} \right. \right. \\
& \left. \left. + \frac{\theta\left(\frac{\Lambda_n^2}{\mathcal{P}^+} - |-D_2|\right) \theta(|D_2| - |D_1|)}{-D_2} \right) \right], \quad (10)
\end{aligned}$$

where  $p_i, k_i$  are light-front three-momenta carried by the constituents;  $\sigma_i, \lambda_i$  are their light-front helicities;  $u(p, \sigma), v(k, \lambda)$  are their spinors [12]; index  $i=1,2$  refers to the initial and final states, respectively;  $D_{\mu\nu}(q) = (q^\perp{}^2/q^+) \eta_\mu \eta_\nu + (1/q^+) (\eta_\mu q^\perp{}_\nu + \eta_\nu q^\perp{}_\mu) - g_{\mu\nu}^\perp$  is the gluon propagator in light-front gauge [13],  $\eta_\mu = (0, \eta_+ = 1, 0, 0)$ ;  $\vec{q} = \vec{p}_1 - \vec{p}_2$  is the exchanged momentum,  $q^- = q^\perp{}^2/q^+$ ;  $D_1, D_2$  are energy denominators:  $D_1 = p_1^- - p_2^- - q^-$  and  $D_2 = k_2^- - k_1^- - q^-$ .

It is convenient to use Jacobi momenta. Setting the total transverse momentum to be zero, the momenta of the constituents are

$$p_i^+ = x_i P^+, \quad p_i^\perp = \kappa_i^\perp;$$

$$k_i^+ = (1-x_i) P^+, \quad k_i^\perp = -\kappa_i^\perp.$$

Let the mass of the constituent with momentum  $p$  be  $m_a$  and the mass of the other constituent be  $m_b$ . The denominators in terms of Jacobi momenta are

$$\begin{aligned}
D_1 &= \frac{1}{P^+} \left( \frac{\kappa_1^{\perp 2} + m_a^2}{x_1} - \frac{\kappa_2^{\perp 2} + m_a^2}{x_2} - \frac{(\kappa_1^\perp - \kappa_2^\perp)^2}{x_1 - x_2} \right), \\
D_2 &= \frac{1}{P^+} \left( \frac{\kappa_2^{\perp 2} + m_b^2}{1-x_2} - \frac{\kappa_1^{\perp 2} + m_b^2}{1-x_1} - \frac{(\kappa_1^\perp - \kappa_2^\perp)^2}{x_1 - x_2} \right). \quad (11)
\end{aligned}$$

When the scale is lowered to  $\Lambda_{n-1}^2/\mathcal{P}^+$  by the similarity transformation, all matrix elements in which the energy jump is larger than the new cutoff are zeroed. The effects of couplings which are removed have to be put directly in the Hamiltonian as new effective interactions. In this case, the new effective interactions according to Eq. (8) are

$$\begin{aligned}
& -g_{\Lambda_{n-1}}^2 \bar{u}(p_2, \sigma_2) \gamma^\mu u(p_1, \sigma_1) \bar{v}(k_2, \lambda_2) \gamma^\nu v(k_1, \lambda_1) \langle T_a T_b \rangle \\
& \times \left[ \frac{\theta(q^+)}{q^+} D_{\mu\nu}(q) \left( \frac{\theta\left(|D_1| - \frac{\Lambda_{n-1}^2}{\mathcal{P}^+}\right) \theta(|D_1| - |D_2|)}{D_1} \right. \right. \\
& \left. \left. + \frac{\theta\left(|D_2| - \frac{\Lambda_{n-1}^2}{\mathcal{P}^+}\right) \theta(|D_2| - |D_1|)}{D_2} \right) \right. \\
& \left. + \frac{\theta(-q^+)}{-q^+} D_{\mu\nu}(-q) \left( \frac{\theta\left(|-D_1| - \frac{\Lambda_{n-1}^2}{\mathcal{P}^+}\right) \theta(|D_1| - |D_2|)}{-D_1} \right. \right. \\
& \left. \left. + \frac{\theta\left(|-D_2| - \frac{\Lambda_{n-1}^2}{\mathcal{P}^+}\right) \theta(|D_2| - |D_1|)}{-D_2} \right) \right]. \quad (12)
\end{aligned}$$

This will repeat as the cutoff is lowered. Once the interaction reproduces itself in form the initial cutoff can be sent to infinity and we can lower the cutoff to the scale of interest. However, as the cutoff decreases, the coupling increases and at some point it becomes invalid to use perturbation theory to further lower the cutoff scale. The interaction in the effective Hamiltonian at the hadronic scale  $\Lambda$  is, thus,

$$\begin{aligned}
& -g_\Lambda^2 \bar{u}(p_2, \sigma_2) \gamma^\mu u(p_1, \sigma_1) \bar{v}(k_2, \lambda_2) \gamma^\nu v(k_1, \lambda_1) \langle T_a T_b \rangle \\
& \times \left[ \frac{1}{q^+} D_{\mu\nu}(q) \left( \frac{\theta\left(|D_1| - \frac{\Lambda^2}{\mathcal{P}^+}\right) \theta(|D_1| - |D_2|)}{D_1} \right. \right. \\
& \left. \left. + \frac{\theta\left(|D_2| - \frac{\Lambda^2}{\mathcal{P}^+}\right) \theta(|D_2| - |D_1|)}{D_2} \right) \right]. \quad (13)
\end{aligned}$$

Here we summed the two terms corresponding to the two time-ordered diagrams in Fig. 1.

Similarly, one can find effective one-body operators (self-energies) (see the top two diagrams in Fig. 1)

$$\begin{aligned}
& \frac{\alpha_\Lambda C_F}{2\pi P^+} \left\{ 2 \frac{P^+}{\mathcal{P}^+} \Lambda^2 \ln \left( \frac{P^+}{\epsilon \mathcal{P}^+} \right) + 2 \frac{P^+}{\mathcal{P}^+} \Lambda^2 \ln \frac{x^2 \frac{P^+}{\mathcal{P}^+} \Lambda^2}{x \frac{P^+}{\mathcal{P}^+} \Lambda^2 + m^2} \right. \\
& \left. - \frac{3}{2} \frac{P^+}{\mathcal{P}^+} \Lambda^2 + \frac{1}{2} \frac{m^2 \frac{P^+}{\mathcal{P}^+} \Lambda^2}{x \frac{P^+}{\mathcal{P}^+} \Lambda^2 + m^2} \right. \\
& \left. + 3 \frac{m^2}{x} \ln \frac{m^2}{x \frac{P^+}{\mathcal{P}^+} \Lambda^2 + m^2} \right\}. \quad (14)
\end{aligned}$$

Here  $m$  and  $x$  stand for either  $m_a$  and  $x_a$ , or  $m_b$  and  $x_b$ . The first term is infrared divergent ( $\epsilon$  is an infinitesimal cutoff on a longitudinal momentum taken to zero at the end of calculation) but it exactly cancels with the infrared divergence in the effective two-body operator (13), if the  $q\bar{q}$  pair is in a color singlet [2].

By finding these counterterms we have completed the first step of the calculation. Let us summarize the effective Hamiltonian:

$$H_{\text{eff}} = H_{\text{free}} + V_1 + V_2 + V_{2\text{eff}}, \quad (15)$$

where  $H_{\text{free}}$  is the kinetic energy;  $V_1$  is  $O(g)$  emission and absorption with nonzero matrix elements only between states with energy difference smaller than the hadronic scale  $\Lambda^2/\mathcal{P}^+$ ;  $V_2$  is  $O(g^2)$  instantaneous interaction and  $V_{2\text{eff}}$  includes the effective interactions, also  $O(g^2)$ , given in previous formulas.

This brings us to the second step: we have to regroup and approximate the Hamiltonian for the purpose of bound state calculations. As mentioned above, the emission and absorption are not included in  $H_0$ . We include kinetic energy, instantaneous fermion exchange, self-energies, and the most infrared divergent piece of the effective interaction arising from one gluon exchange. Even this is still quite complicated as a starting point to gain intuition, so we consider a nonrelativistic limit of this Hamiltonian. As before, we derive in detail results for a specific operator and list the results for the remaining ones.

Let us consider the most infrared divergent piece of the effective interaction arising from one gluon exchange in a color singlet:

$$\begin{aligned} & -g_\Lambda^2 C_F \bar{u}(p_2, \sigma_2) \gamma^+ u(p_1, \sigma_1) \bar{v}(k_2, \lambda_2) \gamma^+ v(k_1, \lambda_1) \\ & \times \left[ \frac{1}{q^+} \frac{q^{\perp 2}}{q^{+2}} \left( \frac{\theta(|D_1| - \frac{\Lambda^2}{\mathcal{P}^+}) \theta(|D_1| - |D_2|)}{D_1} \right. \right. \\ & \left. \left. + \frac{\theta(|D_2| - \frac{\Lambda^2}{\mathcal{P}^+}) \theta(|D_2| - |D_1|)}{D_2} \right) \right]. \quad (16) \end{aligned}$$

Let us denote  $M_{ab} \equiv m_a + m_b$ , introduce a new variable for the longitudinal momentum fraction  $\eta$  such that  $x_a = (m_a/M_{ab}) - \eta$ ,  $x_b = 1 - x_a$  and then make an expansion in powers of  $\eta$ .

To the lowest order in momenta, both energy denominators reduce to

$$D_1 = D_2 = -\frac{1}{\mathcal{P}^+} \left( M_{ab}^2 (\eta_1 - \eta_2) + \frac{(\kappa_1^\perp - \kappa_2^\perp)^2}{(\eta_1 - \eta_2)} \right). \quad (17)$$

Here  $(\kappa_1^\perp - \kappa_2^\perp)^2 = q^{\perp 2}$ . If we further identify  $q_z^2 \equiv M_{ab}^2 (\eta_1 - \eta_2)^2$ , then the expression (16) reduces to

$$4g_\Lambda^2 \sqrt{x_1(1-x_1)x_2(1-x_2)} M_{ab}^2 \frac{q^{\perp 2}}{q_z^2 q^2} (1 - \theta_{\text{below}}), \quad (18)$$

where

$$\theta_{\text{below}} \equiv \theta \left( \frac{\Lambda^2}{\mathcal{P}^+} - \frac{M_{ab} q_z^2}{\mathcal{P}^+ |q_z|} \right).$$

From now on, we drop the omnipresent  $\sqrt{x_1(1-x_1)x_2(1-x_2)}$ , because it cancels exactly with the same factor in the definition of the wave function [see Eq. (30) in the next section].

Adding this to a canonical term of order  $g^2$  which has a similar structure, namely, the instantaneous interaction

$$-4g_\Lambda^2 C_F \frac{1}{(\eta_1 - \eta_2)^2} = -4g_\Lambda^2 C_F \frac{M_{ab}^2}{q_z^2}, \quad (19)$$

leads to the interaction

$$-\frac{4g^2 C_F M_{ab}^2}{q^2} - \frac{4g^2 C_F M_{ab}^2 q^{\perp 2}}{q_z^2 q^2} \theta_{\text{below}}. \quad (20)$$

The first term is the Coulomb potential. The scale dependent part of the interaction (i.e., the second term) leads to a logarithmic confining potential [2,7].

Similarly, one can show that the kinetic energy in the nonrelativistic limit reduces to

$$\frac{\kappa_a^{\perp 2} + m_a^2}{x_a} + \frac{\kappa_b^{\perp 2} + m_b^2}{x_b} \rightarrow 2M_{ab} \frac{k^{\perp 2}}{2m}, \quad (21)$$

where  $m$  is the reduced mass, and that the self-energy produces only a constant shift:

$$\begin{aligned} \Sigma_a + \Sigma_b \rightarrow & \frac{\alpha C_F M_{ab} \mathcal{L}}{\pi} \left[ 2 \ln \left( \frac{\mathcal{P}^+}{\epsilon \mathcal{P}^+} \right) + 2 \ln \left( \frac{\mathcal{L}}{M_{ab}} \right) + \frac{1}{4} \frac{m_a}{\mathcal{L} + m_a} \right. \\ & + \frac{1}{4} \frac{m_b}{\mathcal{L} + m_b} \left( 1 + \frac{3m_a}{4\mathcal{L}} \right) \ln \left( \frac{m_a}{\mathcal{L} + m_a} \right) \\ & \left. + \left( 1 + \frac{3m_b}{4\mathcal{L}} \right) \ln \left( \frac{m_b}{\mathcal{L} + m_b} \right) - \frac{3}{2} \right], \quad (22) \end{aligned}$$

where

$$\mathcal{L} \equiv \frac{\Lambda^2}{\mathcal{P}^+} \frac{\mathcal{P}^+}{M_{ab}}$$

carries dimension of mass and in the nonrelativistic limit replaces the light-front cutoff  $\Lambda^2/\mathcal{P}^+$ . Note that we did not assume anything about the relation between masses and the cutoff  $\mathcal{L}$ . The assumption that momenta are small in comparison to masses is equivalent to assuming that the wave function is peaked at small momenta, which typically requires that  $g_\Lambda$  is sufficiently small.

It is more intuitive to work in position space. The Fourier transform of the potential is

$$\begin{aligned}
2M_{ab}V(\vec{r}) = & -\frac{2M_{ab}C_F\alpha}{r} - \frac{2M_{ab}C_F\alpha\mathcal{L}}{\pi} \\
& \times \int_0^1 dt \frac{1-t}{t} \{ \cos(t\mathcal{L}r_z) [J_0(\sqrt{t-t^2}\mathcal{L}r_\perp) \\
& + J_2(\sqrt{t-t^2}\mathcal{L}r_\perp)] - 1 \} \\
& + \frac{2M_{ab}C_F\alpha\mathcal{L}}{\pi} \int_0^\infty \frac{tdw}{t^2+w} \theta(t-t^2-w) \\
& \times \{ \cos(t\mathcal{L}r_z) J_0(\sqrt{w}\mathcal{L}r_\perp) - 1 \}, \quad (23)
\end{aligned}$$

where  $r_z$  is the  $z$  component of the separation between the quarks and  $r_\perp = |\vec{r}_\perp|$  is the transverse separation between the quarks. Integration variables  $t$  and  $w$  are dimensionless. The first term in the expression is the Coulomb potential, the rest is the confining potential normalized to zero at the origin. As mentioned above, for the color singlet states, the infrared divergence in this two-body operator precisely cancels with the infrared divergence in the self-energies, making the confining potential finite at the origin. Any finite terms required to make the confining potential vanish at the origin are subtracted from the self-energies.

The confining potential is not rotationally invariant, because the cutoff violates rotational symmetry. Also, recall that rotations on the light front are not kinematic, so as long as the gluon emission and absorption is allowed, complete rotational invariance requires states containing arbitrarily large numbers of gluons. The renormalized Hamiltonian restores rotational symmetry, but only to  $O(g^2)$ , while in the bound state calculation the confining interaction is treated to all orders.

The expansion of the confining potential in Legendre polynomials has only even terms:

$$V(\vec{r}) = \frac{\alpha C_F \mathcal{L}}{\pi} \sum_k V_{2k}(r) P_{2k}(\cos\theta), \quad (24)$$

with

$$\begin{aligned}
V_0 = & 2\ln\mathcal{R} - 2\text{Ci}(\mathcal{R}) + 4\frac{\text{Si}(\mathcal{R})}{\mathcal{R}} - 2\frac{(1-\cos\mathcal{R})}{\mathcal{R}^2} \\
& + 2\frac{\sin\mathcal{R}}{\mathcal{R}} - 5 + 2\gamma, \quad (25)
\end{aligned}$$

$$\begin{aligned}
V_2 = & -\frac{5}{3} + \frac{5\text{Si}(\mathcal{R})}{\mathcal{R}} - \frac{10}{\mathcal{R}^2} + \frac{5\text{Si}(\mathcal{R})}{2\mathcal{R}^2} \\
& + \frac{5\cos\mathcal{R}}{\mathcal{R}^2} - \frac{15}{\mathcal{R}^3} + \frac{5\cos\mathcal{R}}{2\mathcal{R}^3} + \frac{5\sin\mathcal{R}}{\mathcal{R}^3} + \frac{5\sin\mathcal{R}}{2\mathcal{R}^4} + \frac{20}{\mathcal{R}^5} \\
& - \frac{20\cos\mathcal{R}}{\mathcal{R}^5}, \quad (26)
\end{aligned}$$

and

$$V_{2k} = \frac{(4k+1)}{2} \int_{-1}^{+1} dx \left[ \frac{\alpha C_F \mathcal{L}}{\pi} \right]^{-1} V(\vec{r}) P_{2k}(x),$$

where  $\mathcal{R} = \mathcal{L}r$ ;  $\text{Ci}(x)$ ,  $\text{Si}(x)$  are cosine and sine integrals, respectively; and  $\gamma$  is the Euler constant.

When the separation between the quarks is purely in the  $z$  direction, i.e.,  $r_\perp = 0$ , the potential has its minimum value with respect to the angle  $\theta$ . Integrals in Eq. (21) can be done analytically leading to

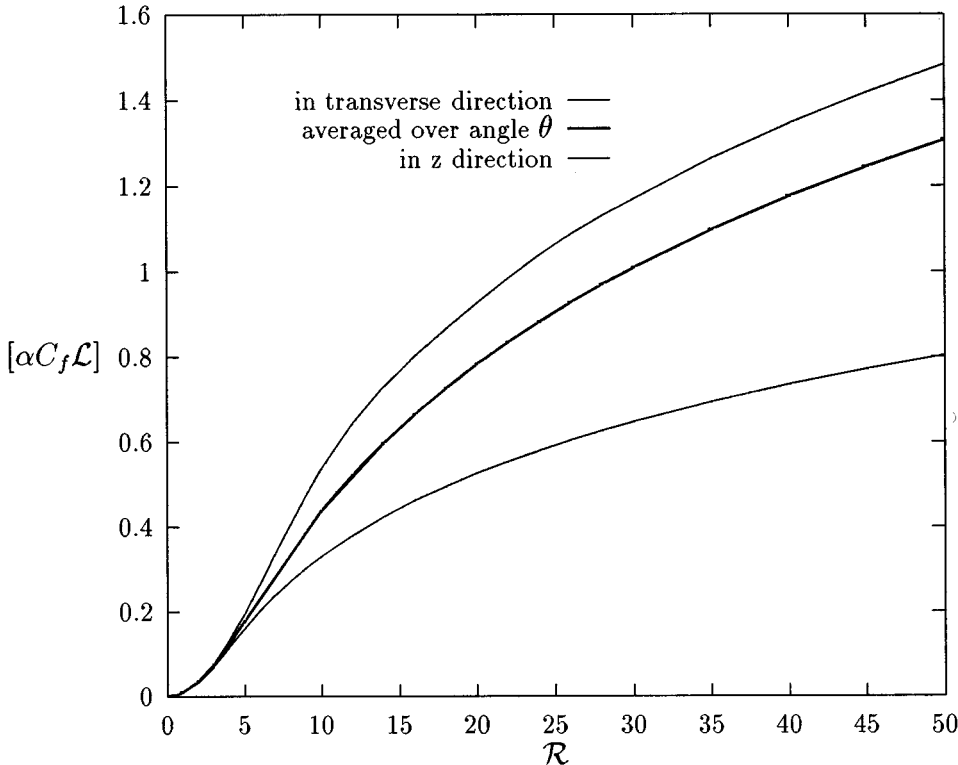


FIG. 2. Violation of rotational symmetry in the confining potential. At small distances, the violation of rotational symmetry is small. At any fixed value of  $\mathcal{R}$ , the confining potential is maximal when quarks are separated in purely transverse direction. It is minimal when the separation between quarks is in purely longitudinal direction. We also show the strength of the potential averaged over the angle  $\theta$ .

$$\frac{2M_{ab}C_F\alpha\mathcal{L}}{\pi}\left[\ln(\mathcal{R})-\text{Ci}(\mathcal{R})+\frac{\sin(\mathcal{R})}{\mathcal{R}}+\frac{\text{Si}(\mathcal{R})}{\mathcal{R}}-(2-\gamma)\right]. \quad (27)$$

The potential is maximal in the purely transverse direction (i.e.,  $r_z=0$ ). For large separations, the difference between the potential in purely transverse and purely longitudinal directions is a factor of 2. At very short distances, even the confining potential is rotationally invariant (see Fig. 2). This is of no help though, because at very short distances, the Coulomb part of the potential dominates.

For the bound state calculation, we want to choose a rotationally invariant confining interaction for  $H_0$ . This is motivated partly by phenomenology and partly by our desire to find a tool to qualitatively analyze the effective operators in the Hamiltonian. Restoration of rotational symmetry in  $H_0$  provides such a tool in the form of a simple scaling analysis.

It is not clear how we should choose the leading rotationally invariant interaction at order  $g^2$ , because we do not know yet how rotational symmetry is restored.

It is clear from Eq. (24) that it is convenient to use the first term in the expansion in Legendre polynomials. It is the only choice that does not lead to any corrections to the ground state in the first order of bound state perturbation theory. The corrections which come from the rotationally noninvariant terms enter at second order in bound state perturbation theory; that is, at the same order as the emission and absorption processes. We have to mention that these corrections are not suppressed by powers of the coupling, and that the problem does not disappear in the low-lying bound states as the coupling decreases. It all returns to the issue of how rotational symmetry is recovered in this approach — an issue that extends beyond our simple leading order calculation.

We will conclude this section with a list of approximations we make in the second step of the calculation.

(1) We do not include emission and absorption in  $H_0$ .

(2) We replace  $x_a$ ,  $x_b=1-x_a$  by  $x_a=(m_a/M_{ab})-\eta$ ,  $1-x_a=(m_b/M_{ab})+\eta$ ; and Taylor expand  $1/x$ ,  $1/(1-x)$  in powers of  $\eta$ .

In energy denominators, we neglect terms that are higher than second order in momenta. The same approximation is made in the arguments of step functions. This leads to a ‘new’ cutoff  $\mathcal{L}$ , which carries only one power of transverse mass. It should be emphasized that the new cutoff arises only in the second step, that is, in the nonrelativistic approximation to the light-front effective Hamiltonian. Only in this context does it replace the original cutoff  $\Lambda^2/\mathcal{P}^+$ . Elsewhere we have to work with  $\Lambda^2/\mathcal{P}^+$  (e.g., the coupling runs with  $\Lambda^2/\mathcal{P}^+$ ). It is  $\Lambda^2$  which has to be at hadronic mass scales. In the self-energies we keep only the leading constant shift, which is independent of momenta. This is because the self-energies are already  $O(g^2)$ .

(3) We introduce the third component of the ‘equal-time’ momentum:  $k_z=\pm M_{ab}\eta$ , and extend the integration limits on  $q_z$  from  $(-M_{ab}, +M_{ab})$  to  $(-\infty, +\infty)$ . It is easy to show that this third component of the equal-time momenta coincides with the  $z$  component of equal-time momentum in the lowest order in  $\alpha$ . As mentioned earlier, it is possible to introduce a third component without nonrelativistic reduction [10], but it does not yield any simplifications if the

masses are not equal. In the nonrelativistic limit they both agree to the leading order in powers of momenta. The main point we want to make here is that in the nonrelativistic limit the dynamics naturally lead to equal-time dynamics.

(4) For the leading order bound state calculation, we include only the rotationally invariant moment of the confining potential [i.e.,  $V_0$  given in Eq. (25)] in  $H_0$ .

This gives  $H_0$  which we choose for the purposes of the bound state calculation:

$$H_0=2M_{ab}\left[-\frac{1}{2m}\vec{\nabla}^2+\tilde{\Sigma}-\frac{C_F\alpha}{r}+\frac{C_F\alpha\mathcal{L}}{\pi}V_0(\mathcal{L}r)\right], \quad (28)$$

where  $V_0(\mathcal{L}r)$  is given in Eq. (25) and  $\tilde{\Sigma}$  contains the finite shift produced by self-energies after subtracting terms needed to make the confining potential vanish at the origin:

$$\begin{aligned} \tilde{\Sigma} &= \frac{\alpha C_F \mathcal{L}}{2\pi} \left[ \left(1 + \frac{3m_a}{4\mathcal{L}}\right) \ln\left(\frac{m_a}{\mathcal{L}+m_a}\right) + \left(1 + \frac{3m_b}{4\mathcal{L}}\right) \ln\left(\frac{m_b}{\mathcal{L}+m_b}\right) \right. \\ &\quad \left. + \frac{1}{4} \frac{m_a}{\mathcal{L}+m_a} + \frac{1}{4} \frac{m_b}{\mathcal{L}+m_b} + \frac{5}{2} \right]. \end{aligned} \quad (29)$$

### A. Schrödinger equation

We now want to find the mass of a bound state and its wave function  $\psi(\kappa^\perp, x)$ :

$$|P\rangle = \int \frac{d^2\kappa^\perp dx}{2(2\pi)^3\sqrt{x(1-x)}} \psi(\kappa^\perp, x) b^\dagger d^\dagger |0\rangle. \quad (30)$$

We use Lorentz-invariant normalization for the states

$$\langle P'|P\rangle = 2(2\pi)^3 P^+ \delta^3(\vec{P}-\vec{P}'),$$

and the wave function is normalized to one:

$$\int \frac{d^2\kappa^\perp dx}{2(2\pi)^3} |\psi(\kappa^\perp, x)|^2 = 1.$$

The bound state satisfies

$$H_0|P\rangle = \mathcal{M}^2|P\rangle, \quad (31)$$

where  $\mathcal{M}^2$  is the invariant mass of the bound state. Let the mass of the bound state be

$$\mathcal{M}^2 = (m_a + m_b)^2 + 2(m_a + m_b)E, \quad (32)$$

which defines  $E$ .

Substituting for  $H_0$  and  $\mathcal{M}^2$  in Eq. (31), after some straightforward algebra one obtains a bound state equation for the wave function  $\psi$ :

$$\begin{aligned} M_{ab} \left( E - \tilde{\Sigma} + \frac{1}{2m} \frac{d^2}{dr^2} \right) \psi(\vec{r}) &= M_{ab} \left[ \frac{-\alpha C_F}{r} \right. \\ &\quad \left. + \frac{\alpha C_F \mathcal{L}}{\pi} V_{\text{conf}}(\mathcal{L}\vec{r}) \right] \psi(\vec{r}), \end{aligned} \quad (33)$$



where  $m$  is the reduced mass and we choose  $V_{\text{conf}}(\mathcal{L}\vec{r}) = V_0(\mathcal{L}r)$  given in Eq. (25). We choose the confining potential for the leading order bound state calculation to be rotationally invariant, but other choices are possible. To remind the reader of this possibility and to keep the discussion as general as possible, we use  $\vec{\mathcal{R}}$  as the argument of the confining potential.

It is convenient to use a dimensionless separation  $\mathcal{R} = \mathcal{L}r$  that naturally arises in the confining piece of the potential, and to absorb  $-\tilde{\Sigma}$  into a definition of the eigenvalue  $\tilde{E}$  of the Schrödinger equation. When extracting the bound state mass,  $-\tilde{\Sigma}$  has to be subtracted. The bound state equation in the dimensionless separation is

$$\left[ -\frac{\mathcal{L}^2}{2m} \frac{d^2}{d\vec{\mathcal{R}}^2} + \mathcal{L}\alpha C_F \left( \frac{1}{\pi} V_{\text{conf}}(\vec{\mathcal{R}}) + V_{\text{Coul}}(\mathcal{R}) \right) \right] \psi(\vec{\mathcal{R}}) = \tilde{E} \psi(\vec{\mathcal{R}}). \quad (34)$$

Multiplying both sides of the equation by  $2m/\mathcal{L}^2$  and introducing a dimensionless coupling and eigenvalue

$$c \equiv \frac{2m\alpha C_F}{\mathcal{L}}, \quad (35)$$

$$e \equiv \frac{2m\tilde{E}}{\mathcal{L}^2}, \quad (36)$$

we obtain a Schrödinger equation, which depends only on dimensionless variables:

$$\left[ -\frac{d^2}{d\vec{\mathcal{R}}^2} + c \left( \frac{1}{\pi} V_{\text{conf}}(\vec{\mathcal{R}}) + V_{\text{Coul}}(\mathcal{R}) \right) \right] \psi(\vec{\mathcal{R}}) = e \psi(\vec{\mathcal{R}}). \quad (37)$$

This form is advantageous for numerical study, but moreover, it is quite general—one obtains an equation of this form for any quark-antiquark systems and any choice of the confining potential in the nonrelativistic limit, regardless of the masses, providing they are nonzero. For different systems  $\mathcal{L}$ ,  $c$ ,  $e$  would differ, but the resulting dimensionless Schrödinger equation will be the same. Thus in the leading order, *qualitative* characteristics of spectra depend only on one particular combination of the masses and the coupling.

### B. Bohr analysis

The purpose of this analysis is to gain a qualitative understanding of the physics described by the Schrödinger equation (37).

Since this is only a qualitative analysis, we will neglect the finite terms in the confining potential, but keep in mind that at small distances the confining potential vanishes. Since the Schrödinger equation is dimensionless, all quantities in this analysis are dimensionless also. The eigenvalue (energy  $e$ ) is given by a sum of the kinetic energy, which in our case is simply  $p^2$ ,  $p$  being dimensionless, and the potential energy. We use the uncertainty principle to replace the momentum by  $1/\mathcal{R}$ .

Let us consider the ground state

$$e_0 \approx p^2 + V(\mathcal{R}) \approx \frac{1}{\mathcal{R}^2} + c \left( 2 \ln \mathcal{R} - \frac{1}{\mathcal{R}} \right). \quad (38)$$

Now we find  $\mathcal{R}$  which minimizes the energy:

$$\frac{de_0}{d\mathcal{R}} = 0.$$

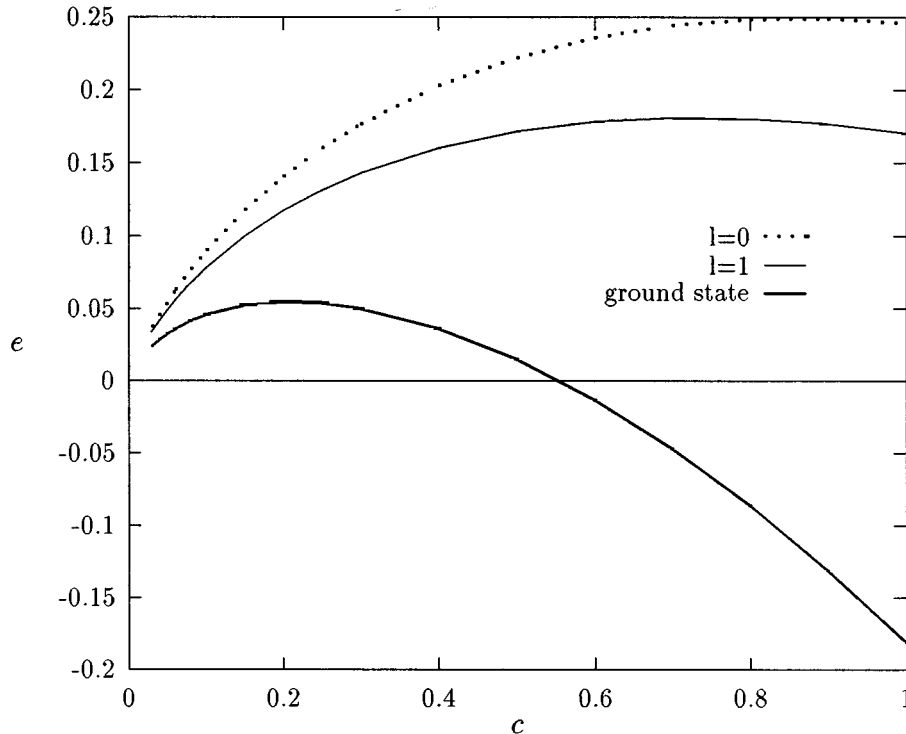


FIG. 3. Dimensionless eigenvalue  $e$  for the ground state, the lowest-lying  $P$  state ( $l=1$ ), and the lowest-lying excited  $S$  state ( $l=0$ ).

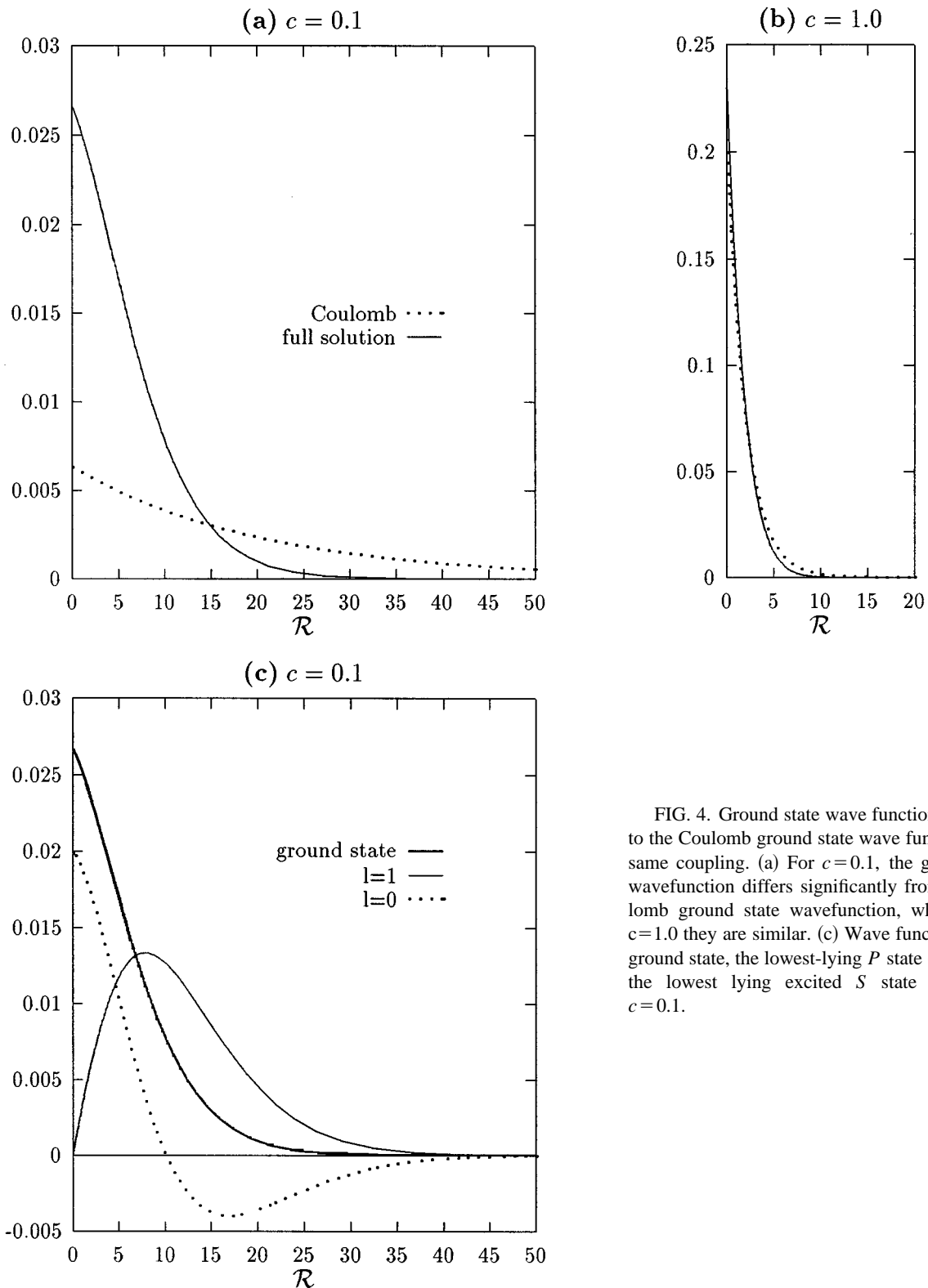


FIG. 4. Ground state wave function compared to the Coulomb ground state wave function at the same coupling. (a) For  $c=0.1$ , the ground state wavefunction differs significantly from the Coulomb ground state wavefunction, while (b) for  $c=1.0$  they are similar. (c) Wave functions of the ground state, the lowest-lying  $P$  state ( $l=1$ ), and the lowest lying excited  $S$  state ( $l=0$ ) for  $c=0.1$ .

The solution is

$$\mathcal{R}_0 = \frac{\sqrt{c^2 + 8c} - c}{2c}.$$

Similarly, we find the  $l=1$  excited state, for which

$$\mathcal{R}_1 = \frac{\sqrt{c^2 + 3 \times 8c} - c}{2c}.$$

We now consider two limiting cases: when  $c$  is small and when  $c$  is large.

*c small.* In this case, in the ground state  $\mathcal{R}_0 \approx 1/\sqrt{c}$ , and the energy  $e_0 \approx c + c \ln(1/\sqrt{c})$ . In the lowest  $l=1$  state

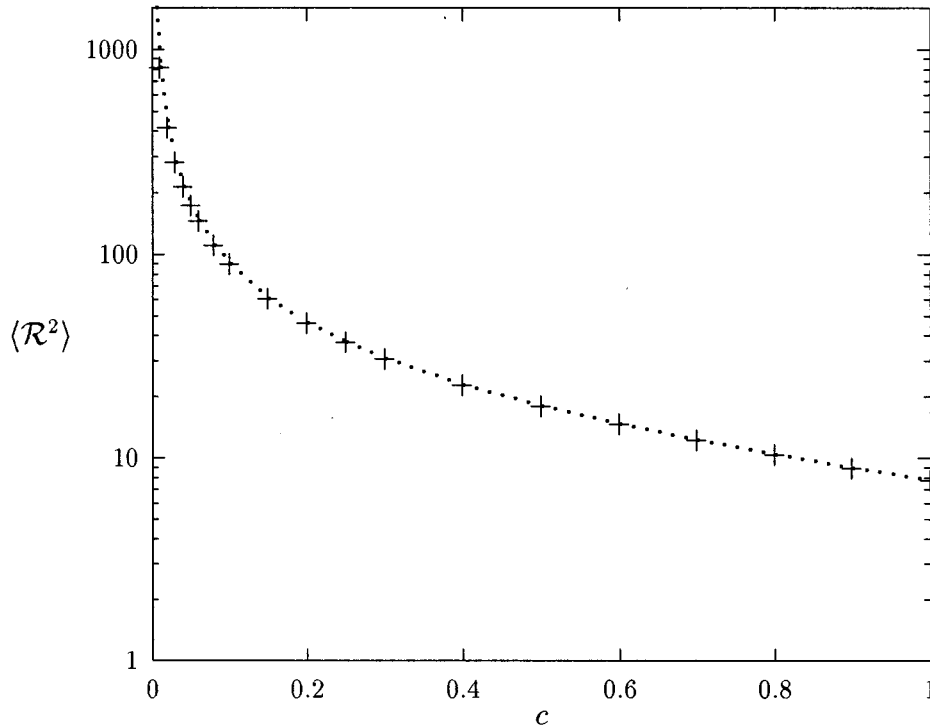


FIG. 5. Expectation value of  $\mathcal{R}^2$  in the ground state as a function of  $c$ . Dotted lines are connecting the data points.

$\mathcal{R}_1/\mathcal{R}_0 \approx \sqrt{3}$ , and the splitting in the energy between these two states is  $e_1 - e_0 \approx c \ln \sqrt{3}$ .

Of course, we are interested in “real world” energies and not the dimensionless results. If we “unwrap” the dimensionless results

$$\begin{aligned} e_1 - e_0 &\approx \frac{m}{\mathcal{L}^2} (E_1 - E_0) \\ &\approx c \approx \frac{m}{\mathcal{L}} \alpha, \end{aligned} \quad (39)$$

which implies that

$$E_1 - E_0 \approx \alpha \mathcal{L}. \quad (40)$$

The splitting between the ground state and the lowest-lying  $P$  state is independent of masses.

Similarly one can show that what sets the scale for individual energies is  $\mathcal{L}$  and that the size of the system,  $\mathcal{R}$ , depends both on  $\mathcal{L}$  and the reduced mass  $m$ .

*c* large. In this case,  $\mathcal{R}_0$  scales like  $2/c$ , the ground state energy  $e_0 \approx -c^2/4$ , while  $\mathcal{R}_1 \approx 6/c$  and  $e_1 \approx -(c^2/4)^{1/6}$ .

Unwrapping shows that the scale  $\mathcal{L}$  drops out:

$$\begin{aligned} e &\approx \frac{m}{\mathcal{L}^2} E \\ &\approx c^2 \approx \left( \frac{m}{\mathcal{L}} \alpha \right)^2, \end{aligned} \quad (41)$$

where  $e$  stands for any of the dimensionless energies under consideration, and  $E$  for any of the “real” bindings:

$$E \approx \alpha^2 m, \quad (42)$$

which depend only on the reduced mass but not on the confining scale.

Similarly,  $\mathcal{L}$  drops out also of the expression for the size.

In summary, when  $c$  is small, the spectra are determined by the logarithmic part of the potential; when  $c$  is large, the spectra are Coulombic. Recall that  $c$  is proportional to the reduced mass. Let us set aside questions on how  $\mathcal{L}$ , or the original light-front  $\Lambda^2$ , depends on the masses of constituents. There is a natural distinction between heavy quarkonia and other mesonic systems. In the case of heavy quarkonia the reduced mass is always proportional to the heavy mass, while in all other systems it is related to the light mass. This means that  $c$  is larger for the heavy quarkonia, and thus these states are substantially more Coulombic, while the other systems, including heavy mesons, are more influenced by the confining potential.

This simple technique can be used to estimate expectation values of various operators in the ground state.

### C. Numerical results

The numerical results are in agreement with the simple Bohr analysis. Figure 3 presents the numerical results for the dimensionless eigenvalue. It confirms that when  $c$  is small, the spectra are dominated by the log potential. As  $c$  increases, the Coulomb potential becomes more important, especially for the ground state.

Note that in the logarithmic regime (i.e., when  $c$  is small) the ground state dimensionless energy is always larger or at least comparable to the splitting between the ground state and the  $l=1$  state. For heavy mesons the splitting is a few hundred MeV, which would imply that the binding in the ground state is also large. However, recall that we had to absorb a constant shift from the self-energies into the definition of the eigenvalue and that the self-energy may be fine-tuned at low energies.

Figure 4 shows a few typical wave functions for the ground state and lowest-lying excited states. In Figs. 4(a) and 4(b) we compare the exact solution of the Schrödinger equation (37) to the ground state wave function of the Coulomb problem at the same value of the coupling  $c$ . When  $c$  is small, the wave function is quite different from the Coulombic wave function. As  $c$  increases, it becomes closer and closer to Coulombic, in agreement with the Bohr analysis presented above. Wave functions of the lowest two excited states for  $c=0.1$ , which we later use to fit data, are shown in Fig. 4(c) together with the ground state wave function.

Knowing the wave functions, one can calculate expectation values of various operators. For example, Fig. 5 shows the expectation value of  $\mathcal{R}^2$ . Even though  $\mathcal{R}^2 \gg 1$  in the entire range of  $c$ , it does not mean that the nonrelativistic approximation is valid. Recall that  $r$  is related to the dimensionless  $\mathcal{R}$  and is measured in units of  $\mathcal{L}$ ; so the expectation value of  $mr$  depends on the relation of  $\mathcal{L}$  to the mass.

We now attempt to unwrap the dimensionless results for  $B$  mesons. There is not enough experimental data, but moreover, the leading order results are too crude to justify a precise fit. Instead we use qualitative arguments and try to find a set of parameters that is reasonable.

We know from the heavy quark effective theory that in heavy mesons the splittings should be independent of the heavy masses [14]. We also know that the spectra are not Coulombic. Both requirements are satisfied if we choose  $c$  small. For example, with  $c=0.1$ , which is on the border of the logarithmic regime, we find that we can fit the splitting between the two lowest-lying doublets with  $m=0.32$  GeV,  $\alpha=0.35$ , and  $\Lambda=0.98$  GeV. At this value of parameters  $\bar{\Sigma} \approx -0.37$  GeV, leading to  $E_0/m \approx 0.8$ . This shows (together with the expectation value of  $mr \approx 1.1$ ) that the ground state is not nonrelativistic, which is not surprising.

For systems consisting of lighter quarks,  $c$  would be smaller because the reduced mass decreases. As  $c$  decreases, the state becomes more and more sensitive to the confining potential at larger distances  $\mathcal{R}$ . At some distances we expect the potential to become stronger than logarithmic, but without a calculation to a higher order in  $g$  we cannot decide whether those distances will manifest themselves in any spectra. Also, as one deals with lighter systems, the question of restoration of rotational symmetry becomes crucial.

#### IV. SUMMARY AND CONCLUSION

Starting with the canonical light-front Hamiltonian with no zero modes, we use a similarity transformation to find an effective Hamiltonian which is band diagonal with respect to a hadronic energy scale. We calculate the effective Hamiltonian to order  $g^2$  for  $q\bar{q}$  color singlet states with massive

quarks of arbitrary masses. Then we split the Hamiltonian to  $H_0$ , treated to all orders, and  $V$ , included perturbatively, choosing the spin-independent and rotationally invariant part of its nonrelativistic reduction for  $H_0$ . In the nonrelativistic limit the bound state equation leads to a dimensionless Schrödinger equation. Its scaling provides a powerful tool to classify the operators and estimate their expectation values.

We solve the leading order problem, and find that our calculation is acceptable for  $B$  mesons, which we can fit with a set of reasonable, self-consistent parameters. We show that heavy mesons are qualitatively different from heavy quarkonia, but there are similarities with other mesonic systems: strange mesons and isospin 1 light mesons. Our approach enables us to relate different mesonic systems and use qualitative features of spectra (e.g., almost constant mass squared splitting in the lowest-lying pseudoscalar and vector states, ordering of the lowest-lying  $0^+$  and  $2^+$  states) as a check of our effective operators. In this paper we present the leading order problem, study of spin and angular momentum-dependent operators generated to this order of the coupling by the similarity transformation will follow. We hope that results for heavy mesons can be generalized to lighter mesons, at least qualitatively, although it is not clear due to violation of rotational symmetry.

We manage to postpone the problem of lack of manifest rotational invariance in the confining potential to an order of bound state perturbation theory where there are additional corrections beyond the current calculation. Nevertheless, the corrections are not suppressed by powers of the coupling and this raises a serious warning that we do not yet know what is needed to recover rotational invariance. There are several options — one might be that in our second order light-front calculation the  $q\bar{q}$  approximation is insufficient. Since these corrections enter at the same order of bound state perturbation theory as the emission and absorption processes, it is possible that rotational invariance requires that a  $q\bar{q}g$  component be included to compensate for a rotationally noninvariant  $q\bar{q}$  component. Another option is that this is not really an issue in the “real world” — the states which would be mixed are well separated in energy. In this case the presence of rotationally noninvariant long range interactions would be merely a nuisance because they make it difficult to construct a simple scaling analysis that reveals the qualitative features of meson spectra. The resolution of these and other issues must await further calculations.

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