Gauge-invariant effective potential: Equilibrium and nonequilibrium aspects

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We propose a gauge-invariant formulation of the effective potential in terms of a gauge-invariant order parameter, for the Abelian Higgs model. The one-loop contribution at zero and finite temperature is computed explicitly, and the leading terms in the high temperature expansion are obtained. The result is contrasted with the effective potential obtained in several covariant gauge-fixing schemes, and the gauge-invariant quantities that can be reliably extracted from these are identified. It is pointed out that the gauge-invariant effective potential in the one-loop approximation is complex for *all values* of the order parameter between the maximum and the minimum of the tree level potential, both at zero and nonzero temperatures. The imaginary part is related to long-wavelength instabilities towards phase separation. We study the real-time dynamics of initial states in the spinodal region, and relate the imaginary part of the effective potential to the growth rate of equal-time gauge-invariant correlation functions in these states. We conjecture that the spinodal instabilities may play a role in nonequilibrium processes *inside* the nucleating bubbles if the transition is first order. [S0556-2821(96)05914-0]

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I. INTRODUCTION AND MOTIVATION

In this article we are concerned with the effective potential in gauge theories. It was recognized very early on that the effective potential is a gauge-dependent quantity [1] and only a limited amount of information extracted from it is actually physically meaningful. This gauge dependence can be understood from several equivalent points of view. The effective potential can be identified with the generating functional of one-particle irreducible Green's functions (the effective action) at zero four-momentum transfer, and therefore it is an off-shell quantity. Alternatively, the effective potential is identified with the energy (or free energy) of a particular state (or ensemble) constrained to have an homogeneous expectation value of the scalar field.

The energy, or the free energy, is usually calculated by fixing a particular gauge in the path integral. In a gauge theory, the (complex) scalar fields transform under gauge transformations and their expectation value in a *gauge-fixed state or ensemble* is obviously a gauge-dependent quantity. Despite this shortcoming it has been recognized that certain quantities are gauge independent. Dolan and Jackiw [1] recognized that the critical temperature is a gauge-invariant quantity and recently Metaxas and Weinberg [2] used the Nielsen identities [3] to prove that the bubble nucleation rate *at zero temperature* is gauge invariant (we are not aware of a similar proof at finite temperature). The gauge invariance of these quantities can be understood from the fact that they are associated with homogeneous and inhomogeneous extrema of the effective action, respectively; these are known to be gauge invariant.

Considerable effort has been devoted to constructing a gauge-invariant effective action and effective potential [4], of which the background field method of Vilkovisky [4] and DeWitt [4] is the most popular [5]. These formulations of effective actions are technically formidable and do not readily lend themselves to a manageable formulation of equilibrium or nonequilibrium descriptions. Furthermore, it has recently been pointed out using the pinch technique, that despite its formal gauge invariance, implementation of the background field method requires a gauge-fixing parameter for the *fluctuations*. This leads to a gauge-parameter dependence in finite parts of self-energies at finite temperature [6,7], which in turn leads to a gauge dependence of the thermal renormalization group β function as discussed in detail by Sasaki [7].

Alternative formulations of effective potentials have been offered in terms of a radial and angular decomposition of the complex scalar fields [8,9] or alternatively in terms of gauge-invariant composite operators [10]. There are several shortcomings in the formulation of the effective potential in terms of the radial field variable [8,9] or composite operators [10]. This variable is understood as the ''square root'' of a composite operator that requires a *subtraction* to be renormalized; shortcomings of this approach had been already recognized [8]. Furthermore, in the path-integral evaluation there is an ambiguous Jacobian arising from the change of variables to radial and angular fields. This Jacobian has to be incorporated in the perturbative expansion to obtain a consistently renormalized effective action [8].

However, even when these technicalities are overcome by some renormalization scheme (such as dimensional regularization), it is conceptually unclear how to interpret symmetry breaking in terms of the radial field. At the operator level, the

radial field variable acquires a ground-state expectation value *even in the symmetric phase* as can be seen with a simple example of a two-dimensional isotropic harmonic oscillator. Composite bilinear operators typically require subtractions to be renormalized, and their expectation value is therefore ambiguous.

Our motivation is to obtain a gauge-invariant description of the effective potential (and eventually the effective action) and to use it to provide some preliminary information on the dynamics of nonequilibrium processes during the phase transition in gauge theories. In view of the above discussion and critique of previous approaches, such an enterprise is clearly worthwhile because only a truly gauge-invariant description of effective potentials can be considered trustworthy in terms of extracting physical quantities such as supercooling temperature, latent heat, and others that are very important in the quantitative description of nonequilibrium features.

Our program for the construction of an effective potential can be summarized in the following steps. (a) Select the gauge-invariant states of the theory, namely, those that are annihilated by the first class constraints (such a selection *will not* involve any gauge fixing), (b) recognize a gauge-invariant order parameter that is invariant under local gauge transformations, but transforms nontrivially under the global symmetry that can be spontaneously broken, and (c) construct the effective potential for this gauge-invariant order parameter. An attempt to establish a finite temperature framework in terms of gauge-invariant states has been reported previously [11] within a different context and with a different goal, but to our knowledge it has not been implemented or attempted within the context of the effective potential.

In this article we focus on such a description for the Abelian Higgs model (scalar electrodynamics) and we expect to generalize the procedure and its quantitative implementation to Yang-Mills theories in the near future.

In Sec. II we implement the first step of the program; that is, we select the gauge-invariant states and order parameter *without* fixing a gauge, by requiring that the physical states be annihilated by the first class constraints of the theory which are recognized as the generators of local gauge transformations. Gauge-invariant operators are then recognized as those that commute with these constraints, out of which we recognize the proper order parameter. In Sec. III we explicitly construct the one-loop effective potential both at zero and nonzero temperature and compare our results with those obtained in popular covariant gauges. From this comparison we establish when the gauge-fixed results lead to physical (gauge-independent) predictions. In this section we also argue that the gauge dependence of the usual (gauge-fixed) effective potential is not relieved by hard-thermal-loop resummation. We also provide the high-temperature expansion of the gauge-invariant effective potential and point out that the ''cubic'' terms which are typically taken as a signal of the strength of a first order transition are in general complex and gauge dependent in fixed-gauge path-integral calculations of the effective potential.

In Sec. IV we use the gauge-invariant effective potential to study the early time behavior of spinodal phase separation and the instabilities associated with the spinodal line in gauge theories. We establish a correspondence between the imaginary part of the one-loop gauge-invariant effective potential and the rate of growth of correlations in the spinodal region. Two appendixes are devoted to some technical details.

II. GAUGE-INVARIANT DESCRIPTION

The focus of our study is scalar electrodynamics or the Abelian Higgs model whose Lagrangian density is

$$\mathcal{L}=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+D^{\mu}\phi^{\dagger}D_{\mu}\phi-\lambda(\phi^{\dagger}\phi-\mu^{2})^{2}, \quad (2.1)$$

$$D_{\mu}\phi=(\partial_{\mu}\phi+ieA_{\mu}\phi). \quad (2.2)$$

The description in terms of gauge-invariant states and operators is best achieved within the canonical formulation, which begins with the identification of canonical field variables and constraints. These will determine the classical physical phase space and, at the quantum level, the physical Hilbert space.

The canonical momenta conjugate to the scalar and vector fields are given by

$$\Pi^{0}=0, \quad (2.3)$$

$$\Pi^{i}=\dot{A}^{i}+\nabla^{i}A^{0}=-E^{i}, \quad (2.4)$$

$$\pi^{\dagger}=\dot{\phi}+ieA^{0}\phi, \quad (2.5)$$

$$\pi=\dot{\phi}^{\dagger}-ieA^{0}\phi^{\dagger}. \quad (2.6)$$

The Hamiltonian is, therefore,

$$H=\int d^{3}x\{\tfrac{1}{2}\vec{\Pi}\cdot\vec{\Pi}+\pi^{\dagger}\pi+(\vec{\nabla}\phi-ie\vec{A}\phi)\cdot(\vec{\nabla}\phi^{\dagger}+ie\vec{A}\phi^{\dagger}) +\tfrac{1}{2}(\vec{\nabla}\times\vec{A})^{2}+\lambda(\phi^{\dagger}\phi-\mu^{2})^{2} +A_{0}[\vec{\nabla}\cdot\vec{\Pi}-ie(\pi\phi-\pi^{\dagger}\phi^{\dagger})]\}. \quad (2.7)$$

There are several different manners of quantizing a gauge theory, but the one that exhibits the gauge-invariant states and operators, originally due to Dirac, begins by recognizing the first class constraints (mutually vanishing Poisson brackets). From here there are several possibilities: (i) The constraints become operators in the quantum theory and are imposed onto the physical states, thus defining the physical subspace of the Hilbert space and gauge-invariant operators. (ii) Introduce a gauge, converting the first class system of constraints into a second class (with nonzero Poisson brackets between the constraints) and introducing Dirac brackets. This is the popular way of dealing with the constraints and leads to the usual gauge-fixed path integral representation [12] in terms of Faddeev-Popov determinants and ghosts.

We will instead proceed with the first possibility that leads to an unambiguous projection of the physical states and operators. Such a method has been previously used by James and Landshoff within a different context [13].

In Dirac's method of quantization [14] there are two first class constraints which are

$$\Pi^{0}=\frac{\delta\mathcal{L}}{\delta A^{0}}=0 \quad (2.8)$$

and Gauss' law

$$\mathcal{G}(\vec{x},t)=\nabla^i\pi^i-\rho=0, \tag{2.9}$$

$$\rho=ie(\phi\pi-\phi^\dagger\pi^\dagger), \tag{2.10}$$

with ρ being the matter (complex scalar) field charge density.

Gauss' law can be seen to be a constraint in two ways: either because it cannot be obtained as a Hamiltonian equation of motion or because, in Dirac's formalism, it is the secondary (first class) constraint obtained by requiring that the primary constraint (2.8) remain constant in time. Quantization is now achieved by imposing the canonical equal-time commutation relations

$$[\pi^0(\vec{x},t),A^0(\vec{y},t)]=-i\delta(\vec{x}-\vec{y}), \tag{2.11}$$

$$[\pi^i(\vec{x},t),A^j(\vec{y},t)]=-i\delta^{ij}\delta(\vec{x}-\vec{y}), \tag{2.12}$$

$$[\pi^\dagger(\vec{x},t),\phi^\dagger(\vec{y},t)]=-i\delta(\vec{x}-\vec{y}), \tag{2.13}$$

$$[\pi(\vec{x},t),\phi(\vec{y},t)]=-i\delta(\vec{x}-\vec{y}). \tag{2.14}$$

In Dirac's formulation, the projection onto the gauge-invariant subspace of the full Hilbert space is achieved by imposing the first class constraints onto the states. Physical operators are those that commute with the first class constraints. With the above equal-time commutation relations it is straightforward to see that the unitary operator

$$U_\Lambda=\exp\left\{i\int[\Pi^0\dot{\Lambda}+\mathcal{G}\Lambda]d^3x\right\} \tag{2.15}$$

performs the local gauge transformations. Thus the first class constraints are recognized as the generators of gauge transformations. In particular, Gauss' law (2.9) is the generator of time-independent gauge transformations. Requiring that the physical states be annihilated by these constraints is tantamount to selecting the gauge-invariant states. Consequently operators that commute with the first class constraints are gauge invariant.

In the Schrödinger representation, in terms of wave functionals, the canonical momenta are represented by Hermitian differential operators, and the constraints applied onto the states become functional differential equations that the wave functionals must satisfy

$$\frac{\delta}{\delta A_0(\vec{x})}\Psi[A,\phi,\phi^\dagger]=0, \tag{2.16}$$

$$\left[\vec{\nabla}_x\frac{\delta}{\delta\vec{A}(\vec{x})}-ie\left(\phi(\vec{x})\frac{\delta}{\delta\phi(\vec{x})}-\phi^\dagger(\vec{x})\frac{\delta}{\delta\phi^\dagger(\vec{x})}\right)\right]\times\Psi[A,\phi,\phi^\dagger]=0. \tag{2.17}$$

The first equation simply means that the Schrödinger wave functional does not depend on A_0, whereas the second equation means that the wave functional is only a functional of the combination of fields that is annihilated by the Gauss' law functional differential operator. It is a simple calculation to prove that the fields

$$\Phi(\vec{x})=\phi(\vec{x})\exp\left[ie\int d^3y\vec{A}(\vec{y})\cdot\vec{\nabla}_yG(\vec{y}-\vec{x})\right], \tag{2.18}$$

$$\Phi^\dagger(\vec{x})=\phi^\dagger(\vec{x})\exp\left[-ie\int d^3y\vec{A}(\vec{y})\cdot\vec{\nabla}_yG(\vec{y}-\vec{x})\right] \tag{2.19}$$

are annihilated by Gauss' law functional differential equation with $G(\vec{y}-\vec{x})$ the Coulomb Green's function that satisfies

$$\nabla^2G(\vec{y}-\vec{x})=0. \tag{2.20}$$

Furthermore, writing the gauge field into transverse and longitudinal components as

$$\vec{A}(\vec{x})=\vec{A}_L(\vec{x})+\vec{A}_T(\vec{x}), \tag{2.21}$$

$$\vec{\nabla}\times\vec{A}_L(\vec{x})=0, \tag{2.22}$$

$$\vec{\nabla}\cdot\vec{A}_T(\vec{x})=0, \tag{2.23}$$

it is clear that

$$\vec{\nabla}_x\frac{\delta}{\delta\vec{A}(\vec{x})}=\vec{\nabla}_x\frac{\delta}{\delta\vec{A}_L(\vec{x})}. \tag{2.24}$$

Therefore the ''transverse component'' $\vec{A}_T(\vec{x})$ is also annihilated by the Gauss' law operator. This analysis shows that the wave-functional solutions of the functional differential equations that represent the constraints in the Schrödinger representation are of the form

$$\Psi[\vec{A},\phi,\phi^\dagger]=\Psi[\vec{A}_T,\Phi,\Phi^\dagger]. \tag{2.25}$$

The fields $\vec{A}_T$, Φ, and $\Phi^\dagger$ are *gauge invariant* as they commute with the constraints. The canonical momenta conjugate to $\Phi\Phi^\dagger$ are found to be

$$\Pi(\vec{x})=\pi(\vec{x})\exp\left[-ie\int d^3y\vec{A}(\vec{y})\cdot\vec{\nabla}_yG(\vec{y}-\vec{x})\right], \tag{2.26}$$

$$\Pi^\dagger(\vec{x})=\pi^\dagger(\vec{x})\exp\left[ie\int d^3y\vec{A}(\vec{y})\cdot\vec{\nabla}_yG(\vec{y}-\vec{x})\right]. \tag{2.27}$$

The momentum canonical to $\vec{A};\vec{\Pi}$, is written in terms of ''longitudinal'' and ''transverse'' components:

$$\vec{\Pi}(\vec{x})=\vec{\Pi}_l(\vec{x})+\vec{\Pi}_T(\vec{x}); \tag{2.28}$$

both components are gauge invariant.

In the physical subspace of gauge-invariant wave functionals, matrix elements of $\vec{\nabla}\cdot\vec{\Pi}$ can be replaced by matrix elements of the charge density ρ. Therefore in all matrix elements between gauge-invariant states (or functionals) one can replace

$$\vec{\Pi}_L(\vec{x})\to ie\vec{\nabla}_x\int d^3yG(\vec{x}-\vec{y})(\Phi\Pi-\Phi^\dagger\Pi^\dagger)(\vec{y}). \tag{2.29}$$

Finally in the gauge-invariant subspace the Hamiltonian becomes

$$H=\int d^3x\{\tfrac{1}{2}\vec{\Pi}_T\cdot\vec{\Pi}_T+\Pi^\dagger\Pi + (\vec{\nabla}\Phi-ie\vec{A}_T\Phi)\cdot(\vec{\nabla}\Phi^\dagger+ie\vec{A}_T\Phi^\dagger) + \tfrac{1}{2}(\vec{\nabla}\times\vec{A}_T)^2+\lambda(\Phi^\dagger\Phi-\mu^2)^2\} + \tfrac{1}{2}\int d^3y\int d^3x\rho(\vec{x})G(\vec{x}-\vec{y})\rho(y). \quad (2.30)$$

Clearly the Hamiltonian is gauge invariant, and it manifestly has the global U(1) gauge symmetry under which Φ transforms with a constant phase, Π transforms with the opposite phase, and $\vec{A}_T$ is invariant.

This Hamiltonian is reminiscent of the Coulomb gauge Hamiltonian, but we emphasize that we have not imposed any gauge-fixing condition. The formulation is fully gauge invariant, written in terms of operators that commute with the generators of gauge transformations and states that are invariant under these transformations.

There is a definite advantage in this gauge-invariant formulation: The (composite) field $\Phi(\vec{x})$ is a candidate for a *locally gauge invariant order parameter*. The point to stress is the following. This operator is *invariant* under local gauge transformations generated by the unitary transformation U_Λ given by Eq. (2.15), that is,

$$U_\Lambda\Phi(\vec{x})U_\Lambda^{-1}=\Phi(\vec{x}), \quad (2.31)$$

whereas it transforms as a charged operator under the *global* gauge transformations generated by $Q=\int d^3x\rho(\vec{x})$, that is,

$$e^{i\alpha Q}\Phi(\vec{x})e^{-i\alpha Q}=e^{ie\alpha}\Phi(\vec{x}). \quad (2.32)$$

Because the gauge constraints annihilate the physical states and these constraints are the generators of local gauge transformations, these states are invariant under the local gauge transformations and any operator that *is not* invariant under these local transformations *must* have zero expectation value. The *local* gauge symmetry cannot be spontaneously broken; this result is widely known in lattice gauge theory as Elitzur's theorem [15]. However, the *global* symmetry generated by the charge Q *can* be spontaneously broken and the expectation value of a charged field signals this breakdown.

From this discussion we clearly see that a trustworthy order parameter must be invariant under the local gauge transformations, thus commuting with the gauge constraints, but must transform nontrivially under the global gauge transformation generated by the charge. The field Φ fulfills these criteria and is the natural candidate for an order parameter.

III. EFFECTIVE POTENTIAL

A. Zero temperature

We are now in condition to define the gauge-invariant effective potential. Consider the gauge-invariant state $|\Psi;\chi\rangle$ such that the expectation value of the gauge-invariant order parameter $\Phi(\vec{x})$ in this state is nonzero and space-time constant:

$$\frac{\langle\Psi;\chi|\Phi(\vec{x})|\Psi;\chi\rangle}{\langle\Psi;\chi|\Psi;\chi\rangle}=\chi. \quad (3.1)$$

The effective potential is defined as the minimum of the expectation value of the Hamiltonian density in this state: namely,

$$V_{\rm eff}(\chi)=\frac{1}{\Omega}\min\left\{\frac{\langle\Psi;\chi|H|\Psi;\chi\rangle}{\langle\Psi;\chi|\Psi;\chi\rangle}\right\}, \quad (3.2)$$

with H being the gauge-invariant Hamiltonian given by Eq. (2.30) and Ω the spatial volume [16]. The state $|\Psi;\chi\rangle$ is chosen to minimize the expectation value of the Hamiltonian subject to the constraint that the expectation value of Φ in this state is χ.

It is convenient to separate the expectation value of Φ as

$$\Phi(\vec{x})=\chi+\eta(\vec{x}). \quad (3.3)$$

The one-loop correction [formally of $O(\hbar)$] to the effective potential is obtained by keeping the *quadratic* terms in the Hamiltonian:

$$H_q=\Omega\lambda(|\chi|^2-\mu^2)^2+\int d^3x\{\tfrac{1}{2}\vec{\Pi}_T^2+\tfrac{1}{2}(\vec{\nabla}\times\vec{A})^2+e^2\vec{A}_T^2|\chi|^2+\Pi^\dagger\Pi+(\vec{\nabla}\eta)(\vec{\nabla}\eta^\dagger)+2\lambda\eta^\dagger\eta(|\chi|^2-\mu^2)+(\eta\chi^\dagger+\eta^\dagger\chi)^2\} + \frac{e^2}{2}\int d^3xd^3y[\Pi(\vec{y})\chi-\Pi^\dagger(\vec{y})\chi^\dagger]G(\vec{y}-\vec{x})[\Pi(\vec{x})\chi-\Pi^\dagger(\vec{x})\chi^\dagger]. \quad (3.4)$$

The transverse components $\vec{A}_T$ describe a field with mass $m_T^2=2e^2|\chi|^2$ and only two polarizations. The phase of χ can be absorbed in Π by a *global* phase transformation under which the Hamiltonian is invariant.

It proves convenient to introduce real fields and canonical momenta as

$$\eta=\frac{1}{\sqrt{2}}(\eta_1+i\eta_2), \quad (3.5)$$

$$\Pi=\frac{1}{\sqrt{2}}(\Pi_1-i\Pi_2), \quad (3.6)$$

with $\{\Pi_{1,2},\eta_{1,2}\}$ being independent canonical pairs. The nonlocal part of the Hamiltonian is best treated in terms of the Fourier transform of the fields and their canonical momenta, in terms of which the quadratic part of the Hamiltonian finally becomes

$$H_q=\Omega V_{\rm cl}(|\chi|)+\frac{1}{2}\sum_k \Big\{ \vec{\Pi}_T(k)\cdot\vec{\Pi}_T(-k)+\omega_T^2(k)\vec{A}_T \times(k)\cdot\vec{A}_T(-k)+\Pi_1(k)\Pi_1(-k)+\omega_H^2(k)\,\eta_1(k)\,\eta_1 (-k)+\Pi_2(k)\Pi_2(-k)\frac{\omega_T^2(k)}{k^2}+\eta_2(k)\,\eta_2(-k)\omega_g^2(k)\Big\}, \tag{3.7}$$

where the frequencies are given in terms of the effective masses as

$$\omega_T^2(k)=k^2+m_T^2,\quad m_T^2=2e^2|\chi|^2, \tag{3.8}$$

$$\omega_H^2(k)=k^2+m_H^2,\quad m_H^2=2\lambda(3|\chi|^2-\mu^2), \tag{3.9}$$

$$\omega_g^2(k)=k^2+m_g^2,\quad m_g^2=2\lambda(|\chi|^2-\mu^2). \tag{3.10}$$

The last two terms can be brought to a canonical form by a Bogoliubov transformation. Define the new canonical coordinate Q and conjugate momentum P as

$$\Pi_2(k)=\frac{k}{\omega_T(k)}P(k), \tag{3.11}$$

$$\eta_2(k)=\frac{\omega_T(k)}{k}Q(k), \tag{3.12}$$

in terms of which the last term of the Hamiltonian (3.7) becomes a canonical quadratic form with the *plasma* frequency

$$\omega_p^2(k)=\omega_g^2(k)\frac{\omega_T^2(k)}{k^2} =[k^2+2\lambda(|\chi|^2-\mu^2)][k^2+2e^2|\chi|^2]/k^2. \tag{3.13}$$

There are four physical degrees of freedom. The modes with frequency $\omega_T(k)$ are the two transverse degrees of freedom, and the mode with frequency $\omega_H(k)$ is identified with the Higgs mode. In absence of electromagnetic interactions ($e=0$) the mode with frequency $\omega_p(k)$ represents the Goldstone mode whereas *in equilibrium*, namely, at the minimum of the tree level potential, when $|\chi|=\mu$, it represents the plasma mode which is identified as the screened Coulomb interaction, and the transverse and plasma modes all share the same mass. However, when the expectation value of the order parameter acquires a nonequilibrium value, away from the minimum of the tree level potential (at this order), this collective mode does not describe a particle with a Lorentz-covariant dispersion relation. The frequency clearly shows the combination of the Goldstone dispersion relation and the long-range Coulomb interaction typical of a description in terms of the dynamical degrees of freedom.

The lack of manifest Lorentz covariance in the dispersion relation can be understood as follows. Although the complex field ϕ in the original Lagrangian density is a Lorentz scalar, the gauge-invariant combination Φ given by Eq. (2.18) is not, although it is a rotational scalar. A particular Lorentz frame has already been chosen in making the transverse and longitudinal decomposition of the vector potential (2.21). The state of lowest available energy is expected to be Lorentz invariant, but for arbitrary χ there is a constraint in the space of functions and these constrained states are not the lowest-energy states in the functional space. In a scalar theory these states are manifestly Lorentz invariant, simply because all fields are Lorentz scalars. With vector fields the situation is more complicated and the constrained, gauge-invariant states are in general not manifestly Lorentz invariant. However, the lowest-energy equilibrium states (at this order corresponding to $\chi=\mu$) *are* manifestly Lorentz invariant. We will see in detail in Sec. IV that for $\chi\neq\mu$ these states are not stationary states of the Hamiltonian; therefore, the lack of Lorentz covariance for these states is reconciled with their nonequilibrium evolution.

The quadratic Hamiltonian is now diagonalized in terms of creation and destruction operators for the quanta of each harmonic oscillator. The ground state is the vacuum for each oscillator and is the state of lowest energy. Therefore the one-loop $[O(\hbar)]$ contribution to the effective potential is obtained from the zero point energy of the oscillators. Therefore accounting for the two polarizations of the transverse components we find

$$V_{\rm eff}(|\chi|)=V_{\rm cl}(|\chi|)+\frac{1}{2}\int\frac{d^3k}{(2\pi^3)}[2\omega_T(k)+\omega_H(k)+\omega_p(k)]. \tag{3.14}$$

The normalized wave functional that satisfies Eq. (3.1) and gives the minimum expectation value of the Hamiltonian, thus determining effective potential via Eq. (3.2) is given by

$$\Psi[A_T,\Phi^\dagger,\Phi] =N\exp\Big\{-\frac{1}{2}\int d^3x\int d^3y\vec{A}_T(\vec{x})\cdot\vec{A}_T(\vec{y})K_T(\vec{x}-\vec{y})\Big\} \times\exp\Big\{-\frac{1}{2}\int d^3x\int d^3y\,\eta_1(\vec{x})\,\eta_1(\vec{y})K_H(\vec{x}-\vec{y})\Big\} \times\exp\Big\{-\frac{1}{2}\int d^3x\int d^3y\,\eta_2(\vec{x})\,\eta_2(\vec{y})K_p(\vec{x}-\vec{y})\Big\}, \tag{3.15}$$

$$N=\Pi_k\left[\frac{\omega_T^2(k)\omega_H(k)\omega_p(k)}{\pi^4}\right]^{1/4}, \tag{3.16}$$

$$K_T(\vec{x}-\vec{y})=\int\frac{d^3k}{(2\pi)^3}\omega_T(k)e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{3.17}$$

$$K_H(\vec{x}-\vec{y})=\int\frac{d^3k}{(2\pi)^3}\omega_H(k)e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{3.18}$$

$$K_p(\vec{x}-\vec{y})=\int \frac{d^3k}{(2\pi)^3}\frac{k^2\omega_p(k)}{\omega_T^2(k)}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}. \qquad (3.19)$$

This Gaussian wave functional is clearly gauge invariant, and it has the correct limits: For $e=0$ ($\omega_T=k$; $\omega_p=\omega_g$) gives the (gauge-invariant) wave functional of free electromagnetism times the Gaussian wave functional of a complex scalar with the U(1) global symmetry spontaneously broken, which for $|\chi|^2=\mu^2$ corresponds to the Higgs and a Goldstone mode. Writing the fluctuation fields η_1 and η_2 in terms of Φ and $\Phi^\dagger$, we clearly see that this wave functional describes a broken symmetry state since under the global U(1) transformation the wave functional is changed into an orthogonal wave functional in the infinite volume limit.

The unbroken phase, with $\chi=0$, $-\mu^2=m^2>0$, corresponds to the ground state wave functional for free electromagnetism times the ground-state wave functional of two free real scalar fields with equal mass $m\sqrt{2\lambda}$. It is straightforward to see that the expectation value of the radial variable $\rho=\sqrt{\Phi^\dagger\Phi}$ is different from zero in this phase and cannot be used as an order parameter to signal spontaneous global symmetry breaking as discussed previously.

The k integrals in the effective potential (3.14) are performed with an upper momentum cutoff Λ. Neglecting a χ-independent term proportional to Λ^4 as well as terms that vanish in the $\Lambda\to\infty$ limit, and introducing a renormalization scale κ we obtain the (unrenormalized) expression

$$V_{\rm eff}(|\chi|)=V_{\rm cl}(|\chi|)+\frac{1}{4\pi^2}\left\{\frac{\Lambda^2}{4}[3m_T^2+m_H^2+m_g^2]+\frac{1}{16}\ln\left(\frac{\kappa^2}{4\Lambda^2}\right)[3m_T^4+m_H^4+m_g^4-2m_T^2m_g^2]\right.$$
$$\left.+\frac{1}{16}\left[2m_T^4\ln\left(\frac{m_T^2}{\kappa^2}\right)+m_H^4\ln\left(\frac{m_H^2}{\kappa^2}\right)+(m_g^2-m_T^2)^2\ln\left(\frac{m_T^2+m_g^2+2\sqrt{m_g^2m_T^2}}{\kappa^2}\right)\right]+\frac{1}{32}[3m_T^4+m_H^4+m_g^4+6m_g^2m_T^2]\right\}. \qquad (3.20)$$

The cutoff-dependent terms can be absorbed in a renormalization of μ^2 [terms of $O(\Lambda^2)$ and $O(\ln(\Lambda))$ proportional to $|\chi|^2$] and the quartic coupling λ [terms of $O(\ln(\Lambda))$ proportional to $|\chi|^4$]. Using this renormalization prescription we find the following result for the renormalized and gauge-invariant one-loop effective potential:

$$V_{{\rm eff},R}(|\chi|)=\lambda(|\chi|^2-\mu^2)^2+\frac{1}{4\pi^2}\left\{\frac{1}{32}[3m_T^4+m_H^4+m_g^4+6m_g^2m_T^2]\right.$$
$$\left.+\frac{1}{16}\left[2m_T^4\ln\left(\frac{m_T^2}{\kappa^2}\right)+m_H^4\ln\left(\frac{m_H^2}{\kappa^2}\right)+(m_g^2-m_T^2)^2\ln\left(\frac{m_T^2+m_g^2+2\sqrt{m_g^2m_T^2}}{\kappa^2}\right)\right]\right\}. \qquad (3.21)$$

In this expression λ and μ and all masses are renormalized with the above prescription.

In the region $|\chi|^2<\mu^2/3$ the Higgs ''mass'' is purely imaginary, whereas for $|\chi|^2<\mu^2, m_g^2<0$. Therefore we see that the logarithmic contributions to the effective potential from Higgs and plasma modes are *imaginary* [the last two logarithms in Eq. (3.21)], whereas the contribution of the gauge boson is real. The region in which the effective potential is imaginary is a region of unstable states [17,18], and the imaginary part of the effective potential disguises a nonequilibrium situation whose dynamics will be addressed in Sec. IV. This region of instabilities for homogeneous configurations is known as the spinodal region. In this region the system is unstable to phase separation, and the imaginary part of the effective potential appears as a result of attempting to describe an intrinsically time-dependent state as a stationary state via analytic continuation.

B. Finite temperature

The finite temperature effective potential is identified with the free energy density under the constraint that the ensemble average of the field be given by a space-time independent configuration χ. That is

$$V_T(\chi)=-T\ln(\mathrm{Tr}\hat{\rho}), \qquad (3.22)$$

$$\chi=\frac{\mathrm{Tr}\Phi(\vec{x})\hat{\rho}}{\mathrm{Tr}\hat{\rho}}, \qquad (3.23)$$

with $\hat{\rho}$ the ensemble density matrix. In equilibrium and when zero conserved charge is considered the density matrix is given by

$$\hat{\rho}=\exp\left(-\frac{H}{T}\right). \qquad (3.24)$$

In a gauge theory, however, the trace over states in Eq. (3.22) must be defined properly in terms of gauge-invariant states. Either the physical states are selected and only these are used in the trace or alternatively a projection operator must be introduced in the definition of the trace [19].

In our approach we select the states as those annihilated by the set of first class constraints, which are therefore gauge invariant as described in the previous section.

To one-loop order, we have seen that the Hamiltonian is quadratic in terms of gauge-invariant operators that describe the physical degrees of freedom. Therefore, to this order the

physical partition function is that of a collection of uncoupled harmonic oscillators for each degree of freedom.

We find the free energy density, which is identified as the finite temperature effective potential, to be ($\beta=1/T$)

$$\mathcal{F}=V_{\rm eff}(|\chi|;T)=V_{\rm eff}(|\chi|;T=0)+\frac{1}{\beta}\int\frac{d^3k}{(2\pi)^3}\times\{2\ln[1-e^{-\beta\omega_T(k)}]+\ln[1-e^{-\beta\omega_H(k)}]+\ln[1-e^{-\beta\omega_p(k)}]\},\tag{3.25}$$

where $V_{\rm eff}(|\chi|;T=0)$ is the zero temperature effective potential given by Eq. (3.14), and arises from the zero point energy of the oscillators.

Just as our gauge-invariant approach in terms of gauge-invariant operators and functionals allowed us to obtain the ground-state constrained wave functional, Eq. (3.15), similarly we can obtain the density matrix elements in the Schrödinger representation $\langle\Phi,\Phi^\dagger,A_T|\hat\rho|A_T',\Phi'^\dagger,\Phi'\rangle$. This representation for the density matrix is very useful to study nonequilibrium aspects and time-dependent phenomena [20], which is the focus of the next section. We find the density matrix elements in the Schrödinger representation to be given by

$$\langle\Phi,\Phi^\dagger,A_T|\hat\rho|A'_T,\Phi'^\dagger,\Phi'\rangle=\langle\Phi,\Phi^\dagger|\hat\rho_\Phi|\Phi'^\dagger,\Phi'\rangle\otimes\langle A_T|\hat\rho_A|A'_T\rangle,\tag{3.26}$$

where the density matrices are of the harmonic oscillator type, the full expression is given in Appendix A.

C. Large T expansion

The contributions to the free energy from the transverse and Higgs modes are straightforward to obtain by applying the methods developed by Dolan and Jackiw [1]. However, the contribution from the ''plasma'' mode is nonstandard and requires a more detailed analysis which is presented in Appendix B. We find the leading high-temperature behavior to the finite temperature contribution to the effective potential to be given by

$$V_{{\rm eff},T}(|\chi|)=-4\frac{\pi^2T^4}{90}+\frac{T^2}{24}[3m_T^2+m_H^2+m_g^2]-\frac{T}{12\pi}[3(m_T^2)^{3/2}+(m_H^2)^{3/2}+(m_g^2)^{3/2}]-\frac{1}{64\pi^2}\left\{(m_g^2-m_T^2)^2\ln\left[\frac{m_g^2+m_T^2+2\sqrt{m_g^2m_T^2}}{T^2}\right]+2m_T^4\ln\left(\frac{m_T^2}{T^2}\right)+m_H^4\ln\left(\frac{m_H^2}{T^2}\right)\right\}+\cdots,\tag{3.27}$$

where the dots stand for terms of $O(T^0)$ or smaller. Remarkably, the logarithmic terms cancel similar terms of the zero temperature part, and although this feature is well known in the standard cases (with standard dispersion relations for the degrees of freedom), it is a new result for the plasma mode. An important feature of this expression is that the terms linear in T, which are nonanalytic, are *complex*. Whereas the term from the gauge boson mass is real, the terms originating in the Higgs couplings given by the contributions from m_H^2 and m_g^2 are purely imaginary on the spinodal regions $|\chi|^2<\mu^2/3$ for m_H^2 and $|\chi|^2<\mu^2$ for m_g^2. These ''cubic'' terms are usually identified as those responsible for a first order phase transition and used to compute quantities relevant to the transition [21,22,10]. In particular these terms determine the supercooling temperature and the latent heat when they are taken as the leading indicators for a first order transition. In their study of the electroweak effective potential Anderson and Hall [23] neglected the terms involving the Higgs self-coupling keeping only the contributions from the gauge boson and top quark Yukawa couplings which are gauge invariant and real to one loop. Arguably such an approximation is justified for very weak Higgs couplings. Boyd *et al.* [24] recognized that the terms arising from the Higgs sector lead to contributions that are imaginary (even after resummation) precisely as pointed out above. Thus using these terms to compute the latent heat, supercooling temperatures, and even approximate dynamics is at best a crude approximation (even when the imaginary parts are ignored) and at worst disguises other nonequilibrium processes which may be equally important (see Sec. IV).

D. Comparison with gauge-fixed results

In this section we compare our gauge-invariant result to the one-loop effective potential obtained in the usual standard path-integral representation for several gauge-fixing procedures. The purpose is to *contrast* our results with the suggestion of Fukuda and Kugo [25] that there is a large class of ''good gauges'' for which the effective potential is gauge invariant. These authors suggested that covariant gauges (including Landau with zero gauge parameter), R_ξ, and other specific gauge-fixing schemes are such ''good'' choices.

In order to make a distinction from the gauge-invariant formulation, we write the original complex field ϕ (not to be confused with the gauge-invariant field Φ) in the Lagrangian density, Eq. (2.1), as

$$\phi(\vec x,t)=\frac{1}{\sqrt2}[\hat\phi_R(\vec x,t)+i\hat\phi_I(\vec x,t)]+\varphi,\tag{3.28}$$

and φ is taken as the (complex) expectation value. To distinguish from the gauge-invariant case we also introduce the masses

$$M_g^2=2\lambda(|\varphi|^2-\mu^2),\tag{3.29}$$

$$M_A^2 = 2e^2|\varphi|^2, \tag{3.30}$$

$$M_H^2 = 2\lambda(3|\varphi|^2 - \mu^2). \tag{3.31}$$

At this point one would be tempted to identify φ with χ, because the effective tree level masses seem to be the same as in the gauge-invariant case under the replacement $\chi \to \varphi$. However, we make a distinction between these two expectation values because χ is a truly gauge-invariant quantity, whereas φ is the expectation value of a gauge-transforming field in a fixed gauge.

The one-loop effective potential is given by

$$V_1 = \sum_j \frac{g_j}{2\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \ln\left[\left(\frac{2\pi n}{\beta}\right)^2 + k^2 + M_j^2\right] = V_{10} + V_{1T},$$

$$V_{10} = \sum_j \frac{g_j}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + M_j^2}, \tag{3.32}$$

$$V_{1T} = \sum_j \frac{g_j}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln\{1 - \exp[-\beta\sqrt{k^2 + M_j^2}]\}, \tag{3.33}$$

where the sum are over all particles j with g_j degrees of freedom ($g_j < 0$ for ghosts) and masses $M_j(|\varphi|)$, and we have used a result given in [1].

The zero temperature contribution is divergent, the k integrals being performed with an ultraviolet cutoff Λ. Discarding a field-independent quartic divergence we find the result

$$V_{10} = \sum_j \frac{g_j}{4\pi^2} \left\{ \frac{M_j^4}{16} \ln\left(\frac{M_j^2}{\Lambda^2}\right) + \frac{\Lambda^2 M_j^2}{4} + \frac{M_j^4}{32} \right\}. \tag{3.34}$$

The finite temperature contribution can be written as

$$V_{1T} = \frac{T^4}{2\pi^2} \sum_j g_j I\left(\frac{M_j}{T}\right). \tag{3.35}$$

The high-temperature expansion of $I(y)$ is given by [1]

$$I(y) = \frac{-\pi^4}{45} + \frac{\pi^2}{12} y^2 - \frac{\pi}{6} (y^2)^{3/2} - \frac{y^4}{32}[\ln(y^2) - \tfrac{3}{2} - \mathcal{C}] + O(y^6), \tag{3.36}$$

where we defined $\mathcal{C} \equiv 2\ln(4\pi) - 2\gamma = 3.9076$. The $\ln(m^2)$ term cancels against the similar term in the zero temperature contribution V_{10}.

1. Lorentz gauge

Dolan and Jackiw [1] calculate the one-loop effective potential both at zero and nonzero temperature in the Lorentz gauge, which is ghost free, with gauge parameter α. The scalar determinant is diagonalized by solving

$$\ln[k^4 + M_g^2 k^2 + \alpha M_T^2 M_g^2] = \ln\{[k^2 + M_+^2(\alpha)][k^2 + M_-^2(\alpha)]\}. \tag{3.37}$$

The resulting masses and effective degrees of freedom are

$$M_\pm^2(\alpha) = \tfrac{1}{2}[M_g^2 \pm \sqrt{M_g^4 - 4\alpha M_T^2 M_g^2}], \quad g_\pm = 1, \tag{3.38}$$

$$M_g^2 = 2\lambda(|\varphi|^2 - \mu^2), \quad g_g = 1, \tag{3.39}$$

$$M_A^2 = 2e^2|\varphi|^2, \quad g_A = 3. \tag{3.40}$$

Thus the zero temperature part of the one-loop effective potential in Lorentz gauge is given by

$$V_{10\mathrm{LG}}(|\varphi|;\alpha) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} [\Omega_H(k) + 3\Omega_A(k) + \Omega_+(k;\alpha) + \Omega_-(k;\alpha)], \tag{3.41}$$

$$\Omega^\pm(k;\alpha) = \sqrt{k^2 + M_\pm^2(\alpha)}, \tag{3.42}$$

$$\Omega_j(k) = \sqrt{k^2 + M_j^2}, \quad j = H, A. \tag{3.43}$$

The gauge dependence (dependence on the gauge parameter α) of the above result is explicit. Even for Landau gauge, that is, $\alpha = 0$, expression (3.41) describes *five* degrees of freedom, rather than the four physical degrees of freedom described by the gauge-invariant result (3.14). With the same renormalization prescription leading to the gauge-invariant result, Eq. (3.21), we find in the Landau gauge the result

$$V(\varphi, \alpha = 0) = \lambda(|\varphi|^2 - \mu^2) + \frac{1}{4\pi^2} \left\{ \frac{1}{32}[3M_A^4 + M_H^4 + M_g^4] + \frac{1}{16}\left[3M_A^4 \ln\left(\frac{m_T^2}{\kappa^2}\right) + M_H^4 \ln\left(\frac{M_H^2}{\kappa^2}\right) + M_g^4 \ln\left(\frac{M_g^2}{\kappa^2}\right)\right]\right\}, \tag{3.44}$$

which is obviously very different from the gauge-invariant result given by Eq. (3.21) if φ is identified with the gauge-invariant order parameter χ.

The effective potential (3.41) becomes independent of the gauge parameter α for the values $\varphi = 0$ and $|\varphi|^2 = \mu^2$. These are the value of the extrema of the *tree level* potential. The gauge dependence appears at one-loop order and is therefore formally of $O(\hbar)$ since the extrema of the effective potential will acquire $O(\hbar)$ corrections. We identify the values of φ at which the gauge dependence cancels out as the extrema of the effective action *to this order*. Up to an irrelevant constant the gauge-invariant effective potential (3.14) and the one-loop effective potential in general covariant gauge (3.41) are the same for $|\chi|^2 = \mu^2$, i.e., at the extrema of the effective action. This equality is a consequence of the known result that the extrema of the effective action are gauge independent. At zero temperature gauge independence at the extrema is also a consequence of the Nielsen identities [3]. Therefore we find that the suggestion of Ref. [25] is *only* valid at the extrema of the effective action.

At finite temperature we find that the $O(T^2)$ is gauge parameter independent and given by

$$V_{1T}^{(2)}=\frac{T^2}{24}[3M_A^2+M_H^2+M_g^2]. \tag{3.45}$$

This term coincides with that from the gauge-invariant effective potential with the identification $|\varphi|\equiv|\chi|$. This contribution determines (to this order) the critical temperature which is therefore a gauge-independent quantity, whereas the $O(T)$ and $O(\ln(T))$ contributions are gauge parameter dependent.

The $O(T)$ contribution is given by

$$V_{1T}^{(1)}=-\frac{T}{12\pi}\{3(M_A^2)^{3/2}+(M_H^2)^{3/2}+[M_+^2(\alpha)]^{3/2}+[M_-^2(\alpha)]^{3/2}\}. \tag{3.46}$$

For $\alpha=0$, $M_+^2=M_g^2$, and $M_-^2=0$ and this $O(T)$ term coincides with the $O(T)$ contribution from the gauge-invariant effective potential given by Eq. (3.27) if $|\varphi|$ is identified with the gauge-invariant order parameter $|\chi|$.

This contribution is of particular importance because it is usually taken as a signal for a first order transition and determines its strength, and sometimes used in phenomenological equations to describe the dynamics [21,22]. In the case of first order phase transitions, this term is sometimes used to compute the latent heat and the supercooling temperature [10]. Clearly, quantities calculated solely from this term would be physically meaningless because of the gauge dependence. Furthermore, this contribution is complex for $|\varphi|^2<\mu^2$; only the contribution from the gauge boson is real and gauge invariant. Therefore our conclusion is that this term is provides the correct gauge-invariant $O(T)$ contribution *only* in the Landau ($\alpha=0$) gauge and with the identification $|\varphi|\equiv|\chi|$. However, this equivalence only holds to leading order in the high-temperature expansion and *is not* a general feature to all orders, as displayed explicitly by higher-order finite temperature corrections and also by the zero temperature part.

2. R_ξ and $\overline{R}_\xi$ gauges

In the R_ξ gauge the following gauge-fixing and ghost terms are added to the Lagrangian density:

$$\mathcal{L}_{\rm GF}=-\frac{1}{2\xi}(\partial_\mu A^\mu+\xi e\varphi\hat{\phi}_I)^2,$$

$$\mathcal{L}_{\rm FPG}=c^\dagger[-\partial^2-\xi e^2\varphi(\varphi+\hat{\phi}_R)]c. \tag{3.47}$$

Kastening [32] describes a useful variant called $\overline{R}_\xi$ gauge, in which ϕ and $\hat{\phi}_R$ are treated more symmetrically. Its gauge-fixing and ghost terms are

$$\mathcal{L}_{\rm GF}=-\frac{1}{2\xi}[\partial_\mu A^\mu+\xi e(\varphi+\hat{\phi}_R)\hat{\phi}_I]^2,$$

$$\mathcal{L}_{\rm FPG}=c^\dagger\{-\partial^2-\hat{\phi}_I e^2[(\varphi+\hat{\phi}_R)^2-\hat{\phi}_I^2]\}c. \tag{3.48}$$

The ghost term is derived as usual by looking at the response of the gauge-fixing functional under a gauge transformation. The corresponding gauge-fixing functional for the $\overline{R}_\xi$ gauge fixing procedure is $f[A^\mu]=\partial_\mu A^\mu+\xi e(i/2)(\phi^\dagger\phi^\dagger-\phi\phi)$. A piece from the photon determinant cancels off half the ghost contribution, and so we will treat the ghosts as having $g_c=-1$. In both R_ξ and $\overline{R}_\xi$ cases we find the following masses and degrees of freedom:

$$M_A^2,\quad g_A=3, \tag{3.49}$$

$$M_c^2=\xi M_A^2,\quad g_c=-1, \tag{3.50}$$

$$M_H^2,\quad g_H=1, \tag{3.51}$$

$$M_\xi^2=M_g^2+\xi M_A^2,\quad g_\xi=1. \tag{3.52}$$

The degree of freedom with M_ξ is a (gauge-dependent) linear combination of the Goldstone mode and the vector boson. At the tree level minimum $|\varphi|^2=\mu^2$ when $M_g^2=0$ this mode cancels the remaining ghost contribution, leaving a ξ-independent result, calculable just from the photon and the Higgs terms. At the tree level maximum $\varphi=0$ the gauge dependence cancels out completely and the ghost decouples. Away from the minimum, however, the one-loop effective potential is ξ dependent.

At finite temperature we find that the $O(T^2)$ term is again gauge parameter independent and given by the same expression as in Eq. (3.45).

However, the next, $O(T)$, term is explicitly gauge parameter dependent and given by

$$\frac{-T}{12\pi}[3(M_A^2)^{3/2}-(\xi M_A^2)^{3/2}+(M_H^2)^{3/2}+(M_g^2+\xi M_A^2)^{3/2}]. \tag{3.53}$$

This term depends on ξ (unless $e=0$ or $|\varphi|^2=\mu^2$) and is complex for $|\varphi|<\mu^2$. In the Landau gauge ($\xi=0$), it is given by

$$\frac{-T}{12\pi}[3(M_A^2)^{3/2}+(M_H^2)^{3/2}+(M_g^2)^{3/2}]\quad\text{(Landau)} \tag{3.54}$$

and coincides with the $O(T)$ contribution to the high-T expansion of the gauge-invariant effective potential (3.27) if φ is identified with the gauge-invariant order parameter χ.

Since to this order only the $O(T^2)$ term enters in the estimate of the critical temperature, we see that T_c is a gauge-invariant quantity (to this order). But comparing the $O(T)$ term with that in Lorentz gauges should convince the reader that this term is gauge dependent and complex in general and one must be very careful in attaching any physical meaning to it, such as a criterion for a first order phase transition, its strength, and the ensuing latent heat and supercooling temperature. Only in the Landau gauge do we find that this term is the same as in the gauge-invariant formulation, although even in this gauge, the higher-order finite temperature corrections and the zero temperature part are gauge parameter dependent.

3. Unitary gauge

In unitary gauge, even the leading term of V_{1T} (calculated to one loop) is incorrect [1]:

$$\frac{T^2}{24}[3(M_A^2)+M_H^2]. \tag{3.55}$$

In this gauge, higher-loop corrections affect the leading term and must be included in the calculation to obtain the correct answer (and gauge independent) for the $O(T^2)$ term. Arnold, Braaten, and Vokos [26] showed that a two-loop calculation restores the correct leading term (3.45).

4. Higher-loop resummation

We have seen that the one-loop effective potential, calculated in various gauge-fixing schemes, is explicitly gauge parameter dependent. One might argue that a resummation of higher-loop diagrams would eliminate this dependence. Here we will show that this is not the case [27].

Although resummation would naturally be invoked to restore gauge invariance in a particular calculation, clearly resummation does nothing to restore gauge parameter independence at $T=0$, since higher-loop diagrams are all higher order in λ or e^2. Thus it is clear that the gauge dependence will remain in any calculation that includes the zero temperature contribution.

At finite (but large) T it is a little less obvious, since there is an additional expansion parameter: M/T (with M any of the masses). In fact, we have already mentioned that resummation in unitary gauge *does* recover the correct, gauge-independent, T^2 term. We now show explicitly in R_ξ and $\overline{R}_\xi$ gauges that the T term remains ξ dependent even after resummation of hard thermal loops [27].

a. Expansion parameters. In this section we take $e^2\sim\lambda$, and M can represent M_H or M_A. Terms in V_T are described as $O(\alpha^a\beta^b\gamma^c)$ with respect to the leading T^2M^2 term, where [27,28]

$$\alpha\equiv\lambda T^2/m^2, \quad \beta\equiv\lambda T/m, \quad \gamma\equiv\varphi^2/T^2. \tag{3.56}$$

We will take $\alpha\approx 1$ and $\gamma\approx 1$ but $\beta<1$, so that $\{e^2,\lambda\}=O(\beta^2)$ and $M/T=O(\beta)$. Our one-loop high-T expansion is then seen to be an expansion in β (not to be confused with inverse temperature here).

We want to resum to $O(\beta)$ (given by daisy diagrams with hard thermal loops). The simplest approach is the tadpole method (for a more comprehensive discussion, see Ref. [27]), replacing the M_j's by thermal masses, which need only be calculated to $O(\beta^0)$. We then integrate with respect to φ, adding constants as needed, to give V_T. The leading (T^2) term is unaffected, and remains gauge independent.

b. Thermal masses. The $O(\beta^0)$ thermal masses are unambiguous and gauge independent. For the scalars we can use the relations $\widetilde{M}_H^2=M_H^2+V''_{1T}(\varphi)$ and $\widetilde{M}_\xi^2=M_\xi^2+V'_{1T}(\varphi)/\varphi$ (the former is true to all orders in R_ξ gauge). We get

$$\widetilde{M}_H^2=M_H^2+\frac{T^2}{24}[6e^2+8\lambda],$$

$$\widetilde{M}_\xi^2=M_g^2+\xi M_A^2+\frac{T^2}{24}[6e^2+8\lambda]. \tag{3.57}$$

Only the longitudinal part of the photon gets a thermal (electric screening, plasma) mass (given by Π_0^0) to leading order:

$$\widetilde{M}_L^2=M_A^2+\frac{e^2T^2}{3}, \quad \widetilde{M}_T^2=M_A^2. \tag{3.58}$$

With these hard-thermal-loop resummed masses, the $O(\beta)$ term in the R_ξ or $\overline{R}_\xi$ gauge becomes

$$\frac{-T}{12\pi}\left[2M_A^3+(1-\xi^{3/2})\left(M_A^2+\frac{e^2T^2}{3}\right)^{3/2}+\left(M_H^2+\frac{T^2}{24}[6e^2+8\lambda]\right)^{3/2}+\left(M_g^2+\xi M_A^2+\frac{T^2}{24}[6e^2+8\lambda]\right)^{3/2}\right]. \tag{3.59}$$

This term is still ξ-dependent. We see that resummation [to $O(\beta)$] does not render the effective potential gauge parameter independent.

IV. NONEQUILIBRIUM ASPECTS

The main motivation for studying an effective potential is to address the issue of symmetry breaking and phase transitions accounting for quantum and thermal corrections. By its very definition, the effective potential is an *equilibrium* quantity because the expectation value of the scalar field, which serves as the order parameter, is space-time independent. At zero temperature the quantum wave functional is taken to be a stationary state; at finite temperature, the calculation is performed with the equilibrium partition function. Thus it is clear that the effective potential is only suitable to describe the equilibrium aspects associated with the phase transition. The only equilibrium states correspond to the extrema of the effective action which for homogeneous configurations coincide with the extrema of the effective potential. Thus quantities such as the critical temperature defined as the value at which the minima of the effective potential become a maximum, as well as expectation values of the scalar field that extremize the effective potential, are meaningful quantities that are useful to determine whether there is a symmetry-breaking phase transition.

For values of the order parameter away from the extrema, the effective potential is simply *not* a reliable tool to describe the situation and by its very definition it is not meant to be. In using the effective potential to address dynamical issues, such as ''the rolling down'' of the order parameter towards the equilibrium configuration, the hope is that the time evolution is rather slow and the use of the ''instantaneous'' effective potential is justified as some sort of adiabatic approximation. This expectation, however, conceals the physics of the dynamical processes during the phase transition [18].

In second order phase transitions, if the order parameter is initially very small and the system is evolving from an initial disordered (high-temperature) phase with short-range correlations towards a final equilibrium broken symmetry state with long-range correlations, the dynamical process is that of phase separation and growth of correlated domains. This dynamics is not captured solely by the time evolution of the expectation value of the scalar field, but the growth of fluctuations will be manifest in the time dependence of the cor-

relation functions of this field. During the early stages of a second order phase transition, when the order parameter is very small (near the maximum of the effective potential), small-amplitude, long-wavelength fluctuations become unstable and grow [29]. The two-point correlation functions of the scalar field reflect this instability and grow exponentially, as the order parameter ''rolls down'' the potential hill.

If the phase transition is of first order, then there are free energy barriers that the system has to overcome to reach the equilibrium state and the phase transition is driven by nucleation, in which large-amplitude configurations become unstable and grow. However, if the first order transition is very weak (that is to say that there is a substantial amount of phase mixing), nucleation and phase separation occur on similar spatial and time scales and the long-wavelength instabilities will still be important.

The information on thermodynamic instabilities that lead to phase separation is *contained* in the effective potential both at zero and finite temperature in the form of an *imaginary part*.

Weinberg and Wu [17] have shown in a beautiful paper that the imaginary part of the effective potential in scalar theories determines the growth rate of the scalar-field two-point correlation function for Gaussian states centered at zero expectation value (in field space). Boyanovsky and de Vega [18] have studied how this growth of correlations affects the time evolution of the order parameter and concluded that the use of the effective potential to describe the time evolution of the expectation value of the scalar field is not only unwarranted but completely misleading and unreliable whenever the initial value of the expectation value is in the ''spinodal region'' (the region in the tree level potential where the second derivative is negative).

A. Imaginary part

We notice that in the ''classical spinodal'' region, where $m_H^2<0$ or $0<|\chi|<\mu/\sqrt{3}$, the Higgs mode has a band of unstable wave vectors with imaginary frequencies for $k^2<\mu^2-3|\chi|^2$. The plasma mode has a band of unstable wave vectors with imaginary frequencies for $k^2<\mu^2-|\chi|^2$ in the spinodal region $0<|\chi|^2<\mu^2$. This new spinodal line ranges from the maximum to the minima of the tree level potential.

When the gauge coupling is switched off, in the absence of long-range forces, this is recognized as the ''spinodal'' region for the would be Goldstone modes [20]. However, in the presence of long-range forces, the instabilities are much more severe for long-wavelength fluctuations, as can be seen from the infrared behavior of the plasma frequency, Eq. (3.13).

As a consequence of these unstable modes, both the zero and finite temperature effective potentials acquire an imaginary part for *all values* of $|\chi|$ between the maximum and the minima of the tree level potential. Therefore we see that unlike the case of a scalar order parameter in which there is a thermodynamically stable region in the phase diagram (between the classical spinodal and the coexistence line), in this case the unstable region covers the whole of the phase diagram below the coexistence curve.

Thus we here obtain one of the important conclusions of this work: The gauge-invariant effective potential to one-loop order is *complex* from the maximum to the minimum of the tree level potential. There are no homogeneous configurations corresponding to quasiequilibrium states away from the minimum. Explicitly, the imaginary part of the gauge-invariant effective potential at zero temperature is given by

$$\mathrm{Im}(V_{\mathrm{eff}})=\pm\frac{1}{4\pi^2}\left\{\int_0^{k_1(|\chi|)}k^2dk\,\sqrt{k_1^2(|\chi|)-k^2}\;\Theta(\mu^2-3|\chi|^2)\right.$$
$$\left.+\int_0^{k_2(|\chi|)}kdk\,\sqrt{k_2^2(|\chi|)-k^2}\,\sqrt{2e^2|\chi|^2+k^2}\;\Theta(\mu^2-|\chi|^2)\right\},$$
$$k_1^2(|\chi|)=2\lambda(\mu^2-3|\chi|^2),\quad k_2^2(|\chi|)=2\lambda(\mu^2-|\chi|^2),\tag{4.1}$$

where the $\pm$ is determined by the direction of the analytic continuation in the frequencies.

The presence of an imaginary part of the effective potential has been sometimes justified as a failure of the loop expansion and ''corrected'' by the Maxwell construction which yields a convex free energy. However, just as the van der Waals loop in the equation of state for liquid-gas systems is a signal of thermodynamic instabilities, the imaginary part of the effective potential signals the presence of similar instabilities in the quantum system.

The flat region of the Maxwell constructed free energy indicates that the system will be found in a coexistence of phases but offers no information on the nonequilibrium processes leading to phase separation. In this respect the effective potential with its imaginary part at least provides some restricted information on dynamical processes. As discussed in Refs. [17,18] this imaginary part determines the decay rate of unstable Gaussian states.

In the one-loop approximation, the Hamiltonian for the modes with wave vectors in the unstable bands corresponds to *inverted harmonic oscillators* for which the analysis of Refs. [17,18,20,30] can be applied. The effective potential (3.14) is complex because the modes corresponding to inverted harmonic oscillators were treated as ordinary harmonic oscillators, i.e., by *analytically continuing* the zero point energy for these oscillators. The Gaussian wave functionals and density matrix have complex kernels for the fields $\eta_{1,2}$ reflecting this analytic continuation.

In terms of the shifted fields η_1 and η_2 with zero expectation value, the Gaussian wave functional (3.15) and the

density matrix (see Appendix A) are ''centered'' at the origin in field space, and for the wave vectors in the unstable bands, they are not stationary states of the Hamiltonian.

In this one-loop approximation we can use the results of previous investigations of similar situations [17,18,20,30] to study the early time evolution of initially prepared nonequilibrium states.

B. Early time evolution of nonequilibrium states

From the previous discussion it is clear that the imaginary part of the effective potential conceals a dynamical nonequilibrium situation associated with the instabilities towards phase separation. This is a dynamical situation that *must* be treated as a time-dependent problem. The quantum states that lead to the effective potential (and free energy) are *not* stationary states.

Since only the Higgs and plasma modes have unstable bands, we will focus only on these. Consider the situation in which an initial Gaussian density matrix has been prepared such that the gauge-invariant scalar field Φ has initial expectation value in this ensemble given by χ, and that the kernels have positive frequencies. That is, the density matrix in the Schrödinger representation is given by the expression in Appendix A but with the real kernels

$$K_H^{(1)}(k,t=0)=-\frac{1}{2}W_H(k)\coth\left[\frac{W_H(k)}{T}\right],$$

$$K_H^{(2)}(k,t=0)=\frac{W_H(k)}{\sinh[W_H(k)/T]}, \tag{4.2}$$

$$K_p^{(1)}(k,t=0)=-\frac{1}{2}W_P(k)\coth\left[\frac{W_P(k)}{T}\right],$$

$$K_H^{(2)}(k,t=0)=\frac{W_P(k)}{\sinh[W_P(k)/T]} \tag{4.3}$$

$$W_H^2(k)>0, \quad W_P^2(k)>0. \tag{4.4}$$

The frequencies $W_H(k)$ and $W_P(k)$ determine the initial conditions on the state as discussed in Refs. [17,18,20,30]. We can now use the results of Refs. [17,18,20,30] to determine the time evolution of this initial state. The early time evolution of the expectation value χ will be determined by the classical equations of motion with small quantum and thermal corrections. However, at longer times the growth of the unstable modes makes the one-loop approximation to the dynamics unreliable as in the problem of domain growth in scalar theories [20,29]. Clearly, the most important dynamics is described by the evolution of the unstable modes and determined by the time evolution of the kernels. We now use the method discussed in Ref. [20] to obtain the following time evolution of the kernels:

$$K_H^{(1)}(k,t)=\frac{K_H^{(1)}(k,t=0)}{|U_k(t)|^2}-i\frac{d\ln[|U_k(t)|^2]}{dt}, \tag{4.5}$$

$$K_H^{(2)}(k,t)=\frac{K_H^{(2)}(k,t=0)}{|U_k(t)|^2}, \tag{4.6}$$

$$K_p^{(1)}(k,t)=\frac{K_p^{(1)}(k,t=0)}{|V_k(t)|^2}-i\frac{d\ln[|V_k(t)|^2]}{dt}, \tag{4.7}$$

$$K_p^{(2)}(k,t)=\frac{K_p^{(2)}(k,t=0)}{|V_k(t)|^2}, \tag{4.8}$$

where the mode functions $U_k(t)=U_{1k}(t)+iU_{2k}(t)$ and $V_{1k}(t)+iV_{2k}(t)$ obey the evolution equations [20]

$$\left[\frac{d^2}{dt^2}+k^2+2\lambda(3|\chi_{\rm cl}|^2(t)-\mu^2)\right]U_k(t)=0,$$

$$U_{1k}(0)=1, \quad U_{2k}(0)=0, \quad \dot{U}_{1k}(0)=0, \quad \dot{U}_{2k}(0)=W_H(k), \tag{4.9}$$

$$\left[\frac{d^2}{dt^2}+[k^2+2\lambda(|\chi_{\rm cl}|^2(t)-\mu^2)][k^2+2e^2|\chi_{\rm cl}|^2(t)]/k^2\right]V_k(t)=0,$$

$$V_{1k}(0)=1, \quad V_{2k}(0)=0, \quad \dot{V}_{1k}(0)=0, \quad \dot{V}_{2k}(0)=W_P(k), \tag{4.10}$$

where $\chi_{\rm cl}$ is the classical evolution of the ''zero mode'' [20]. Obviously the mode functions $U_k(t)$ and $V_k(t)$ grow almost exponentially for wave vectors in the respective unstable bands when $|\chi_{\rm cl}|(t)$ is in the corresponding spinodal regions.

Consider the equal-time two-point correlation functions of the *gauge-invariant* operators η and $\eta^\dagger$:

$$\langle\eta(\vec{x},t)\,\eta^\dagger(\vec{y},t)\rangle=\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left\{\frac{|U_k(t)|^2}{2W_H(k)}\coth\left[\frac{W_H(k)}{T}\right]+\frac{|V_k(t)|^2}{2W_P(k)}\coth\left[\frac{W_P(k)}{T}\right]\right\}. \tag{4.11}$$

When $\chi_{\rm cl}(t)$ remains in the spinodal, it is clear that the *growth rate* of $\langle\eta(\vec{0},t)\,\eta^\dagger(\vec{0},t)\rangle$ is related to the imaginary part of the effective potential given by Eq. (4.1) at zero temperature.

At this point we recognize an important payoff of the gauge-invariant description. In order to compute gauge-invariant correlation functions from the gauge-variant operator $\phi(\vec{x},t)$ we would have to append a line integral of the (time-dependent) gauge field, with the ensuing path ambiguities and complications. The formulation in terms of gauge-invariant order parameters from the start overcomes these difficulties and allows one to extract physically meaningful correlation functions that provide dynamical information on nonequilibrium processes.

Just as in the case of a scalar field theory undergoing spinodal decomposition [29] this equal-time correlation function will grow exponentially at early times because of the unstable modes. This growth of is the hallmark of the dynamics of phase separation [17,18,29].

If the phase transition is weakly first order in the sense that a considerable amount of phase separation and mixing occurs during nucleation, then the growth of correlations described by the dynamics of the unstable modes will be also a dominant process.

As noted before, unlike the case of a scalar field transforming under a discrete symmetry, in this case the spinodal region reaches all the way to the minimum of the potential (at least in the one-loop approximation) and the long-wavelength instabilities are enhanced by the long-range forces. This feature provides interesting possibilities if the phase transition is strongly first order and nucleation is the dominant mechanism. Typically after nucleation of a critical bubble, the order parameter inside the bubble is very close to its equilibrium value. If this value is smaller in magnitude to that of the equilibrium configuration, spinodal instabilities may be still present *inside* the bubble.

If the critical droplet has rather small radius, it will have to grow to a size $\approx 1/(|\chi|-\mu)$, with $|\chi|$ the value of the order parameter inside the bubble, before any instability develops inside the bubble. Therefore the fact that the spinodal line reaches to the minimum may lead to the tantalizing possibility that for large critical bubbles there may still be nonequilibrium processes *inside* the bubble if the order parameter inside is smaller in magnitude than the equilibrium value.

Clearly all these possibilities will have to be studied in deeper detail and we expect a strong dependence on the values of the gauge and Yukawa couplings (in the case of fermions). Furthermore, our analysis of the time evolution of the spinodal instabilities only holds at very early times after the preparation of the initial state. At longer times a nonperturbative gauge-invariant scheme will have to be used to determine the dynamics. We hope to be able to implement the variational schemes of Ref. [31] to the gauge-invariant formulation.

V. CONCLUSIONS

In this article we have presented a formulation of the effective potential in terms of a gauge-invariant order parameter for the case of the Abelian Higgs model. The gauge-invariant states of the theory are those annihilated by the first class constraints and gauge-invariant operators are those that commute with these constraints. We recognized an order parameter that is invariant under the local gauge transformations but transforms as a charged operator under global phase rotations; its expectation value in the lowest-energy state therefore signals the breakdown of the global U(1) symmetry.

We evaluated the one-loop contribution to this gauge-invariant effective potential both at zero and nonzero temperature and obtained its high-temperature expansion. We found that the effective potential is complex both at zero and nonzero temperature and that the spinodal (thermodynamically unstable) region extends from the maximum to the minimum of the tree level potential (in this approximation). The gauge-invariant effective potential was compared to the effective potential obtained in several covariant gauges, and we found that the dependence on gauge parameter cancels only at the extrema, but for all other values of the order parameter the gauge-fixed effective potentials are gauge parameter dependent at zero and nonzero temperature. In particular in a high-temperature expansion only the $O(T^2)$ contribution is gauge invariant, whereas the $O(T), O(\ln(T))$, . . . depend on the gauge parameter and are complex. In general these cannot be taken as trustworthy quantities to extract information on the strength of the transition and its features, such as latent heat and supercooling temperature, although the contributions of the gauge boson masses (we did not study fermions) are gauge independent and real to this order.

The imaginary part of the one-loop effective potential determines the spinodal lines, which identify the region of long-wavelength instabilities. Unlike the case of a scalar field with discrete symmetries, the spinodal line encompasses all values of the homogeneous order parameter ranging between the maximum and the minimum of the tree level potential in the one-loop approximation.

Nonequilibrium dynamical aspects were then studied in terms of gauge-invariant correlation functions. It is in these dynamical situations out of equilibrium that the gauge-invariant formulation has the greatest impact, since, in general, correlation functions involve nonlocal line integrals of the gauge field. We obtained the early time behavior of the gauge-invariant correlation functions for initial conditions in which the order parameter is in the spinodal region; these correlation functions grow exponentially at early times as a consequence of the instabilities.

We conjectured that these spinodal instabilities may play an important role in first order phase transitions, in that they may be responsible for nonequilibrium dynamics *inside* the nucleating bubbles. Clearly this possibility will have to be studied further.

There are several important avenues to pursue: higher-order calculations and the implementation of variational or Hartree-like approximations in the gauge-invariant formulation to address the long time behavior and to obtain a more solid understanding of nonequilibrium processes. Furthermore, it is important to generalize the resummation program to the gauge-invariant effective potential, and to generalize these new methods and results to non-Abelian gauge theories, in particular, the electroweak theory. Study of these issues is underway.

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APPENDIX A

As shown in Sec. II, in the one-loop approximation the Hamiltonian becomes a sum of independent harmonic oscillators for the gauge-invariant normal modes and in this case the explicit form of the density matrix in the Schrödinger representation is available (see [20] and references therein). This form of the density matrix is particularly convenient to study real-time dynamics, since the Liouville equation becomes a functional differential equation, which in this case can be solved explicitly for the time dependence [20].

The *equilibrium* density matrix that describes the system at temperature $T=1/\beta$ is given in the Schrödinger representation by

$$\langle\Phi,\Phi^{\dagger},A_T|\hat{\rho}|A_T',\Phi'^{\dagger},\Phi'\rangle = N[T]\exp\left\{\int d^3x\int d^3y[\vec{A}_T(\vec{x})\cdot\vec{A}_T(\vec{y})+\vec{A}_T'(\vec{x})\cdot\vec{A}_T'(\vec{y})]K_T^{(1)}(\vec{x}-\vec{y})\right.$$
$$\left.+\vec{A}_T'(\vec{x})\cdot\vec{A}_T(\vec{y})K_T^{(2)}(\vec{x}-\vec{y})\right\}$$
$$\times\exp\left\{\int d^3x\int d^3y[\,\eta_1(\vec{x})\,\eta_1(\vec{y})+\eta_1'(\vec{x})\,\eta_1'(\vec{y})]K_H^{(1)}(\vec{x}-\vec{y})+\eta_1'(\vec{x})\,\eta_1(\vec{y})K_H^{(2)}(\vec{x}-\vec{y})\right\}$$
$$\times\exp\left\{\int d^3x\int d^3y[\,\eta_2(\vec{x})\,\eta_2(\vec{y})+\eta_2'(\vec{x})\,\eta_2'(\vec{y})]K_p^{(1)}(\vec{x}-\vec{y})+\eta_2'(\vec{x})\,\eta_2(\vec{y})K_p^{(2)}(\vec{x}-\vec{y})\right\}, \tag{A1}$$

$$N[T]=N[0]\Pi_k\left[\tanh^2\left(\frac{\omega_T(k)}{T}\right)\tanh\left(\frac{\omega_H(k)}{T}\right)\tanh\left(\frac{\omega_p(k)}{T}\right)\right]^{1/2}, \tag{A2}$$

$$K_T^{(1)}(\vec{x}-\vec{y})=-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\omega_T(k)\coth\left[\frac{\omega_T(k)}{T}\right]e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{A3}$$

$$K_T^{(2)}(\vec{x}-\vec{y})=\int\frac{d^3k}{(2\pi)^3}\,\frac{\omega_T(k)}{\sinh[\,\omega_T(k)/T]}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{A4}$$

$$K_H^{(1)}(\vec{x}-\vec{y})=-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\omega_H(k)\coth\left[\frac{\omega_H(k)}{T}\right]e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{A5}$$

$$K_H^{(2)}(\vec{x}-\vec{y})=\int\frac{d^3k}{(2\pi)^3}\,\frac{\omega_H(k)}{\sinh[\,\omega_H(k)/T]}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{A6}$$

$$K_p^{(1)}(\vec{x}-\vec{y})=-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\frac{k^2\omega_p(k)}{\omega_T^2(k)}\coth\left[\frac{k^2\omega_p(k)}{T\omega_T^2(k)}\right]e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{A7}$$

$$K_p^{(2)}(\vec{x}-\vec{y})=\int\frac{d^3k}{(2\pi)^3}\,\frac{\omega_p(k)k^2}{\omega_T^2(k)\sinh[k^2\omega_p(k)/T\omega_T^2(k)]}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}. \tag{A8}$$

The trace of this density matrix gives the gauge-invariant one-loop partition function, whose logarithm (divided by $-\beta$) gives the one-loop effective potential at finite temperature.

APPENDIX B

In this appendix we provide the essential ingredients to obtain the high-temperature expansion of the contribution of the plasma mode to the finite temperature effective potential.

The first two terms in the free energy density given by Eq. (3.25) (the transverse and Higgs modes) are of the usual form with dispersion relations of the form $\omega(k)=\sqrt{k^2+m^2}$ and their asymptotic high-temperature expansion can be obtained by following the steps described in [1]. The third term requires special attention because of the unusual dispersion relation of the plasma mode. This term can be written as

$$\int\frac{d^3k}{(2\pi)^3}\ln[1-e^{-\beta\omega_p(k)}]=\frac{T^4}{2\pi^2}I(y_1^2,y_2^2),$$

$$I(y_1^2,y_2^2)=\int_0^{\infty}x^2dx\ln[1-e^{-(1/x)\sqrt{(x^2+y_1^2)(x^2+y_2^2)}}],$$

$$y_1^2=\frac{m_g^2}{T^2},\quad y_1^2=\frac{m_T^2}{T^2}. \tag{B1}$$

The function $I(y_1^2,y_2^2)$ can be written as

$$I(y_1^2,y_2^2)=\int_0^{y_1^2}\int_0^{y_2^2}dadb\,\frac{d}{da}\,\frac{d}{db}I(a,b)$$
$$+I(0,y_2^2)+I(y_1^2,0)-I(0,0). \tag{B2}$$

The last three terms are of standard form and can be calculated following the procedures of Ref. [1]. Using the identity (3.11) of Ref. [1] we find

$$\frac{d}{da}\,\frac{d}{db}I(a,b)=-\frac{1}{2}\,\frac{d}{da}\,\frac{d}{db}\int_0^{\Lambda}xdx\sqrt{(x^2+a)(x^2+b)}+\int_0^{\Lambda}x^2dx$$
$$\times\sum_{l=1}^{\infty}\left[\frac{2\pi lx}{(x^2+a)(x^2+b)+(2\pi lx)}\right]^2, \tag{B3}$$

where we introduced an upper momentum cutoff to regulate the individual terms. The integral in the first term is the same

as the plasma contribution to the zero temperature effective potential in Eq. (3.14) but in terms of the variables a,b. This integral has a logarithmic cutoff dependence. The second term does not have infrared divergences for $a,b=0$ and can be expanded in power series of a,b. In this series, the first term, with $a=b=0$, has a logarithmic cutoff dependence that exactly cancels (upon integration on a,b) that of the first term. The remaining series is both ultraviolet and infrared safe. Finally only the finite terms from the *zero temperature* effective potential but in terms of the variables y_1^2 and y_2^2 contribute to $I(y_1^2,y_2^2)$ and we obtain the result quoted in Eq. (3.27).

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