Critical behavior near the singularity in a scalar field collapse

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The critical behavior of a field near the singularity in the spherical symmetric scalar field collapse is studied. We consider the supercritical case in which a black hole singularity is formed by the strong imploding scalar field at some advanced time v. We find that the field evolution near the singularity can be divided into the following two stages. The spacetime structure near the onset of the first singularity is shown to be well approximated by a self-similar solution. In this self-similar stage the horizon mass linearly increases with v. After the self-similar stage ends, the logarithmic behavior becomes remarkable and the system evolves toward the Schwarzschild spacetime with the advanced time. According to this evolution, the strength of the curvature singularity decreases as $I \simeq \chi^{-2(f+2)/(1+f)}$, where χ is the circumference radius and f runs from 0 to 1 with the advanced time v. In the final stage of gravitational collapse ($v \to \infty$), the scalar field dies away as $\exp(-kv)$ inside the apparent horizon and the system smoothly approaches the static Schwarzschild spacetime. We find that the power-law behavior of the black hole mass is crucially related to the logarithmic behavior of the field. We also propose our main idea that the critical exponent β of the mass power law is a decreasing function of v, which is due to the area law of the apparent horizon. [S0556-2821(96)01814-0]

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I. INTRODUCTION

Recently, critical behavior has been shown to occur near the zero mass black hole formation in gravitational collapses of various massless fields. The known examples are spherical symmetric scalar waves [1–3], axisymmetric gravitational waves [4], and radiation fluids [5,6]. These systems have a field strength parameter $p \simeq M_0/L_0$ where M_0 and L_0 are the mass-energy of initial wave packet and the width of matter fields, respectively. The final states of these collapses are characterized by a critical parameter value p^*: Only strong fields ($p > p^*$) can form a black hole. Near the critical limit $p \to p^*$, these systems exhibit critical phenomena that are characterized by a universal power-law behavior and a scaling relation. The black hole mass m_0 serves as an order parameter with the power-law dependence such as $m_0 \simeq |p - p^*|^\beta$. The critical exponent β is universal in the sense of being independent of the details of the initial data in gravitational collapses. A scaling relation is also observed near the critical point. For example, the scalar field approximately satisfies a discrete scaling relation $Z(\rho - \Delta, \tau - \Delta) \simeq Z(\rho, \tau)$, where Z is the derivative of the scalar field and $\Delta \simeq 3.4$ is a universal constant such as the critical exponent β. ρ and τ are logarithmic scales of the proper radius r and of the proper time of a central observer T: $\rho = \ln r$ and $\tau = \ln(T^* - T)$. The time T^* is a finite accumulation time of the scaling relation. The similar critical behavior of black hole formation has been also found in a complex scalar field collapse [7] and in a radiation fluid plus shell system [8].

These critical phenomena are important issues in general relativity, because one is able to create a black hole of arbitrarily small mass by tuning the parameter p to the critical value p^* [9]. This means that arbitrarily high spacetime curvature can develop from regular Cauchy data. Then, the threshold for black hole formation is an interesting subject in the context of cosmic censorship hypotheses. Furthermore, if an observer could treat a controllable source of gravitational wave, he would be able to produce an arbitrarily small black hole and observe its Hawking radiation.

Though the numerical works have succeeded in pointing out such interesting features, any analytic approach to the critical phenomena will also be useful for understanding its essential dynamical processes. Several arguments have been given to explain this critical behavior, in particular, the value of the critical exponent [6,10,11]. However, dynamical processes in the critical phenomena are still unclear. For example, the numerical works show that the self-similar evolution (the scaling relation) near the regular center and the critical exponent β are ''universal,'' but the relation between both universality is not clarified by recent work. The analytical works [6,10] claim that the latter should be due to the process which induces the breakdown of the self-similarity. In fact, the numerical calculations based on the spacelike [1] and null [2] foliations of the spacetime give the same result such that $\beta \simeq 0.37$, which means the critical exponent for a final black hole mass. We must consider the field evolution toward the final stage near the apparent horizon which will contain some universality, to assure the universal for universality of β. The analysis near the singular center is very helpful for understanding the nonself-similar evolution, because the apparent horizon is very close to the singularity in the critical limit. The field behavior near the singularity is also an important subject in the study of the gravitational collapse. For example, the field behavior at the point A drawn in Fig. 1 is relevant to the problem that the singularity is naked or not [12,13]. Further, the singularity structure can clearly show a dynamical aspect of the critical black holes. In this paper we limit our consideration to the scalar field collapse. Our main purpose is to clarify the critical behavior of the field evolution near the singularity and make an attempt to understand some new universality in the critical process.

The organization of this paper is as follows. In Sec. II we discuss the behavior of the field near the singularity and de-

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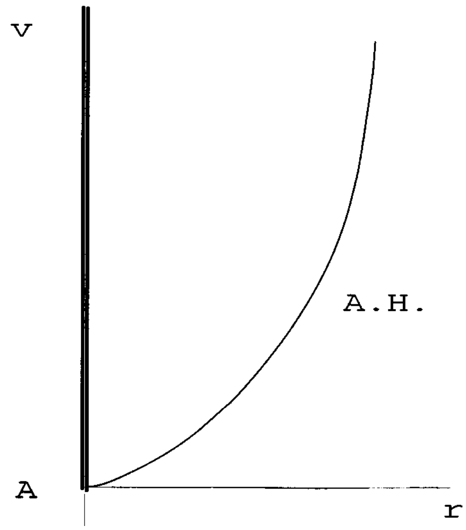

FIG. 1. Schematic process of the spherical gravitational collapse of a scalar field. A singularity at the center of symmetry begins at A (where we set $v=r=0$). The apparent horizon is schematically shown.

rive the evolution equations by reducing the Einstein equations. (Note that the black hole singularity is spacelike. Hence, we consider the evolution measured in an advanced time v.) To analyze the singularity structure, we can introduce the notion of a scalar charge which is related to the strength of the Riemann curvature. The strong field behavior in the critical limit is discussed in Sec. III. The field equations become simple by virtue of the critical limit. The field evolution can be divided into the following two stages. In the first stage, the system is assumed to exhibit the self-similar property and the apparent horizon mass linearly increases with the advanced time v. In the second stage, a logarithmic behavior appears as a result of the breakdown of the self-similarity, which makes the system approach the Schwarzschild spacetime. In Sec. IV, we discuss the final stage of gravitational collapse ($v\to\infty$). The scalar field falls off at the stage, and the system is shown to smoothly approach the Schwarzschild spacetime. Since the local mass and the strength of singularity are closely related to each other, we can find that the strong field behavior is directly related to a power-law behavior of the black hole mass. In Sec. V we propose the idea that the critical exponent is a decreasing function of the advanced time v. This dependence of the critical exponent is due to the increasing area of the apparent horizon. The final section is devoted to a summary of our result. In this paper we use $G=c=1$ unit.

II. APPROXIMATION NEAR THE SINGULARITY

To discuss the strong field evolution near the singularity, we use the single-null time coordinate system

$$ds^2=F(v,r)dv^2+2dv\,dr+\chi(v,r)d\Omega^2, \tag{2.1}$$

where v is an advanced time, r is an affine parameter, and we set r to be zero at $\chi=0$. Since we analyze the inside of the apparent horizon, the component $g_{vv}=F$ is chosen to be positive. A singularity at the center of symmetry begins at the point A in Fig. 1, and we set $v=r=0$ at this point. Since we set $r=0$ at $\chi=0$, we assume that near the singular center (i.e., regions I, II, and III in Fig. 2) the leading term of χ have the form

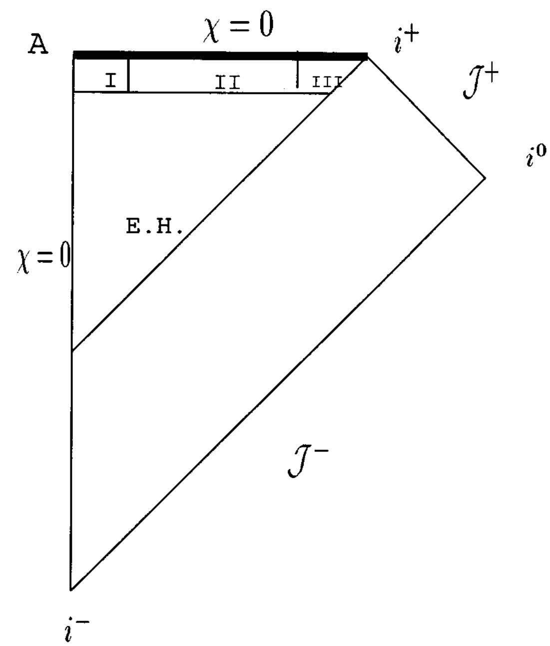

FIG. 2. Penrose diagram for black hole formation due to a wave-packet collapse. A singularity at the center of symmetry begins at A in advanced time v. Region I exhibits the self-similar evolution, while in region II the spacetime structure changes from the self-similar type to the Schwarzschild one. An asymptotically Schwarzschild geometry appears in the region III.

$$\chi\simeq\chi_0(v)r^{1+f}, \tag{2.2}$$

where χ_0 and f are functions of v and the exponent f should be restricted to the range $f>-1$ to require that $\chi(r=0)=0$. We use this approximation only in the $v>0$ region.

From the $G_{\theta\theta}$ component of the Einstein equation, the metric F can be determined by χ as

$$\chi' F=2\dot{\chi}-2r+h(v), \tag{2.3}$$

where the prime and overdot denote derivatives with respect to r and v, respectively, and $h(v)$ is arbitrary function of v. The function $h(v)$ is related to the apparent horizon. The apparent horizon is a surface where the light waves are momentarily ''frozen'' [14], and it is given by the equation

$$g^{\mu\nu}\chi_\mu\chi_\nu=\chi'(2\dot{\chi}-F\chi')=\chi'\{2r-h\}=0. \tag{2.4}$$

Hence, the apparent horizon is located at $r=h(v)/2\equiv r_h$. Therefore h plays an important role in the supercritical case. Since $h\neq0$, from Eqs. (2.2) and (2.3), we find

$$F\simeq\frac{h}{(1+f)\chi_0}r^{-f}. \tag{2.5}$$

The exponent f determines the divergence of the curvature singularity, because the curvature invariant $I=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ behaves as $I\simeq r^{-4-2f}\simeq\chi^{-2(2+f)/(1+f)}$ near the singular center. For example, for the Schwarzschild spacetime, we obtain $f=1$ and $I\simeq\chi^{-3}$, and for the Robert self-similar solution [15], we obtain $f=0$ and $I\simeq\chi^{-4}$.

When we assume Eq. (2.2), we obtain a scalar field ψ as (see appendix)

$$\psi\simeq\frac{1}{2}\sqrt{1-f^2}\ln r+O(1). \tag{2.6}$$

From Eqs. (2.2) and (2.6) we find that f is restricted by $-1<f\leqslant 1$. Note that the scalar field is proportional to $\sqrt{1-f^2}$, so we can interpret the quantity $\sqrt{1-f^2}$ to be a scalar charge. This interpretation gives an important property of f. Since the singularity is formed by a strong imploding scalar field, $f\neq 1$ at the point A in Fig. 1. However, after the first singular point A, the system must relax to the Schwarzschild spacetime and then the scalar field falls off with v. Hence, $f\to 1$ in the limit $v\to\infty$, as a function of v, i.e., f changes during the gravitational collapse in the region II (see Fig. 2). It follows that the divergence of the curvature singularity also changes with v as $I\simeq\chi^{-2(2+f)/(1+f)}$. Thus, the function f plays an important role in the strong filed evolution near the singularity.

One of the interesting questions is whether the value of f at the point A is unique or not in the critical limit. Recent numerical calculations suggest that the initial value of f is also universal in the critical limit. The behavior of the scalar curvature $\mathcal{R}$ is given by $\mathcal{R}\simeq(T^*-T)^{-2}$ at the center, where T is a central proper time and T^* is a finite accumulation time of the scaling relation [3]. Because the accumulation number of echoes becomes infinite in the critical limit $(T\to T^*)$ [16], the scalar curvature $\mathcal{R}$ diverges such as $\mathcal{R}\simeq(T^*-T)^{-2}$ at the first singular point. This proper time dependence of the scalar curvature has the same form with the Roberts solution which is given by

$$ds^2=(p-1)dV^2+2dVdR+R(pV+R)d\Omega^2, \tag{2.7}$$

where p is a constant. From the line element (2.7), we find the scalar curvature to be $\mathcal{R}=\mathcal{R}_0(\xi)/V^2$, where $\xi=R/V$ is self-similar variable and V is proportional to the central proper time T^*-T. Since the center is given by a line of $\xi=$const, the Roberts solution gives $\mathcal{R}\simeq(T-T^*)^{-2}$. This suggests that the spacetime near the point A can be well approximated by the self-similar solution. Another justification of the self-similarity at the first singular point is as follows: We can express a function which exhibits the known discrete scaling relation by the form $Z(\tau,\rho)=X(\tau-\rho)+Y(\tau-\rho)\exp(i2\pi\rho/\triangle)$ with logarithmic scale ρ of r. The second term representing the discrete scaling rapidly oscillates in with the variable ρ near the first singular point A ($|\rho|\to\infty$ at $r\to 0$). Then, the mean value of the second term vanishes. Therefore, the scalar field approximately satisfies a continuous scaling relation at the point A. If we assume the continuous self-similar Roberts solution near the point A in the critical limit, we can set the initial value to be $f=0$.

To analyze the field evolution (in term of the advanced time) near the singularity, let us extend Eqs. (2.2) and (2.5) to the terms of higher orders. When we assume the Eq. (2.2), we obtain higher correction terms as (see appendix)

$$\chi\simeq A_0(2\eta)^2\left(\frac{r}{2\eta}\right)^{1+f}\left\{1-(1-f)\left(\frac{r}{2\eta}\right)-\frac{f}{2}(1-f)\left(\frac{r}{2\eta}\right)^2+\cdots+a_1\left(\frac{r}{2\eta}\right)^{1+f}\ln\kappa r+a_2\ln 2\kappa\eta\left(\frac{r}{2\eta}\right)^{1+f}+\cdots\right\}, \tag{2.8}$$

$$F\simeq A_0^{-1}\left(\frac{r}{2\eta}\right)^{-f}\left\{1-f\left(\frac{r}{2\eta}\right)-\frac{f}{2}(1-f)\left(\frac{r}{2\eta}\right)^2+\cdots+2(1+f)a_1\left(\frac{r}{2\eta}\right)^{1+f}\ln\kappa r+b_2\ln 2\kappa\eta\left(\frac{r}{2\eta}\right)^{1+f}+\cdots\right\}, \tag{2.9}$$

where $\eta\equiv h/2(1+f)$ (which has dimension of length) and $A_0\equiv\chi_0(2\eta)^{f-1}$, a_1, a_2, b_2 (are dimensionless) are functions of v and κ is a constant which is determined by a scale L_0 of the initial data. The final black hole mass m_0 is assumed to be $m_0<(2\kappa)^{-1}$ in the critical limit.

For $f=1$, we obtain $\chi\simeq A_0 r^2$ and $F\simeq 2\eta/A_0 r$. To obtain the Schwarzschild metric at the final stage, A_0 should be equal to unity at $f=1$ and η gives a final black hole mass m_0. Therefore, we can interpret η as a mass function, and the limit $\eta\to 0$ corresponds to the critical limit. Note that the mass scale is given by η only, and A_0 is chosen to be of order of unity $[A_0\sim O((\eta)^0)\sim O(1)]$. Further, f is assumed to change from 0 to 1 as a function of order of unity. These choices of orders are important when we analyze the critical limit $\eta\to 0$.

According to the expansions (2.8) and (2.9), the Einstein equations give the equations (see the derivation given in the appendix)

$$(2\eta)A_0\dot{f}=(1+f)(2+f)a_1, \tag{2.10}$$

$$4\eta\dot{A}_0-4\eta A_0\dot{f}\ln 2\kappa\eta+4A_0(1-f)\dot{\eta}=a_1+(1+f)(2a_2+b_2)\ln 2\kappa\eta, \tag{2.11}$$

$$2\eta A_0\dot{f}\ln 2\kappa\eta+4A_0 f\dot{\eta}=-(4+5f+2f^2)a_1+(1+f)\times(fa_2-b_2)\ln 2\kappa\eta, \tag{2.12}$$

and the scalar filed ψ is given by

$$\psi=\pm\frac{1}{2}\sqrt{1-f^2}\left\{\ln\kappa r+\psi_0(v)+\frac{r}{2\eta}+O(r^{1+f}\ln r)\right\}. \tag{2.13}$$

Here, the overdot denotes the derivative with respect to v. The evolution equation for ψ_0 in Eq. (2.13) has a very complicated form. Because it has no relevance to the leading behavior of the system, we do not write it here. In the next section we will discuss the solution of these equations in the critical limit $\eta\to 0$.

III. DYNAMICAL BEHAVIOR IN THE CRITICAL LIMIT

Since the critical limit is given by the condition $\eta\to 0$ and both A_0 and f are of order of unity, from Eq. (2.10) we find a_1 to be of order of $O(\eta)$. Therefore, we obtain the following simplified equations from Eqs. (2.10)–(2.12) in the critical limit:

$$(2\eta)A_0\dot{f}=(1+f)(2+f)a_1, \tag{3.1}$$

$$2A_0\dot{\eta}=\frac{1}{2}(1+f)(2+f)(a_1+a_2)\ln 2\kappa\eta, \tag{3.2}$$

$$b_2=-(1+f)(2+f)a_1-f(1+f)a_2. \tag{3.3}$$

For sufficiently small f ($f\simeq 0$), the above equations will exhibit the self-similar evolution, which is valid in the neighborhood of the first singular point A (i.e., the region I in Fig. 2). In this region the horizon mass M_h increases in proportion to v. Since f increases with the advanced time v, self-similar approximation breaks down at some time v_0 which depends on the initial data. Then, the system begins to evolve toward the Schwarzschild spacetime with advance time v in the region II drawn in Fig. 2. The strength of the curvature singularity will change in this transition stage. Before discussing the transition stage, we give a brief comment on the self-similar stage.

A. Self-similar stage

As mentioned in the previous section, f should vanish at the point A. To verify the approximate expansions (2.8) and (2.9) near the self-similar region, we compare them with the Roberts metric (2.7). Then, under the condition $f\simeq 0$, we obtain $a_1\simeq b_2\sim O(\eta)$ and the relations

$$2\eta=\sqrt{(p-1)/A_0}pv, \tag{3.4}$$

$$a_2\ln 2\kappa\eta=p, \tag{3.5}$$

where the limit $p\rightarrow 1$ corresponds to the critical limit. We set $v=0$ at the first singular point A. It is easy to confirm that Eqs. (3.4) and (3.5) can also satisfy the evolution equations (3.1)–(3.3). In this first stage of collapse, the apparent horizon has a simple structure. The apparent horizon is given by the equation

$$g^{\mu\nu}\chi_\mu\chi_\nu=\chi'(2\dot{\chi}-F\chi')=2\chi'\{r-(1+f)\eta\}=0. \tag{3.6}$$

This means that the apparent horizon is located at $r=(1+f)\eta\rightarrow\eta$. The local mass defined by

$$M=\frac{1}{2}\sqrt{\chi}(1-g^{\mu\nu}\partial_\mu\sqrt{\chi}\partial_\nu\sqrt{\chi}), \tag{3.7}$$

is estimated to be

$$M_h=\frac{1}{4}p\sqrt{p-1}v, \tag{3.8}$$

at the apparent horizon [10]. Then the apparent horizon mass M_h at this stage should be $M_h\simeq(p-1)^{1/2}$ in the critical limit $p\rightarrow 1$. Since the numerical and analytical calculations have estimated the horizon mass at different regions, the disagreement of the critical exponent is not surprising. The relation between them will be seen in Sec. V.

Since f increases with the advanced time v, self-similar approximation breaks down at some time v_0. The breakdown of self-similarity is also necessary for producing a finite black hole mass, because the local mass at the apparent horizon linearly increases in v in this region. The time scale v_0 will correspond to the initial width L_0 of the wave packet, because first strong imploding waves continue during this time scale.

B. Transition stage

At the final stage of the evolution, the mass parameter η must satisfy the condition $\dot{\eta}\simeq 0$ for a finite black hole mass, because the horizon scale is given by η. Hence, from Eqs. (3.2) and (3.3), we require the relation $a_1+a_2\sim O(\eta)$ at the final stage. This means that after breakdown of the self-similarity, a_2 becomes of order of $O(\eta)$ which is the same order with a_1. Recall that in the region I, a_2 is of order of $1/\ln 2\kappa\eta$ by virtue of Eq. (3.5). In the transition stage (toward the Schwarzschild spacetime), we will find a dynamical evolution different from the self-similar one. The field behavior in the region II (see Fig. 2) is determined by the equation

$$2\dot{\eta}=(1-\gamma)\eta\dot{f}\ln 2\kappa\eta, \tag{3.9}$$

which is derived from Eqs. (3.1) and (3.2) with the unknown function $\gamma(v)$ defined by the ratio of a_1 to a_2 as

$$\gamma=-\frac{a_2}{a_1}. \tag{3.10}$$

We must require that $\gamma\rightarrow\infty$ in the self-similar region where $a_1\simeq\eta$ and $a_2\simeq 1/\ln\eta$. Further, γ should also satisfy the condition $\gamma=1$ at $f=1$, because the spacetime approaches the Schwarzschild one. Let us analyze the evolution of the apparent horizon in this region. Since the apparent horizon is located at $r=(1+f)\eta$, and both a_1 and a_2 are of order of η, the expansion (2.8) of χ and the local mass (3.7) lead to

$$M_h\simeq C(f)\eta+O(\eta^2), \tag{3.11}$$

where $C(f)$ is a function of f and is of order of unity. From Eq. (3.11), we obtain

$$\dot{M}_h\simeq\dot{C}(f)\eta+C(f)\dot{\eta}\simeq C(f)\dot{\eta}. \tag{3.12}$$

Here we used the fact that $\dot{f}$ is of order of unity and $\dot{\eta}$ is of order of $\eta\ln\kappa\eta$, [see Eq. (3.9)]. The relation $\dot{M}_h\simeq\eta\ln\eta\simeq 0$ guarantees to keep the black hole mass finite.

It is important to find the following relation between η and f form (3.9):

$$2\kappa\eta(v)=(2\kappa\eta_i)^{\overline{\beta(v)}}, \tag{3.13}$$

where the critical exponent $\overline{\beta}$ is given by

$$\overline{\beta(v)}=\exp\left[\frac{1}{2}\int_{v_0}^{v}(1-\gamma)\dot{f}dv\right]=\exp\left[\frac{1}{2}\int_0^{f(v)}(1-\gamma)df\right], \tag{3.14}$$

and

$$\eta_i\equiv\eta(v=v_0)=\tfrac{1}{2}\sqrt{p-1/A_0}pv_0. \tag{3.15}$$

If we obtain the dependence of γ on f, $\overline{\beta}$ and η are given as functions of f. We expect these functions $\overline{\beta}(f)$ and $\eta(f)$ to be universal to obtain universal power law of M_h in the critical limit. Although our local analysis cannot give the explicit form of γ, we will roughly estimate Eq. (3.14) in the following.

Since the area of apparent horizon should increase with the advanced time v [17], we find $\dot{\eta}\geqslant 0$ from Eq. (3.12). Furthermore, $\dot{f}>0$ because the spacetime should approach the Schwarzschild spacetime during gravitational collapse. Therefore, from Eq. (3.9) we find $\gamma\geqslant 1$ in the critical limit. Hence, $\overline{\beta}$ is restricted to the range $0<\overline{\beta}\leqslant 1$.

We can now express the field quantities in terms of f. Near the singularity, the invariant curvature I and the local mass are given by

$$I\simeq \frac{1}{64\eta^4}A_0^{2/(1+f)}(1+f)^2(3+2f+7f^2)\left(\frac{\chi}{4\eta^2}\right)^{-2(f+2)/(f+1)}, \tag{3.16}$$

$$M\simeq \frac{1}{4}A_0^{1/(1+f)}\eta(1+f)^2\left(\frac{\chi}{4\eta^2}\right)^{-(1-f)/2(1+f)}. \tag{3.17}$$

Thus, the evolution of the scalar field (2.13), the metric components (2.8) and (2.9), and the local mass (3.17) are mainly determined by the single function f. It seems that the field evolution is universal in the meaning of the dependence on f. Another interesting result of our analysis is the power-law behavior of the black hole mass. The local mass at the apparent horizon is given by

$$M_h\simeq C(f)\,\eta\simeq(\eta_i)^{\overline{\beta}}. \tag{3.18}$$

The form of Eq. (3.18) claims that the power-law behavior of the black hole mass is related to the strong field evolution near the singularity. This relation will be discussed in Sec. V. Before discussing it, we will analyze the decay process in the static limit $f\to 1$.

IV. FINAL STAGE OF GRAVITATIONAL COLLAPSE

In this section we show that the strong fields near the singularity smoothly relax to the perturbation of the Schwarzschild background in the limit $f\to 1$. We start with determining the form of the scalar field at the final stage. From Eq. (2.13), we obtain the derivative with respect to v of the scalar field as

$$\dot{\psi}\simeq \frac{f\dot{f}}{\sqrt{1-f^2}}\ln\kappa r. \tag{4.1}$$

In the final stage of the gravitational collapse, $\dot{\psi}$ must go to zero as f approaches to unity. This means that at $v\to\infty$, $\dot{f}$ behaves as

$$\dot{f}\simeq(1-f)^{\epsilon}, \tag{4.2}$$

where $\epsilon\geqslant 1$. From the Klein-Gordon equation for ψ, we find that only the case $\epsilon=1$ is allowed as a solution near the origin. Thus, the scalar field has the form

$$\psi=e^{-kv}\phi(r), \tag{4.3}$$

at the final stage. Here, k is a damping factor depending on the final black hole and $\phi(r)$ is a function of r. Note that the v dependence (4.3) of the scalar field differs from the power-law tail behavior which was given by Price [18]. The power-law tail behavior is valid outside the horizon while our result gives the behavior inside the horizon. The two results will not be applicable across. Therefore, to study the relation between them, we need a careful discussion near the horizon.

The Klein-Gordon equation $\psi^{;\mu}{}_{;\mu}=0$ in the Schwarzschild background is given by

$$x(1-x)\phi''+(1-2x+2\lambda x^2)\phi'+2\lambda\phi=0, \tag{4.4}$$

where $\lambda\equiv 2\eta k=2m_0k$ is a constant, $x\equiv r/2m_0$, and the prime denotes the derivative with respect to x. Near the origin $(x\sim 0)$, ϕ can be expanded into

$$\phi=\sum_{n=0}(\alpha_n\ln x+\beta_n)x^n, \tag{4.5}$$

where the coefficients α_n, β_n are determined by the Klein-Gordon equation as

$$\alpha_1=0,\quad \alpha_2=-\frac{1}{2}\lambda\alpha_0,\ldots, \tag{4.6}$$

$$\beta_1=\alpha_0,\quad \beta_2=\frac{1}{2}(\alpha_0-\lambda\beta_0),\ldots. \tag{4.7}$$

Equations (4.6) and (4.7) show that the expansion (2.13) can smoothly match the expansion (4.5) in the limit $f\to 1$. To determine the relation between the parameter $\{\alpha_0,\,\beta_0\}$ and $\{a_1,\,b_2\}$, let us estimate the back reaction of the scalar perturbation to the metric. We express the perturbation of χ as

$$\sqrt{\chi}=r+2m_0\delta,$$

where δ denotes a perturbation of χ from the background proper radius r. From the (r,r) component of the Einstein equation $R_{rr}=2(\psi_r)^2$, we obtain

$$\delta=-e^{-2kv}\alpha_0^2\left\{c_2+x\ln x+(1-c_1)x+x^2+\frac{1}{2}x^3+\lambda\left[-\frac{1}{3}x^3\ln x+\left(\frac{1}{9}-\frac{\beta_0}{3\alpha_0}\right)x^3\right]\right\}, \tag{4.8}$$

where c_1 and c_2 are integral constants. On the other hand, using Eq. (2.8), we can express δ in terms of a_1 and a_2 as

$$\delta=\frac{\sqrt{\chi}-r}{2m_0}=-\frac{1}{2}(1-f)\left\{x\ln x+x^2+\frac{1}{2}x^3\right\}+\frac{1}{2}a_1x^3\ln\kappa x+\frac{1}{2}(a_1+a_2)x^3\ln 2\kappa m_0, \tag{4.9}$$

in the limit $f\to 1$. The important point is that the perturbation analysis gives the same result as the expansion near the singularity. From Eqs. (4.9) and (4.8), we find

$$c_1 = 1, \quad c_2 = 0, \tag{4.10}$$

$$a_1 = \frac{2}{3}\lambda e^{-2\kappa v}\alpha_0^2 = \frac{\lambda}{3}(1-f), \tag{4.11}$$

$$(1-\gamma)\ln 2\kappa m_0 = -\frac{1}{3} + \frac{\beta_0}{\alpha_0}. \tag{4.12}$$

[These results are also obtained from Eqs. (2.13), (4.5), (4.6), and (4.7).] Since $\gamma \to 1$ in this stage (see Sec. III), we obtain $\beta_0 = \alpha_0/3$. The above analysis claims that the strong fields near the singularity smoothly decay to the Schwarzschild spacetime at the final stage of gravitational collapse. Note that the analysis in this section is valid not only in the critical limit but also in the noncritical case ($p > p^*$). In fact, we did not use the limit $\eta \to 0$ in this section. The advantage of the critical limit is that we obtain the simple equation (3.9) to give the solution (3.13).

V. POWER-LAW BEHAVIOR OF BLACK HOLE MASS

An interesting application of the strong field behavior near the singularity is to derive a power law of the black hole mass. We now show that the critical exponent β is a decreasing function of v. From Eqs. (3.13), (3.15), and (3.11), we obtain the power-law behavior of the apparent horizon mass as

$$M_h(v) \simeq (p-1)^{\beta(v)}, \tag{5.1}$$

where $\beta(v)$ is given by

$$\beta(v) = \frac{1}{2}\exp\left[\frac{1}{2}\int_{v_0}^{v}(1-\gamma)\dot{f}\,dv\right] = \frac{1}{2}\exp\left[\frac{1}{2}\int_0^{f(v)}(1-\gamma)\,df\right]. \tag{5.2}$$

Note that the detail of $C(f)$ as a function of f has no relevance to the power-law behavior of horizon mass. This power-law behavior is crucially related to the evolution of strong field near the singularity through the function $\gamma(f)$. Especially, the logarithmic behavior of the field is found to be important for producing the power law of black hole mass.

The remarkable property of our results is a (advanced) time dependence of the critical exponent β. As we have seen in Sec. III, we obtain $\gamma \geqslant 1$. Then, Eq. (5.2) shows that the critical exponent β is a decreasing function of v. The decreasing property of the critical exponent is related to the increasing area of the apparent horizon, because the condition $\gamma \geqslant 1$ is derived from the condition $\dot{M}_h \geqslant 0$. The evolution of the critical exponent is completely due to the evolution of the apparent horizon. We can now understand the difference between numerical result $\beta \simeq 0.37$ [1] and analytic result $\beta = 1/2$ [10]. The latter is estimated in the self-similar stage. The later evolution, i.e., evolution in the transition stage, is essential to obtain a smaller value of β.

Before closing this section, let us estimate roughly the critical exponent β. Since a_1, a_2, and b_2 are related by the simple equation (3.3), we expect that a_2 and a_1 are also related by a simple function of f. Although our analysis lacks the clincher to determine the critical exponent, we can assume a simple function γ which satisfies the condition $\gamma \to \infty$ at $f=0$ and $\gamma = 1$ at $f=1$ (see Sec. III). For example, the function $\gamma = -3 + 3/\sqrt{f} - 2/(1+f)$ leads to the result $\beta = 1/e$, which is suggested in Ref. [1]

VI. SUMMARY

Let us summarize our results. In this paper we have analyzed the critical behavior of the strong field near singularity in the critical black hole formation due to the scalar field collapse. The strong field evolution near the singularity is roughly divided into the two stages according to the value of f. For sufficiently small f ($f \simeq 0$), the system exhibits the self-similar evolution. This self-similar region appears in the neighborhood of the first singular point A (region I in Fig. 2). In this region the horizon mass M_h increases in proportion to v. Since f increases with the advanced time v, the self-similar approximation breaks down at finite time v_0 which depends on initial data. After the self-similar stage ends, system evolves toward the Schwarzschild spacetime with the advance time v (in the region II drawn in Fig. 2). In this stage, the field behavior denoted by Eqs. (2.8), (2.9), and (2.6) is mainly determined by a single function f. In this meaning, the filed evolution near the singularity shows a universal feature. According to this evolution, the strength of the curvature singularity decreases as $I \simeq (\chi/\eta)^{-2(f+2)/(1+f)}$, where f runs from 0 to 1 with the advanced time v. This evolution of the singularity may be understood in term of scalar charge $\sqrt{1-f^2}$. During the gravitational collapse, the scalar charge goes to zero ($f \to 1$). Then the curvature strength decreases and the system relaxes to the Schwarzschild spacetime. In the final stage of gravitational collapse ($v \to \infty$), scalar field dies away as $\exp(-kv)$ inside the apparent horizon. The exponential falloff is due to the logarithmic behavior $\ln r$ of the scalar field near the singularity. The power-law behavior of the black hole mass is also related to the strong field evolution. By virtue of the increasing property of the apparent horizon area, the critical exponent β becomes a decreasing function of v. Equation (5.2) suggests that the critical exponent β is crucially related to the evolution of the scalar field in the neighborhood of the origin. Numerical calculations have shown that the β is universal, which will mean that the f dependence of γ is also universal. We expect that some universal evolution of the strong field will continue after the self-similar stage near the regular origin gives the universality of the critical exponent.

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APPENDIX: DERIVATION OF EVOLUTION EQUATIONS

The purpose of this appendix is to derive the leading behavior of the field. Now, Einstein-scalar equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ is equivalent to

$$R_{\mu\nu}=2\psi_\mu\psi_\nu. \tag{A1}$$

For a line element of the form

$$ds^2=Fdv^2+2dvdr+\chi d\Omega^2, \tag{A2}$$

the Einstein equation (A1) is written in the form

$$-\frac{1}{\sqrt{\chi}}(\sqrt{\chi})''=\psi'^2, \tag{A3}$$

$$-\frac{2}{\sqrt{\chi}}(\dot{\sqrt{\chi}})'+\frac{1}{2\chi}(\chi F')'=2\psi'\dot{\psi}, \tag{A4}$$

$$-\frac{2}{\sqrt{\chi}}(\dot{\sqrt{\chi}})^{\cdot}+\frac{F}{2\chi}(\chi F')'-\frac{1}{2\chi}(\dot{\chi}F'-\chi'\dot{F})=2\dot{\psi}^2, \tag{A5}$$

$$1+\frac{1}{2}(\chi'F)'-\dot{\chi}'=0, \tag{A6}$$

where prime and overdot denote derivatives with respect to r and v, respectively. From Eq. (A6), we obtain

$$\chi'F=2\dot{\chi}-2r+h(v). \tag{A7}$$

Since the local mass is given by

$$M=\frac{\sqrt{\chi}}{2}\left\{1-\frac{\chi'}{4\chi}(2\dot{\chi}-F\chi')\right\}=\frac{\sqrt{\chi}}{2}\left\{1-\frac{\chi'}{4\chi}(2r-h)\right\}, \tag{A8}$$

the apparent horizon is located at $r=h(v)/2\equiv r_h$, and then the horizon mass is given by

$$M_h=\frac{1}{2}\sqrt{\chi_h}. \tag{A9}$$

Therefore, h plays an important role in supercritical case. Since $h(v)\neq 0$, from Eqs. (2.2) and (A7), we find $F\simeq r^{-f}$ for singular center.

Let us determine the field behavior near the origin. Since χ behaves such as Eq. (2.2), from Eq. (A3) we find

$$\psi\simeq\frac{1}{2}\sqrt{1-f^2}\ln\kappa r, \tag{A10}$$

where κ is a constant.

Next, let us consider the Klein-Gordon equation which has the form

$$(\chi\psi')^{\cdot}+(\chi\dot{\psi})'-(\chi F\psi')'=0. \tag{A11}$$

Since $(\chi\dot{\psi})'$ and $(\chi\psi')^{\cdot}$ behave as

$$(\chi\psi')^{\cdot}\simeq\frac{A}{2}\sqrt{1-f^2}\dot{f}r^f\ln\kappa r, \tag{A12}$$

$$(\chi\dot{\psi})'\simeq-\frac{A}{2\sqrt{1-f^2}}(1+f)f\dot{f}r^f\ln\kappa r, \tag{A13}$$

for $\dot{f}\neq 0$ the term χF must have logarithmic term as

$$\chi F\simeq O(r^{2+f}\ln\kappa r)+ABr+\cdots. \tag{A14}$$

For this reason, the higher correction terms of χ and F are given by

$$\chi=A(v)r^{1+f}+C(v)r^{2+f}+E(v)r^{2+2f}\ln\kappa r, \tag{A15}$$

$$F=B(v)r^{-f}+D(v)r^{1-f}+G(v)r\ln\kappa r, \tag{A16}$$

and from Eqs. (A5) and (A7), we obtain

$$AD=-f, \tag{A17}$$

$$BC=f-1. \tag{A18}$$

$$AG=2(1+f)BE, \tag{A19}$$

$$\dot{f}=(1+f)(2+f)\frac{BE}{A}. \tag{A20}$$

Therefore, from Eqs. (A17), (A18) and (A19), we obtain

$$\chi=A_0(2\eta)^2\left(\frac{r}{2\eta}\right)^{1+f}\left\{1-(1-f)\frac{r}{2\eta}+a_1\left(\frac{r}{2\eta}\right)^{1+f}\ln r/2\eta\dots\right\}, \tag{A21}$$

$$F=A_0^{-1}\left(\frac{r}{2\eta}\right)^{-f}\left\{1-f\frac{r}{2\eta}+2(1+f)a_1\left(\frac{r}{2\eta}\right)^{1+f}\ln r/2\eta\dots\right\}, \tag{A22}$$

where $2\eta=AB$ (which has dimension of length), $A_0=A(2\eta)^{f-1}$ (which is dimensionless), and $a_1=E/A$. From Eqs. (A7), (A21), and (A22) we find $h=2(1+f)\eta$. Thus, the apparent horizon is located at $r=(1+f)\eta=r_h$. For $f=1$, we obtain $\chi\simeq A_0r^2$ and $F\simeq 2\eta/A_0r$. Hence, at $f=1$, $A_0=1$ and η gives a final black hole mass m_0. Therefore, we can interpret η as a mass function and the limit $\eta\to 0$ corresponds to the critical limit. Note that because the mass scale is given by η, A_0 remains to be of order of unity. This is a reason why we introduce the functions A_0 and η into Eqs. (A21) and (A22).

To obtain the equation for the mass function η, we need the higher correction terms of metric. Since the form of Eq. (A21) suggests that χ can be expanded into $\chi\simeq A_0(2\eta)^2(r/2\eta)^{1+f}\{\Sigma\alpha_nr^n+r^{f+1}\Sigma(\mu_m\ln r+\nu_m)r^m\}$, we obtain Eqs. (2.10)–(2.12).

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