From Euclidean to Lorentzian general relativity: The real way

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We study in this paper a new approach to the problem of relating solutions to the Einstein field equations with Riemannian and Lorentzian signatures. The procedure can be thought of as a ''real Wick rotation.'' We give a modified action for general relativity, depending on two real parameters, that can be used to control the signature of the solutions to the field equations. We show how this procedure works for the Schwarzschild metric and discuss some possible applications of the formalism in the context of signature change, the problem of time, and black hole thermodynamics. [S0556-2821(96)05714-1]

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I. INTRODUCTION

This paper is devoted to the study of the relationship between Riemannian (referred to in the following as ''Euclidean'') and Lorentzian signature solutions to the Einstein field equations (EFE's).

The Schwarzschild metric is a good example for discussing the importance of having the possibility of relating Euclidean and Lorentzian metrics, and illuminates why the usual approach fails. For Lorentzian signatures it is

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{1}$$

In this case there is a simple way to obtain a Euclidean solution to the EFE's: introduce a complex time variable $\tau=it$, rewrite Eq. (1) in the new coordinate system (τ,r,θ,ϕ), and take advantage of the fact that the components of the metric do not have any explicit time dependence. We get then

$$ds_E^2=\left(1-\frac{2M}{r}\right)d\tau^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{2}$$

which has Euclidean signature. The properties of Eq. (2) are useful in the study of the thermodynamics of a Schwarzschild black hole (see for example [1,2]), and in particular to understand the thermal nature of the Hawking radiation.

The strategy of analytic continuation used in the previous example consists, more precisely, in trying to find a four-dimensional complex manifold endowed with a complex metric solution to the EFE's, such that different real sections (real four-dimensional manifolds) exist, some of them with Euclidean and others with Lorentzian signature. The problem with this approach is that it can only be made to work in some very special examples, static space-times, for which it is always possible to find coordinates such that the g_{0i} components of the metric are zero and the rest of them time independent. It is known that this method cannot be used in general and hence an alternative must be found.

Another context in which the use of Euclidean metrics (and even Euclidean metrics that evolve into Lorentzian ones) is useful is the Euclidean path integral approach to quantum cosmology and quantum gravity [3]. Some of the reasons for this are technical. In many cases, the Euclidean action is exponentially damped instead of being oscillatory as its Lorentzian counterpart, thus improving the convergence properties of the path integral. Lorentzian propagators and related objects are obtained as analytic continuations of the Euclidean ones. From the point of view of quantum cosmology some of the most appealing models for the very early universe describe its origin as a quantum tunneling from a Euclidean regime to a Lorentzian one, so the general problem of understanding how to ''connect'' Euclidean and Lorentzian space-times, both at the quantum and the classical level, seems to be an important one.

The main result of this paper is a prescription to find families of metrics parametrized by two real numbers α,β satisfying the following properties: (i) All of them are solutions to the vacuum EFE's (except for a zero measure set of values of α,β); (ii) Some members of the family have Euclidean signature whereas the others have Lorentzian signature.

The layout of the paper is the following. After this introduction we discuss in Sec. II a modified action principle for general relativity that describes both Euclidean and Lorentzian vacuum solutions for general relativity in terms of an auxiliary metric field. We study this by considering the Hamiltonian formulation of the theory. In Sec. III we discuss the interpretation of the constraints derived from the new action. Section IV concentrates on the study of the new field equations, and especially on how to construct solutions to the EFE's from them. We discuss in Sec. V a particular, and physically relevant, example: Schwarzschild space-times; we show that we recover the usual Euclidean and Lorentzian solutions (1) and (2). We end the paper in Sec. VI with some comments, conclusions, and suggestions for future work within this new approach.

II. THE MODIFIED ACTION FOR GENERAL RELATIVITY

The main point of this paper is the discussion of a new action principle for general relativity. In what follows we will use the Hamiltonian formulation of the theory extensively. As is well known the Arnowitt-Deser-Misner (ADM) [4] formulations for Euclidean and Lorentzian general relativity share the same phase space (coordinatized by three-metrics q_{ab} and their canonically conjugate momenta $\tilde{p}^{ab}$) and the same symplectic structure (i.e., Poisson brackets). The difference between both theories reduces ''only'' to the relative sign between the potential and kinetic terms in the Hamiltonian constraint. As it is important to be able to track the minus signs, and for the benefit of the reader, we give some details about the derivation of the ADM Hamiltonian; we will also use some of these results to study the new action. For most of this part we follow [1].

The notation used in the paper is the following. Tangent space indices are denoted as $a,b,c\ldots$ going from 0 to 3. We will not use three-dimensional indices; tensor fields in three dimensions will be naturally described by their projection properties onto three-dimensional manifolds. Let us consider a four-dimensional manifold $\mathcal{M}=\Sigma\times\mathbb{R}$ where the three-dimensional Σ is either compact without a boundary or, if not compact, the falloff of the fields is taken such that the possible surface terms that appear do not give any contribution. We introduce in $\mathcal{M}$ a metric g_{ab} with Euclidean signature $(++++)$. $\mathcal{M}$ can be foliated by three-dimensional surfaces defined by the constant value of a certain scalar function t. We take the foliation in the ''noncompact direction'' corresponding to $\mathbb{R}$. Given this foliation we can write the gradient one-form[1]

$$dt=(\partial_a t)dx^a. \tag{3}$$

With the help of the inverse four-metric g^{ab} we can define a unit, ''future directed,'' normal to the foliation

$$n_a=\frac{\partial_a t}{(g^{bc}\partial_b t\partial_c t)^{1/2}},\quad n^a\equiv g^{ab}n_b,\quad n_a n^a=1. \tag{4}$$

The three-metric q_{ab} (first fundamental form) induced in the sheets of the foliation by the four-dimensional g_{ab} is

$$q_{ab}\equiv g_{ab}-n_a n_b,\quad q^{ab}\equiv g^{ab}-n^a n^b,$$
$$q_a{}^b\equiv\delta_a{}^b-n_a n^b=q_{ac}q^{cb}=q_{ac}g^{bc}, \tag{5}$$

where we have made use of the fact that $q_{ab}n^b=0$. Notice that $q_a{}^b$ is a projector onto the three-dimensional sheets of the foliation. We give now a mapping that relates points in different sheets by using a congruence of curves filling the manifold $\mathcal{M}$ and never tangent to them. Two points in different sheets are ''connected'' if they are on the same curve of the congruence. In a rather loose sense the foliation allows us to say if two points of $\mathcal{M}$ are ''simultaneous'' and the congruence if they are ''at the same space point'' (see [5]). We parametrize the curves in the congruence with the help of t. If one such curve is given by the functions $x^a(t)$ we have

$$dt=(\partial_a t)dx^a\Rightarrow 1=\partial_a t\frac{dx^a}{dt}\equiv t^a\partial_a t, \tag{6}$$

where t^a is the tangent vector to the congruence. We define ''partial time derivatives'' as Lie derivatives along the direction given by t^a. Notice that this is consistent because Eq. (6) implies

$$\dot{t}\equiv\mathcal{L}_{\vec{t}}t=t^a\partial_a t=1. \tag{7}$$

Another way to see this is by building a coordinate system in $\mathcal{M}$ using t as the x^0 coordinate, giving coordinates (x^1,x^2,x^3) to one sheet of the foliation and Lie-dragging them to the rest of $\mathcal{M}$ by using the congruence of curves. It is obvious that by proceeding in this manner, ∂_t is the same as $\mathcal{L}_{\vec{t}}$.

At each point P of $\mathcal{M}$ the vector field t^a can be uniquely written as a vector tangent to the foliation passing through P and a vector normal to the foliation at P:

$$t^a=Nn^a+N^a. \tag{8}$$

N and N^a are known as the lapse and the shift, respectively, and satisfy the properties

$$N^a n_a=0,\quad N=t^a n_a,\quad N=\frac{1}{n^a\partial_a t}=\frac{1}{(g^{ab}\partial_a t\partial_b t)^{1/2}}. \tag{9}$$

Without loss of generality we can take $N>0$. The extrinsic curvature of the foliation K_{ab} (second fundamental form) is defined by

$$K_{ab}=q_a{}^c q_b{}^d\nabla_c n_d=q_a{}^c\nabla_c n_b \tag{10}$$

and is a symmetric tensor that ''lives'' on the sheets of the foliation, i.e., $K_{ab}n^a=0$, $q_a{}^b K_{bc}=K_{ac}$. In the previous expressions ∇_a denotes the (covariant) derivative operator compatible with the four-metric g_{ab}. It is possible to define a three-dimensional covariant derivative on objects tangent to the foliation $S_{a_1\cdots}{}^{b_1\cdots}$ as

$$\mathcal{D}_a S_{a_1\cdots}{}^{b_1\cdots}=q_a{}^f q_{a_1}{}^{f_1}\ldots q_{g_1}{}^{b_1}\nabla_f S_{f_1\cdots}{}^{g_1\cdots}. \tag{11}$$

It is straightforward to show that this is the unique operator compatible with q_{ab}. With the help of $\mathcal{D}_a$ we can build the intrinsic curvature of the sheets of the foliation and obtain the Gauss-Codazzi equations

$${}^3R_{abc}{}^d=q_a{}^e q_b{}^f q_c{}^g q_h{}^{d\,4}R_{efg}{}^h+K_{ac}K_b{}^d-K_{bc}K_a{}^d, \tag{12}$$

$$\mathcal{D}_a K_b{}^a-\mathcal{D}_b K={}^4R_{fe}n^f q_b{}^e. \tag{13}$$

Contracting (12) with $q^{ac}q^{bd}$ we obtain

$$2G_{ab}n^a n^b=-{}^3R+K^2-K_{ab}K^{ab}, \tag{14}$$

where $G_{ab}\equiv R_{ab}-1/2g_{ab}R$ is the four-dimensional Einstein tensor and $K\equiv q^{ab}K_{ab}$. Before we introduce the new action we give an additional identity that we will use in order to obtain the momenta canonically conjugate to q_{ab} in the Hamiltonian formulation

$$\dot{q}_{ab}\equiv\mathcal{L}_{\vec{t}}q_{ab}=2NK_{ab}+2\mathcal{D}_{(a}N_{b)}. \tag{15}$$

Let us consider now the following action defined in $\mathcal{M}$ as

[1] ∂_a is any torsion-free derivative in $\mathcal{M}$.

$$S[g_{ab};\alpha,\beta]=\int_{\mathcal{M}}d^4x\sqrt{g}[\alpha G^{ab}+\beta R^{ab}]\eta_a\eta_b, \quad (16)$$

where α and β are constant real parameters and

$$\eta_a=\frac{\partial_a\Phi}{(g^{bc}\partial_b\Phi\partial_c\Phi)^{1/2}}. \quad (17)$$

Φ is a *fixed* scalar function monotonically growing in the $\mathbb{R}$ direction that can be interpreted as an external time variable.[2] The previous action is noncovariant in the sense that there is a fixed field related to the structure of space-time that appears in the action and, hence, in the field equations (see, for example, the discussion in [1]). Before we continue, and with the object of making the reader less suspicious with respect to Eq. (16), we make here the following comment (a longer discussion will appear at the end of the paper). In the following we are going to perform a particular Legendre transform on Eq. (16). In order to do that we first write Eq. (16) as the time integral (or rather the integral in Φ) of some Lagrangian whose variables are defined with the help of a particular foliation of $\mathcal{M}$: that given by the level surfaces of the scalar function Φ. After that we perform a Legendre transform to obtain the Hamiltonian and derive the constraints. The dynamics defined by them can be shown to be that of general relativity (for most values of α and β). We will see that we can choose the space-time signature simply by changing the value of these parameters. Of course foliations other than the one given by Φ can be used. In that case the Legendre transform is more complicated to perform but the dynamics is the same as the one obtained by using the particular foliation given by Φ because they are derived by performing Legendre transforms on the same action functional.

By using Eq. (14) we get

$$\alpha\int_{\mathcal{M}}d^4x\sqrt{g}G^{ab}\eta_a\eta_b=\frac{\alpha}{2}\int_{\mathcal{M}}d^4x\sqrt{g}[-{}^3R+K^2-K_{ab}K^{ab}], \quad (18)$$

where all the three-dimensional quantities refer to the foliation defined by Φ. The second term in Eq. (16) can be written as

$$\beta\int_{\mathcal{M}}d^4x\sqrt{g}R^{ab}\eta_a\eta_b=\beta\int_{\mathcal{M}}d^4x\sqrt{g}\{\nabla_a[\eta^b\nabla_b\eta^a] -\nabla_b[\eta^b\nabla_a\eta^a]+(\nabla_b\eta^b)(\nabla_a\eta^a) -(\nabla_a\eta^b)(\nabla_b\eta^a)\}. \quad (19)$$

The first two terms are surface terms. If Σ is compact they are not present. If Σ is noncompact they are not zero but do not give any contribution when varying the action because the variations of the fields are zero at the boundaries.[3] With these remarks in mind we see that

$$\beta\int_{\mathcal{M}}d^4x\sqrt{g}R^{ab}\eta_a\eta_b=\beta\int_{\mathcal{M}}d^4x\sqrt{g}[K^2-K_{ab}K^{ab}]. \quad (20)$$

Combining Eqs. (18) and (20) we finally get

$$S[g_{ab};\alpha,\beta]=\int_{\mathcal{M}}d^4x\sqrt{g}\left[-\frac{\alpha}{2}{}^3R+\left(\frac{\alpha}{2}+\beta\right)\times(K^2-K_{ab}K^{ab})\right]. \quad (21)$$

The measure in the previous integral can be written as

$$d^4x\sqrt{g}=d\Phi d^3xN\sqrt{q}. \quad (22)$$

Notice that $d^3x\sqrt{q}$ is the measure in the three-dimensional slices of the foliation defined by the induced metric q_{ab}. We have then

$$S[g_{ab};\alpha,\beta]=\int_{\mathbb{R}}d\Phi\int_{\Sigma_\Phi}d^3x\sqrt{q}N\left[-\frac{\alpha}{2}{}^3R+\left(\frac{\alpha}{2}+\beta\right)\times(K^2-K_{ab}K^{ab})\right]. \quad (23)$$

The Lagrangian L can be read off directly from the previous formula. In order to obtain the corresponding Hamiltonian we perform a Legendre transform in the usual way. We first define the momenta canonically conjugate to the configuration variable q_{ab}:

$$\tilde{p}^{ab}(x)=\frac{\delta L}{\delta\dot{q}_{ab}(x)}=\int_{\Sigma_\Phi}d^3y\frac{\delta L}{\delta K_{cd}(y)}\frac{\delta K_{cd}(y)}{\delta\dot{q}_{ab}(x)} =\sqrt{q}\left(\frac{\alpha}{2}+\beta\right)(Kq_{ab}-K_{ab}), \quad (24)$$

which implies

$$K_{ab}=\frac{2}{\alpha+2\beta}\frac{1}{\sqrt{q}}\left[\frac{1}{2}q_{ab}\tilde{p}-\tilde{p}_{ab}\right],\quad \alpha+2\beta\neq 0,\quad \tilde{p}\equiv q^{ab}\tilde{p}_{ab}. \quad (25)$$

When $\alpha+2\beta=0$ we find the primary constraint $\tilde{p}^{ab}=0$ and the theory is very different from the ones with $\alpha+2\beta\neq 0$. We will not discuss this here although understanding its meaning may be relevant for the study of signature change. The Hamiltonian is now

[2]The possibility of taking Φ as a dynamical field will be explored below.

[3]Surface terms are important for the consistency of the Hamiltonian formulation; the usual Einstein-Hilbert action must be modified with the addition of surface integrals at the boundaries. None of these will change the arguments presented here.

$$H=\int_{\Sigma_\Phi} d^3x\left\{N\left[\frac{\alpha}{2}\sqrt{q}\,{}^3R+\frac{2}{\alpha+2\beta}\frac{1}{\sqrt{q}}\left(\frac{1}{2}\tilde{p}^2-\tilde{p}^{ab}\tilde{p}_{ab}\right)\right]\right.$$
$$\left.-2N_a\mathcal{D}_b\tilde{p}^{ab}\right\}, \tag{26}$$

and the constraints are

$$\frac{\alpha}{2}\sqrt{q}\,{}^3R+\frac{2}{\alpha+2\beta}\frac{1}{\sqrt{q}}\left(\frac{1}{2}\tilde{p}^2-\tilde{p}^{ab}\tilde{p}_{ab}\right)=0, \tag{27}$$

$$\mathcal{D}_b\tilde{p}^{ab}=0. \tag{28}$$

III. DISCUSSION AND INTERPRETATION OF THE NEW CONSTRAINTS

As is well known the ADM constraints for general relativity in the Euclidean and Lorentzian formulations differ only in the relative signs between the kinetic and potential terms, that is,

$$\text{Euclidean}\quad -\sqrt{q}\,{}^3R+\frac{1}{\sqrt{q}}\left(\frac{1}{2}\tilde{p}^2-\tilde{p}_{ab}\tilde{p}^{ab}\right)=0, \tag{29}$$

$$\text{Lorentzian}\quad +\sqrt{q}\,{}^3R+\frac{1}{\sqrt{q}}\left(\frac{1}{2}\tilde{p}^2-\tilde{p}_{ab}\tilde{p}^{ab}\right)=0. \tag{30}$$

We can see that we can get either Eq. (29) or Eq. (30) from Eq. (27) by a suitable choice of the parameters α and β. If we take $\alpha=-2$, $\beta=+2$ we get Eq. (29) whereas $\alpha=+2$, $\beta=0$ gives Eq. (30). We arrive at the conclusion then, that the action (16) describes both Euclidean and Lorentzian general relativity depending on the values chosen for the parameters. A question arises now: What is the meaning of the action for values of α and β other than the ones considered above? In particular we would like to know if the constraints (27) and (28) are first class for arbitrary values of the parameters. To answer these questions let us consider the following canonical transformation with $k\in\mathbb{R}$ and $k>0$:

$$q_{ab}\rightarrow kq_{ab},$$
$$\tilde{p}^{ab}\rightarrow k^{-1}\tilde{p}^{ab}. \tag{31}$$

We impose $k>0$ in order to keep the signature of the three-metric q_{ab} positive. Under this transformation we have

$${}^3R\rightarrow k^{-1}\,{}^3R,$$
$$\sqrt{q}\rightarrow k^{3/2}\sqrt{q}, \tag{32}$$

so Eqs. (27) and (28) will become

$$\frac{\alpha}{2}k^{1/2}\sqrt{q}\,{}^3R+\frac{2}{(\alpha+2\beta)k^{3/2}}\frac{1}{\sqrt{q}}\left(\frac{1}{2}\tilde{p}^2-\tilde{p}^{ab}\tilde{p}_{ab}\right)=0, \tag{33}$$

$$\frac{1}{k}\mathcal{D}_b\tilde{p}^{ab}=0. \tag{34}$$

As we can see, we cannot change the relative sign between the kinetic and potential terms in the Hamiltonian constraint, but we can choose k in such a way that we get a Hamiltonian constraint proportional either to Eq. (29) or Eq. (30) (see Fig. 1).

This proves that Eq. (27) and (28) are first class constraints for arbitrary values of the parameters with the exception of $\alpha+2\beta=0$ (for which they are not defined). Notice that the constraints are also first class for $\alpha=0$ although the theory is neither Euclidean nor Lorentzian general relativity. It is important to point out, also, that the canonical transformation introduced cannot be thought of as a coordinate transformation: first of all because the coordinates are really kept fixed and second, and more importantly, because there is no coordinate transformation that can simultaneously change the components of the covariant tensor q_{ab} and the contravariant tensor density $\tilde{p}^{ab}$ as in Eq. (31).

The evolution of initial data satisfying the constraints (27) and (28) is given by the Hamiltonian (26). The evolution equations for q_{ab} and $\tilde{p}^{ab}$ are

$$\dot{q}_{ab}=\frac{2N}{\alpha+2\beta}\frac{1}{\sqrt{q}}(\tilde{p}q_{ab}-2\tilde{p}_{ab})+2q_{c(a}\mathcal{D}_{b)}N^c,$$

$$\dot{\tilde{p}}^{ab}=\frac{\alpha}{2}\sqrt{q}N\left[R^{ab}-\frac{1}{2}q^{ab}\right]-\frac{N}{\alpha+2\beta}\frac{1}{\sqrt{q}}q^{ab}\left[\tilde{p}^{cd}\tilde{p}_{cd}-\frac{1}{2}\tilde{p}^2\right]$$
$$+\frac{2N}{\alpha+2\beta}\frac{1}{\sqrt{q}}[2\tilde{p}_c{}^a\tilde{p}^{cb}-\tilde{p}\tilde{p}^{ab}]$$
$$+\frac{\alpha}{2}\sqrt{q}[q^{ab}\Box N-\mathcal{D}^a\mathcal{D}^bN]+\mathcal{L}_{\vec{N}}\tilde{p}^{ab}. \tag{35}$$

In the spirit of the canonical transformation (31) we introduce now

$$h_{ab}\equiv\kappa q_{ab},$$
$$\tilde{\pi}^{ab}\equiv\kappa^{-1}\tilde{p}^{ab}. \tag{36}$$

The evolution equations for h_{ab} and $\tilde{\pi}^{ab}$ are then

$$\dot{h}_{ab}=\frac{2N\kappa^{3/2}}{\alpha+2\beta}\frac{1}{\sqrt{h}}(\tilde{\pi}h_{ab}-2\tilde{\pi}_{ab})+2h_{c(a}\mathcal{D}_{b)}N^c,$$

$$\dot{\tilde{\pi}}^{ab}=\frac{\alpha}{2}\sqrt{h}\kappa^{-1/2}N\left[R^{ab}(h)-\frac{1}{2}R(h)h^{ab}\right]$$
$$-\frac{N\kappa^{3/2}}{\alpha+2\beta}\frac{1}{\sqrt{h}}h^{ab}\left[\tilde{\pi}^{cd}\tilde{\pi}_{cd}-\frac{1}{2}\tilde{\pi}^2\right]$$
$$+\frac{2N\kappa^{3/2}}{\alpha+2\beta}\frac{1}{\sqrt{h}}[2\tilde{\pi}_c{}^a\tilde{\pi}^{cb}-\tilde{\pi}\tilde{\pi}^{ab}]$$
$$+\frac{\alpha}{2}\kappa^{-1/2}\sqrt{h}[h^{ab}\Box_hN-\mathcal{D}_h{}^a\mathcal{D}_h{}^bN]+\mathcal{L}_{\vec{N}}\tilde{\pi}^{ab}, \tag{37}$$

where both R_{ab} and the covariant derivatives are built with h_{ab}. From every solution to Eq. (35) we can obtain a solution to the EFE's for both Euclidean and Lorentzian gravity in the following way. Choose κ and define new lapse and shift ($\mathcal{N}$, and $\mathcal{N}^a$ respectively) such that Eqs. (37) have exactly the form of the equations derived from the Hamiltonian constraints (29) or (30) but written in terms of h_{ab}, $\tilde{\pi}^{ab}$, $\mathcal{N}$, and $\mathcal{N}^a$. In particular, let us take Eq. (36) with $\kappa=\frac{1}{2}\sqrt{|\alpha(\alpha+2\beta)|}$ and define

$$\mathcal{N}^a=N^a, \quad \mathcal{N}=\frac{2N\kappa^{3/2}}{|\alpha+2\beta|}. \tag{38}$$

Note that, in general, the q_{ab} from which h_{ab} is built depends on α and β in a nontrivial way. From these objects we can build the four-dimensional metric g_{ab}^{Eins} solution to the EFE's for both Euclidean and Lorentzian signatures if we know q_{ab}, N, and N^a [that we can get, for example, by solving the field equations derived from Eq. (16) after a certain foliation has been introduced]. Although it is possible to give a concrete prescription to build g_{ab}^{Eins} in terms of $\mathcal{N}$, $\mathcal{N}^a$, h_{ab} and the foliation given by Φ it is better to do this directly in four dimensions. This is the scope of the next section.

IV. THE FIELD EQUATIONS

In this section we return to the four-dimensional form of the action (16) and derive the field equations for a fixed Φ. The solutions to them are always four-metrics with Euclidean signature. These metrics are taken as auxiliary fields from which we can build solutions to the EFE's. We show how this is done in this section and discuss some specific solutions in the next. In order to show all the dependence on g_{ab} of the action (16), and in order to simplify the field equations it is convenient to rewrite it as

$$S[g_{ab};\alpha,\beta]=\int_{\mathcal{M}} d^4x\sqrt{g}\left[(\alpha+\beta)R^{ab}\frac{\partial_a\Phi\partial_b\Phi}{g^{cd}\partial_c\Phi\partial_d\Phi}-\frac{1}{2}\alpha R\right]. \tag{39}$$

Notice that this action reduces to the usual Euclidean Einstein-Hilbert action if $\alpha+\beta=0$, so the field equations must reduce exactly to the Euclidean EFE's in this case. Varying Eq. (39) with respect to g_{ab} we get

$$\frac{\alpha}{2}G^{ab}+(\alpha+\beta)\left\{\left[\frac{1}{2}g^{ab}+\eta^a\eta^b\right]R_{cd}\eta^c\eta^d+\nabla_c\nabla^{(a}(\eta^{b)}\eta^c)\right.$$
$$\left.-\frac{1}{2}g^{ab}\nabla_c\nabla_d(\eta^c\eta^d)-2\eta_d\eta^{(a}R^{b)d}-\frac{1}{2}\Box(\eta^a\eta^b)\right\}=0, \tag{40}$$

where η_a is given by Eq. (17). According to the discussion in the previous sections, for any given η_a the solutions to these equations give rise to solutions to the EFE's either for Euclidean or Lorentzian general relativity. For $\alpha\neq 0$ and $\alpha+2\beta\neq 0$ all the solutions to Eq. (40) can be interpreted in this way and no new solutions are introduced in the theory. Notice that, with the exception of $\alpha+\beta=0$, for which Eq. (40) actually reduces to the Euclidean EFE's, the solutions to the previous equations (even if the values of α and β correspond to Euclidean gravity) are not solutions to the EFE's themselves; we need a prescription to build them from the solutions to Eq. (40) and the scalar field Φ. From the argument in Sec. III, and especially from Eqs. (36) and (38), we find out that g_{ab}^{Eins} can be written in terms of g_{ab} and Φ as

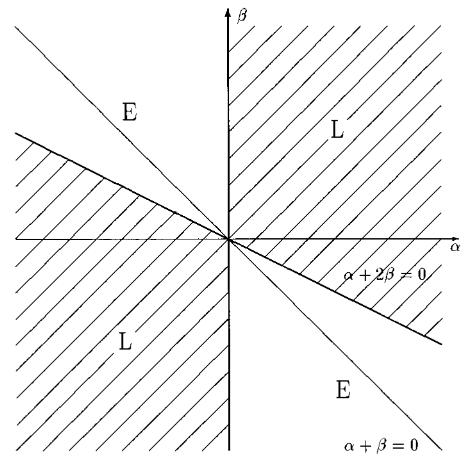

FIG. 1. Regions in the (α,β) plane corresponding to Euclidean and Lorentzian signatures (E and L, respectively). The metrics in the line $\alpha+\beta=0$ are actual solutions to the Euclidean EFE's. If $\alpha+2\beta=0$ the theory is singular and if $\alpha=0$ it is not general relativity.

$$g_{ab}^{\mathrm{Eins}}=\frac{1}{2}\sqrt{|\alpha(\alpha+2\beta)|}\left(g_{ab}-2\frac{\alpha+\beta}{\alpha+2\beta}\eta_a\eta_b\right). \tag{41}$$

In order to show that this is the case we will compute $\mathcal{N}$ and h_{ab}, the lapse and the three-metric defined by g_{ab}^{Eins} and the foliation, and write them in terms of N and q_{ab} (defined by g_{ab} and Φ). To this end we need

$$g^{ab}_{\mathrm{Eins}}=\frac{2}{\sqrt{|\alpha(\alpha+2\beta)|}}\left(g^{ab}-2\frac{\alpha+\beta}{\alpha}\eta^a\eta^b\right), \tag{42}$$

$$\eta_a^{\mathrm{Eins}}=\frac{|\alpha|}{2^{1/2}|\alpha(\alpha+2\beta)|^{1/4}}\eta_a, \tag{43}$$

where $g^{ab}_{\mathrm{Eins}}\eta_a^{\mathrm{Eins}}\eta_b^{\mathrm{Eins}}=-\mathrm{sgn}[\alpha(\alpha+2\beta)]\equiv\zeta$.[4] Taking into account that

$$\mathcal{N}=\frac{1}{|g^{ab}_{\mathrm{Eins}}\partial_a\Phi\partial_b\Phi|^{1/2}}, \tag{44}$$

we obtain

$$\frac{\mathcal{N}}{N}=\frac{(g^{ab}\partial_a\Phi\partial_b\Phi)^{1/2}}{|g^{ab}_{\mathrm{Eins}}\partial_a\Phi\partial_b\Phi|^{1/2}}=\frac{1}{|g^{ab}_{\mathrm{Eins}}\eta_a\eta_b|^{1/2}}=\frac{|\alpha|}{2^{1/2}|\alpha(\alpha+2\beta)|^{1/4}} \tag{45}$$

which coincides with the result given by Eq. (38). In the same fashion we find

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[4] $\zeta=+1$ and $\zeta=-1$ for Lorentzian and Euclidean signatures, respectively.

$$h_{ab}=g^{ab}_{\rm Eins}+\zeta\,\eta_a^{\rm Eins}\eta_b^{\rm Eins}=\frac{1}{2}\sqrt{|\alpha(\alpha+2\beta)|}(g_{ab}-\eta_a\eta_b)$$
$$=\frac{1}{2}\sqrt{|\alpha(\alpha+2\beta)|}q_{ab}. \qquad (46)$$

Notice that $g_{ab}^{\rm Eins}$ will depend, in general, on α and β, both through the explicitly parameter-dependent factors and the (α,β) dependence of g_{ab} and η_a.

In view of Eq. (41) there is an alternative way to understand what we are doing. Let us *define*

$$\hat{g}_{ab}\equiv\frac{1}{2}\sqrt{|\alpha(\alpha+2\beta)|}\left(g_{ab}-2\frac{\alpha+\beta}{\alpha+2\beta}\eta_a\eta_b\right) \qquad (47)$$

and compute

$$\hat{S}[g_{ab};\alpha,\beta]=-\mathrm{sgn}[\alpha]\int_{\mathcal{M}}d^4x\sqrt{|\hat{g}|}R[\hat{g}_{ab}], \qquad (48)$$

where $\hat{g}\equiv\det\hat{g}_{ab}$. Throwing away surface terms we get, precisely, Eq. (39) if $\alpha(\alpha+2\beta)\neq 0$, so we conclude that our action (16) reduces to the Einstein-Hilbert action for metrics with either signature and Eq. (40) are the Einstein field equations. Notice, however, that if $\alpha(\alpha+2\beta)=0$ then Eq. (42) is not defined because $\hat{g}=0$. In this case the action (39) and the field equations (40) provide a generalization of general relativity for degenerate metrics. The degrees of freedom of this degenerate theory are contained in the auxiliary Euclidean metric g_{ab} even if $\alpha=0$ although in this case we do not have the possibility of interpreting them as describing a space-time metric [it would be zero, according to Eq. (47)].

Once we know that our action is the Einstein-Hilbert action we can couple matter very easily, just by adding the usual matter terms written with the help of $\hat{g}_{ab}$. We will not discuss this issue further here.

In the canonical framework used in Secs. II and III we derived the field equations for a fixed Φ and argued that the noncovariance of the action should play no role as we are free to choose any foliation when performing the Legendre transform from the Lagrangian to the Hamiltonian formulation. Now it is clear why this is so: we are just using the Einstein-Hilbert action written in the form given by Eq. (48).

Let us consider now the case in which Φ is taken as a dynamical field (and thus we have a covariant action). As we showed above, the action (16) is really the Einstein-Hilbert action for the metric $\hat{g}_{ab}$ defined by Eq. (47). If the scalar field is dynamical, i.e., not fixed, Eq. (16) has a new gauge symmetry: the invariance under local transformations of Φ that do not change the combination of g_{ab} and Φ defined by Eq. (47) (modulo four-dimensional diffeomorphisms). The choice of a particular Φ should be considered as a gauge fixing of this new symmetry; given two solutions to the field equations (40) $(g_{ab}^{(1)},\Phi^{(1)})$ and $(g_{ab}^{(2)},\Phi^{(2)})$ it is always possible, if $\alpha\neq 0$, to use the new gauge freedom in order to have $\Phi^{(1)}=\Phi^{(2)}$, i.e., refer them to the same scalar field. In this sense the particular choice of Φ is irrelevant. It is also obvious that the additional field equation obtained by varying the action with respect to Φ is redundant, i.e., identically satisfied for any g_{ab} and Φ that solve Eq. (40) because by varying Φ we only generate a particular variation in $\hat{g}_{ab}$.

V. SOME SOLUTIONS TO THE NEW FIELD EQUATIONS

Equation (40) looks, arguably, more complicated than the usual Einstein field equations; nevertheless some simple solutions to it can be found, and Eq. (41) can be used to obtain some familiar space-time metrics. We will concentrate on obtaining a solution to Eq. (40) that describes both Euclidean and Lorentzian Schwarzschild black holes. Let us consider

$$g_{ab}=\mathrm{Diag}\left[1-\frac{2M}{r},\left(1-\frac{2M}{r}\right)^{-1},r^2,r^2\sin^2\theta\right], \qquad (49)$$

$$\eta_a=\left(1-\frac{2M}{r}\right)^{1/2}[1,0,0,0]. \qquad (50)$$

Notice that the previous η_a is obtained from $\Phi=x^0$ and has unit length. As it is well known the Ricci tensor computed from the previous g_{ab} is zero so it is enough to show that g_{ab} and η_a satisfy

$$\nabla_c\nabla^{(a}(\eta^{b)}\eta^c)-\frac{1}{2}g^{ab}\nabla_c\nabla_d(\eta^c\eta^d)-\frac{1}{2}\Box(\eta^a\eta^b)=0. \qquad (51)$$

A straightforward computation shows that this is indeed the case. Plugging Eqs. (49) and (50) in Eq. (41) we get the following solution to the EFE's:

$$g_{ab}^{\rm Eins}=\frac{1}{2}\sqrt{|\alpha(\alpha+2\beta)|}\,\mathrm{Diag}\left[-\frac{\zeta|\alpha|}{|\alpha+2\beta|}\times\left(1-\frac{2M}{r}\right),\left(1-\frac{2M}{r}\right)^{-1},\ r^2,r^2\sin^2\theta\right]. \qquad (52)$$

By suitably rescaling the x^0 and r coordinates we can write Eq. (52) as the Schwarzschild solution with mass $\bar{M}=2^{-1/2}|\alpha(\alpha+2\beta)|^{1/4}M$ and either Euclidean or Lorentzian signature. Hence we see how the Schwarzschild solution is recovered in our formalism. Some features of this solution are worthy of discussion at this point. First of all we notice that, in this case, g_{ab} is independent of α and β. This is not expected to be a feature of general solutions to Eq. (40) but only of those already satisfying $R_{ab}=0$. Second, it is worthwhile pointing out that Eq. (52) is not well defined for $\alpha+2\beta=0$ and is zero for $\alpha=0$, but Eq. (49) is always well defined, so there are solutions to the field equations in the $\alpha=0$ case that do not admit a space-time interpretation. Their degrees of freedom are contained in the auxiliary object g_{ab}. Third, the solution to Eq. (40) describes a family of Schwarzschild black holes with a mass that depends on α and β, and both signatures.

VI. COMMENTS AND CONCLUSIONS

The main result presented in this paper is the introduction of a modified action principle for general relativity that has the merit of describing space-times of Lorentzian or Euclidean signature by introducing some free parameters in the action. The solutions to the field equations can be interpreted as families of solutions to the EFE's whose signature can be chosen at will; some members of these families are Lorentzian and some others are Euclidean. The problem of finding a Euclidean solution associated to a certain Lorentzian one can be solved in this framework by taking members of a certain two parametric family of solutions to the new field equations and choosing two of them with different signatures. At least in some simple cases, like Schwarzschild, one gets the same result as applying the usual Wick rotation (obtained by introducing a purely imaginary time coordinate). The action used in the paper can be useful in order to obtain generalizations of general relativity and offers some interesting possibilities. For example, the fact that Eq. (16) is written in terms of a Euclidean signature metric allows us to use compact internal symmetry groups if we want to use a tetrad formalism. In this respect it should be pointed out that the Ashtekar formalism for general relativity [6] can be derived from the, so-called, Samuel-Jacobson-Smolin [7] action that is written in terms of a tetrad and a self-dual SO(3,1) connection. The fact that the internal group is SO(3,1) has the consequence of requiring the use of complex fields thus complicating the, otherwise, very elegant formalism. Although there are some actions that lead to real Ashtekar-like formulations [8,9] it is interesting to explore alternative formulations in the spirit of the present paper.

Another interesting consequence of the analysis presented here is the realization of the fact that one can describe a covariant theory with a noncovariant action; just notice that for a fixed Φ the action (16) may be written in the obviously noncovariant form

$$S=\int_{\mathcal{M}} d^4x\left[(\alpha+\beta)\frac{R^{00}}{g^{00}}-\frac{1}{2}\alpha R\right], \tag{53}$$

where $\Phi=x^0$. Of course the fact that the field equations of a covariant theory (or, rather, the solutions to them) can be derived from a noncovariant action should not be considered as very surprising. A finite function, for example, does not have to be periodic in order to have periodic extrema; just consider $f(x)=e^{-x}\sin^2 x$. Not even the field equations have to be periodic but only have periodic solutions as it happens in this case. We briefly discuss now some applications of this formalism.

(i) *The problem of time*. The problem of understanding the meaning of time in general relativity is a difficult one. For example, one can try to find some function of the configuration variable (a three-metric in the ADM formalism or the Ashtekar connection) such that evolution can be defined or described with respect to it. In the asymptotically flat case, where the ADM energy is different from zero, one can think of time as the object canonically conjugate to it. In a rather loose sense it can be said that this time is some kind of boundary condition for certain operators that must be supplied by hand [10]. The scalar field used in this paper has a natural interpretation as time, not only because it provides slicings of the space-time and is used to define ''evolution,'' but also because it gives us the time direction that we need in order to get Lorentzian metrics from the auxiliary Euclidean metric used in Eq. (41). In the asymptotically flat case a function of Φ might be interpreted as the time canonically conjugate to the ADM energy. This issue will be considered in future work.

(ii) *Quantum gravity*. As suggested in the introduction, having the possibility of associating to any Lorentzian solution to the EFE's a Euclidean one can be used to derive some interesting properties of black holes, for example the thermal nature of the Hawking radiation in the Schwarzschild case. If the new field equations can be solved for families containing more general black hole solutions (with or without matter) it would be possible to study if the arguments valid for the Schwarzschild black hole still hold in more general cases. In a different context, the possibility of writing actions for general relativity with different kinds of fields may be useful in order to find extensions of the theory that, considered from a perturbative point of view, may behave better that the usual Einstein-Hilbert action or its higher derivative extensions.

(iii) *Signature change*. Signature change has become a popular subject since the suggestion by Hawking that the present Lorentzian regime of gravity could have derived from a Euclidean space-time via quantum tunneling. The issue of finding solutions to the EFE's with signature change and, especially, the joining conditions at the hypersurface of signature change has received much attention in recent years [11]. If the parameters of the action (16) can be provided with some dynamics the metric (41) can change signature while still being a solution to the EFE's. The issue of junction conditions may be looked at not in the solutions to the EFE's but in the auxiliary Euclidean metric. Notice that in the Schwarzschild case considered above the auxiliary metric is independent of the parameters and so, even if α and β ''evolve'' in such a way that g_{ab}^{Eins} changes signature, g_{ab} remains perfectly regular. We do not know if this is true in more general situations. In our opinion this issue deserves a closer look and will be explored in the future.

The main open question that remains to be answered is the issue of the regularity (analyticity?) of the solutions to the new field equations in the parameters α and β. In particular it would be desirable to see if the following statements are true.

(i) If $g_{ab}^{(1)}(\alpha,\beta)$ and $g_{ab}^{(2)}(\alpha,\beta)$ are solutions to Eq. (40) corresponding to the same Φ, and α_0, β_0, α_1, β_1 exist such that $g_{ab}^{(1)}(\alpha_0\beta_0)=g_{ab}^{(2)}(\alpha_1\beta_1)$ (modulo four-dimensional diffeomorphisms), then for any α and β it is possible to find α' and β' such that $g_{ab}^{(1)}(\alpha,\beta)=g_{ab}^{(2)}(\alpha'\beta')$ (modulo four-dimensional diffeomorphisms). In other words, if the $g_{ab}^{(1)}(\alpha,\beta)$ and $g_{ab}^{(2)}(\alpha,\beta)$ share an element then they are the same family of solutions to Eq. (40).

(ii) Every solution to Eq. (40) is sufficiently regular (for example, analytic) in the parameters α and β.

These issues will be explored in future work.

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[1] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[2] R. M. Wald, Commun. Math. Phys. **70**, 221 (1979); *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago 1994).

[3] S. W. Hawking, in *Three Hundred Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987); J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983); S. W. Hawking, in *General Relativity, an Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[4] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[5] K. Kuchař, in *Quantum Gravity 2: A Second Oxford Symposium*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon, Oxford, 1981), p. 329.

[6] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. D **36**, 1587 (1987); *Non Perturbative Canonical Gravity* (Notes prepared in collaboration with R. S. Tate) (World Scientific, Singapore, 1991).

[7] J. Samuel, Pramana, J. Phys. **28**, L429 (1987); T. Jacobson and L. Smolin, Class Quantum Grav. **5**, 583 (1988).

[8] S. Holst, Phys. Rev. D **53**, 5966 (1996).

[9] J. F. Barbero G., Phys. Rev. D **51**, 5507 (1995).

[10] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).

[11] T. Dray, C. A. Manogue, and R. W. Tucker, Gen. Relativ. Gravit. **23**, 967 (1991); G. F. R. Ellis, A. Sumeruk, D. Coule, and C. Hellaby, Class. Quantum. Grav. **9**, 1535 (1992); G. F. R. Ellis, Gen. Relativ. Gravit. **24**, 1047 (1992); S. A. Hayward, Class. Quantum Grav. **9**, 1851 (1992); T. Dray, C. A. Manogue, and R. W. Tucker, Phys. Rev. D **48**, 2587 (1993); F. Embacher, *ibid.* **51**, 6764 (1995); S. A. Hayward, ‘‘Junction Conditions for Signature Change,’’ Report No. PRINT-93-0451 (MPI, GARCHING), gr-qc/9303034 (unpublished).