Energy losses by gravitational radiation in inspiraling compact binaries to 5/2 post-Newtonian order

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(Received 21 July 1995)

This paper derives the total power or energy loss rate generated in the form of gravitational waves by an inspiraling compact binary system to the 5/2 post-Newtonian (2.5PN) approximation of general relativity. Extending a recently developed gravitational-wave generation formalism valid for arbitrary (slowly moving) systems, we compute the mass multipole moments of the system and the relevant tails present in the wave zone to 2.5PN order. In the case of two point masses moving on a quasicircular orbit, we find that the 2.5PN contribution in the energy loss rate is entirely due to tails. Relying on an energy balance argument we derive the laws of variation of the instantaneous frequency and phase of the binary. The 2.5PN order in the accumulated phase is significantly large, being grossly of the same order of magnitude as the previous 2PN order, but opposite in sign. However, finite mass effects at 2.5PN order are small. The results of this paper should be useful when analyzing the data from inspiraling compact binaries in future gravitational-wave detectors such as VIRGO and LIGO. [S0556-2821(96)03012-3]

PACS number(s): 04.25.Nx, 04.30.Db

I. INTRODUCTION

Compact binaries in their late stage of evolution are very relativistic systems in which the two compact objects (neutron stars or black holes) orbit around each other with velocities as large as 30% of that of light. The gravitational radiation these systems emit during the inspiral phase preceding the coalescence of the two objects is expected to be routinely analyzed in future detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) and VIRGO (see [1–3] for reviews). Hundreds to tens of thousands of gravitational-wave cycles should be monitored in the detectors' sensitive frequency bandwidth. The combination of high orbital velocities and a large number of observed rotations, together with the fact that the emitted waves are highly predictable, implies that high-order relativistic (or post-Newtonian) effects should show up in the gravitational signals observed by VIRGO and LIGO [4–10]. Alternatively, this means that high-order post-Newtonian effects should be known in advance so that they can be included in the construction of theoretical filters (templates) to be cross correlated with the outputs of the detectors.

The relevant model for describing most of the observed inspiral phase is a model of two point masses moving on a circular orbit. Radiation reaction forces tend to circularize the orbit very rapidly. On the other hand, point masses can be used in the case of nonrotating and (initially) spherically symmetric compact objects up to a very high precision [11]. This is due to a property owned by general relativity of ''effacing'' the internal structure. Even in the case of stars with intrinsic rotations the dynamics of the binary is likely to be dominated by post-Newtonian gravitational effects [12].

High-order post-Newtonian effects that are measurable are mainly those affecting the orbital phase evolution of the binary, which in turn is determined using a standard energy balance argument by the total power emitted in the form of gravitational waves by the system at infinity, or total luminosity in the waves. To what level in a post-Newtonian expansion we should know the gravitational luminosity (or energy loss) in order to guarantee an optimal detection of the signal (given some power spectral density of the noise in a detector) is still unclear, but the theory of black-hole perturbations can be used to gain insights in this problem. Black-hole perturbations, which deal with the special case of a test mass orbiting a massive black hole, have recently been the focus of intense activity [13–17]. It emerges from this field that neglecting even such a high approximation as the third post-Newtonian (3PN) one, i.e., neglecting the relativistic corrections in the luminosity which are of relative order c^{-6} (or below), is likely to yield unacceptable systematic errors in the data analysis of binary signals [14,15,18,19]. This shows how relativistic are inspiraling compact binaries, as compared, for instance, to the binary pulsar for which the Newtonian approximation in the luminosity (Einstein quadrupole formula) is adequate. The post-Newtonian theory is presently completed through the second post-Newtonian (2PN) approximation, i.e., through relative order c^{-4} (both in the wave form and in the associated energy loss). Two computations were performed to this order, one by Blanchet, Damour, and Iyer [20,21] based on a post-Minkowski matching formalism, and one by Will and Wiseman [22] using an approach initiated by Epstein and Wagoner [23] and generalized by Thorne [24]. The common result of these two computations for the energy loss was summarized in Ref. [12], and the wave form can be found in [25].

In the present paper we develop the post-Minkowski matching formalism one step beyond the work of Refs. [20,21] by computing the 2.5PN order in the energy loss of an inspiraling binary. This entails extending both Ref. [20] on the general formalism valid for an arbitrary (slowly moving) source, and Ref. [21] dealing with the specific application to the binary. The computation of the wave form of the binary to 2.5PN order (the square of which should give back the 2.5PN energy loss) will be left for future work.

The post-Minkowski matching formalism is a wave generation formalism which is especially suited for ''semirelativistic'' sources whose internal velocities can reach 0.3c at most (say), as in the case of inspiraling compact binaries (see [26] for a review). The formalism combines (i) an analytic post-Minkowskian approximation scheme for the computation of the gravitational field in the exterior of the source where multipole expansions can be used to simplify the problem, (ii) a direct post-Newtonian approximation scheme for the resolution of the field equations inside the near zone of the source, and (iii) an asymptotic matching between both types of solutions which is performed in the exterior part of the near zone. The necessity of using a post-Minkowskian approximation scheme first is because its validity extends up to the regions far away from the system where the observer is located, contrary to the post-Newtonian approximation whose validity is limited to the near zone. The exterior field is computed using an algorithm developed in Ref. [27] which set on a general footing previous investigations by Thorne [24] and Bonnor [28]. The implementation of the wave generation formalism (i)–(iii) was done at first with 1PN accuracy in Ref. [29], which obtained the 1PN correction terms in the mass-type quadrupole moment of the source (and in fact in all the mass-type multipole moments). The dominant nonlinear contribution in the radiation field was added in Ref. [30] and shown to be due to the contribution in the wave zone of the well-known ''tail'' effect. The inclusion of this nonlinear contribution pushed the accuracy of the formalism to 1.5PN order in the energy loss. The 2PN precision in both the energy loss and wave form was reached in Ref. [20] where the second (2PN) correction terms in the mass-type multipole moments and the first (1PN) ones in the current-type multipole moments were obtained. An equivalent expression of the 1PN current-type moments had been derived earlier [31] in a different form which is very useful in applications [21].

The main result of the present paper is the expression of the 2.5PN-accurate energy loss by gravitational radiation from a general (semirelativistic) source, and from an inspiraling compact binary. In the latter case of application, the 2.5PN contribution in the energy loss is found to be entirely due to tails in the wave zone (this is like the 1.5PN contribution), and to reduce in the test-body limit to the known result of perturbation theory [15–17].

With the energy loss one can derive the laws of variation of the inspiraling binary's orbital frequency and phase using an energy balance equation. However, note that this is a weak point of the analysis because the latter energy balance equation has been proved to hold only at the Newtonian order (see [32–36] for general systems, and [37] for binary systems), and more recently at the 1PN order [38,39]. It is also known to hold for the specific effects of tails at 1.5PN order [40,30]. To prove this equation at the 2.5PN order as we would need below, one should in principle obtain the equations of motion of the binary up to the very high 5PN order (or order c^{-10}) beyond the Newtonian acceleration. Indeed the radiation reaction forces which are responsible for the decrease of the binding energy of the binary are themselves dominantly of order 2.5PN beyond the Newtonian term. The only method which is available presently in order to deal with this problem is to *assume* that the 2.5PN-accurate radiation reaction forces are such that there is exact agreement between the loss of 2.5PN-accurate binding energy of the binary (as computed from the Damour-Deruelle equations of motion [37]) and the 2.5PN-accurate energy flux we shall compute below. This assumption is verified at the 1.5PN order and sounds reasonable, but will have to be justified in future work.

As an indication of the quantitative importance of the 2.5PN approximation in the orbital phase of the binary, we compute the contribution of the 2.5PN term to the number of gravitational-wave cycles between the entry and exit frequencies of some detector. Essentially we find that the 2.5PN approximation is of the same order of magnitude as the 2PN term (computed in [12]), but opposite in sign. In the case of two neutron stars of mass $1.4M_\odot$ and of the frequency bandwidth [10 Hz, 1000 Hz], the 2PN term contributed $+9$ units to the total number of cycles [12]. We find here that the 2.5PN term contributes -11 cycles in the same conditions. This shows the importance of the 2.5PN approximation for constructing accurate theoretical templates. [However, we find that the contribution of the finite mass effects in the 2.5PN term (which cannot be obtained in perturbation theory) is numerically small.]

In the present paper we shall make a thorough investigation (see Secs. II, III, and IV below) of all the relativistic corrections in the multipole moments of the system which contribute to the 2.5PN-accurate energy loss. However, when we are interested only in the application to inspiraling compact binaries, this investigation can be seen *a posteriori* to be unnecessary. Indeed, the orbit of an inspiraling binary is *circular*, and we shall prove in this case that the 2.5PN relativistic corrections in the multipole moments give in fact no contribution in the energy loss. As we said above, the only contribution is that of the tails present at this order (see Sec. VI). A simple argument (concerning the result at 2.5PN order of a contracted product of tensors made of the relative separation and velocity of the bodies) could be used beforehand to see that this is true. But because inspiraling compact binaries may not constitute the only sources for which the 2.5PN approximation in the energy loss is needed, or simply because one may need in the future to consider the case of a binary moving on an eccentric orbit, we have chosen in this paper to compute systematically all the terms which enter the 2.5PN-accurate energy loss for general systems. This permits us to show explicitly that all the terms but the tail terms give zero in the energy loss for (circular) inspiraling binaries. The simple argument mentioned above may be used in future work to simplify the investigation of higher post-Newtonian orders.

The plan of this paper is as follows. In the following Sec. II, and in the next one, III, we follow step by step the derivation done in Ref. [20] of the near zone gravitational field and the corresponding matching equation, and show how this derivation can be extended to the 2.5PN order. In Sec. IV we obtain the explicit 2.5PN corrections arising in the mass-type multipole moments. Section V deals with the derivation of the energy loss formula valid for general systems (however some coefficients are left unspecified in the formula). Finally, these results are applied in Sec. VI to inspiraling compact binaries. Appendix A derives a useful integration formula and Appendix B presents a relevant summary of the 2.5PN equations of motion.

Throughout this paper we refer to Ref. [20] as paper I and to Ref. [21] as paper II.

II. THE GRAVITATIONAL FIELD IN THE NEAR ZONE

Following the plan of paper I we first investigate the gravitational field generated by a slowly moving isolated source in its *near zone*, which is defined in the usual way as being a zone whose size is of small extent with respect to a typical wavelength of the emitted radiation. Two distinct methods are used. The first one is a direct post-Newtonian iteration (speed of light $c\to+\infty$) of the field equations inside the source, and is valid all over the near zone. The second method consists of reexpanding when $c\to+\infty$ a solution of the vacuum field equations obtained by means of the multipolar and post-Minkowskian iteration scheme of Ref. [27], and is valid only in the exterior part of the near zone.

We denote the small post-Newtonian parameter by $\varepsilon\sim v/c$, where v is a typical velocity in the source (e.g., the relative orbital velocity of the two bodies in the case of a binary system). A remainder term of order $O(\varepsilon^n)$ is abbreviated by $O(n)$. In a vector A^μ or a tensor $B^{\mu\nu}$, the remainder term is denoted by $O(n,p)$ or $O(n,p,q)$, by which we mean a term of order $O(n)$ in A^0 or B^{00}, of order $O(p)$ in A^i or B^{0i} or B^{i0}, and of order $O(q)$ in B^{ij}. (Greek indices range from 0 to 3, and latin indices range from 1 to 3.) Most of the notations used here are as in paper I.

A. The inner gravitational field

$T^{\alpha\beta}$ are the contravariant components of the stress-energy tensor (with dimension of an energy density) of the material source in some inner coordinate system $(\mathbf{x},t)$. The densities of mass σ, of current σ_i, and of stress σ_{ij} in the source are defined by

$$\sigma=\frac{T^{00}+T^{ii}}{c^2}, \tag{2.1a}$$

$$\sigma_i=\frac{T^{0i}}{c}, \tag{2.1b}$$

$$\sigma_{ij}=T^{ij}, \tag{2.1c}$$

where T^{ii} denotes the spatial trace $\Sigma\,\delta_{ij}T^{ij}$. These definitions are such that σ, σ_i, and σ_{ij} have a finite nonzero limit as $c\to+\infty$. From these matter densities one defines the retarded potentials

$$V=-4\pi G\Box_R^{-1}\sigma, \tag{2.2a}$$

$$V_i=-4\pi G\Box_R^{-1}\sigma_i, \tag{2.2b}$$

$$W_{ij}=-4\pi G\Box_R^{-1}\left[\sigma_{ij}+\frac{1}{4\pi G}\left(\partial_i V\partial_j V-\frac{1}{2}\delta_{ij}\partial_k V\partial_k V\right)\right], \tag{2.2c}$$

where G is Newton's constant, and where $\Box_R^{-1}$ denotes the retarded integral operator

$$(\Box_R^{-1}f)(\mathbf{x},t)=-\frac{1}{4\pi}\int\int\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}f\left(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right). \tag{2.3}$$

Contrary to the sources of V and V_i which are of compact support, $\Box V=-4\pi G\sigma$ and $\Box V_i=-4\pi G\sigma_i$ where $\Box$ is the d'Alembertian operator, the source of the potential W_{ij} is not of compact support, $\Box W_{ij}=-4\pi G\sigma_{ij}-\partial_i V\partial_j V+\frac{1}{2}\delta_{ij}\partial_k V\partial_k V$. Indeed, we have included in W_{ij} the stress density of the (Newtonian) gravitational field itself since it is of the same order as σ_{ij} when $c\to+\infty$. (Note that paper II used the notation W_{ij} for a closely related but different potential; here we do not follow paper II but stick to the notation of paper I.) To Newtonian order the densities and potentials so defined satisfy the equations of continuity and of motion:

$$\partial_t\sigma+\partial_i\sigma_i=O(2), \tag{2.4a}$$

$$\partial_t\sigma_i+\partial_j\sigma_{ij}=\sigma\partial_i V+O(2). \tag{2.4b}$$

From these dynamical equations one deduces the differential identities

$$\partial_t V+\partial_i V_i=O(2), \tag{2.5a}$$

$$\partial_t V_i+\partial_j W_{ij}=O(2). \tag{2.5b}$$

With the introduction in paper I of the retarded potentials V, V_i, and W_{ij}, a simple expression of the gravitational field $h^{\alpha\beta}$ inside the source which is valid to some intermediate accuracy $O(6,5,6)$ was written: namely,

$$h^{00}=-\frac{4}{c^2}V+\frac{4}{c^4}(W_{ii}-2V^2)+O(6), \tag{2.6a}$$

$$h^{0i}=-\frac{4}{c^3}V_i+O(5), \tag{2.6b}$$

$$h^{ij}=-\frac{4}{c^4}W_{ij}+O(6). \tag{2.6c}$$

The field variable is $h^{\alpha\beta}\equiv\sqrt{-g}g^{\alpha\beta}-\eta^{\alpha\beta}$, where g and $g^{\alpha\beta}$ are the determinant and inverse of the usual covariant metric $g_{\alpha\beta}$, and where $\eta^{\alpha\beta}$ is the Minkowski metric (signature $-+++$). Note the important fact that there are no *explicit* terms in Eqs. (2.6) involving powers of c^{-1} which are "odd" in the post-Newtonian sense (e.g., a term of order $\sim c^{-5}$ in h^{00} or h^{ij}). This is because we have kept the potentials V, V_i, and W_{ij} in retarded form, without expanding the retardation they contain when $c\to+\infty$. The "odd" terms in Eqs. (2.6) could be easily computed using the post-Newtonian expansions of the retarded potentials as given by Eqs. (4.4) below.

Paper I iterated the inner field (2.6) from this intermediate post-Newtonian order to the next order with the result that the field to the higher precision $O(8,7,8)$ could be written as

$$h^{\alpha\beta}=\Box_R^{-1}\left[\frac{16\pi G}{c^4}\overline{\lambda}(V,W)T^{\alpha\beta}+\overline{\Lambda}^{\alpha\beta}(V,W)\right]+O(8,7,8) \tag{2.7}$$

where $\overline{\lambda}$ and $\overline{\Lambda}^{\alpha\beta}$ denote some explicit combinations of the inner gravitational potentials V, V_i, and W_{ij} and their derivatives. First, we have

$$\overline{\lambda}(V,W)=1+\frac{4}{c^2}V-\frac{8}{c^4}(W_{ii}-V^2) \tag{2.8}$$

which represents in fact the post-Newtonian expansion of (minus) the determinant of the metric, with terms $O(6)$ suppressed. Second, the components of $\overline{\Lambda}^{\alpha\beta}$ read

$$\overline{\Lambda}^{00}(V,W)=-\frac{14}{c^4}\partial_k V\partial_k V+\frac{16}{c^6}\Big\{-V\partial_t^2 V-2V_k\partial_t\partial_k V-W_{km}\partial^2_{km}V+\frac{5}{8}(\partial_t V)^2+\frac{1}{2}\partial_k V_m(\partial_k V_m+3\partial_m V_k)+\partial_k V\partial_t V_k+2\partial_k V\partial_k W_{mm}-\frac{7}{2}V\partial_k V\partial_k V\Big\}, \tag{2.9a}$$

$$\overline{\Lambda}^{0i}(V,W)=\frac{16}{c^5}\Big\{\partial_k V(\partial_i V_k-\partial_k V_i)+\frac{3}{4}\partial_t V\partial_i V\Big\}, \tag{2.9b}$$

$$\overline{\Lambda}^{ij}(V,W)=\frac{4}{c^4}\Big\{\partial_i V\partial_j V-\frac{1}{2}\delta_{ij}\partial_k V\partial_k V\Big\}+\frac{16}{c^6}\Big\{2\partial_{(i}V\partial_t V_{j)}-\partial_i V_k\partial_j V_k-\partial_k V_i\partial_k V_j+2\partial_{(i}V_k\partial_k V_{j)}-\frac{3}{8}\delta_{ij}(\partial_t V)^2-\delta_{ij}\partial_k V\partial_t V_k+\frac{1}{2}\delta_{ij}\partial_k V_m(\partial_k V_m-\partial_m V_k)\Big\}, \tag{2.9c}$$

and represent the expansion of the effective nonlinear gravitational source of Einstein's equations in harmonic coordinates, with $O(8,7,8)$ terms suppressed. The overbar on λ and $\Lambda^{\alpha\beta}$ reminds us that these quantities are only determined up to a certain post-Newtonian order. The (approximate) harmonic coordinate condition is

$$\partial_\beta h^{\alpha\beta}=O(7,8). \tag{2.10}$$

As we shall see, the post-Newtonian accuracy of the inner field (2.7)–(2.10) is sufficient for our purpose.

B. The external gravitational field

In the exterior we use a solution of the vacuum field equations which has in principle sufficient generality for dealing with an arbitrary source of gravitational radiation. This solution is given as a (nonlinear) functional of two infinite sets of time-varying multipole moments, $M_L(t)$ and $S_L(t)$. The index L carried by these moments represents a multi-index formed with ℓ spatial indices: $L=i_1i_2\cdots i_\ell$. The "order of multipolarity" ℓ goes from zero to infinity for the "mass-type" moments $M_L(t)$ and from 1 to infinity for the "current-type" moments $S_L(t)$. The moments M_L and S_L are symmetric and trace-free (STF) in their ℓ indices. The mass monopole M is simply the total mass of the source or Arnowitt-Deser-Misner (ADM) mass, the mass dipole M_i is the position of the center of mass (in units of total mass), and the current dipole S_i is the total angular momentum. M, M_i, and S_i are constant. Furthermore, we shall choose $M_i=0$ by translating the spatial origin of the coordinates to the center of mass.

Some external potentials playing a role analogous to the inner potentials but differing from them in both structural form and numerical values are introduced. First, the potentials $V^{\rm ext}$, $V_i^{\rm ext}$, and $V_{ij}^{\rm ext}$ are given by their multipolar series parametrized by the multipole moments $M_L(t)$ and $S_L(t)$:

$$V^{\rm ext}=G\sum_{\ell=0}^{\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}M_L\left(t-\frac{r}{c}\right)\right], \tag{2.11a}$$

$$V_i^{\rm ext}=-G\sum_{\ell=1}^{\infty}\frac{(-)^\ell}{\ell!}\partial_{L-1}\left[\frac{1}{r}M^{(1)}_{iL-1}\left(t-\frac{r}{c}\right)\right]-G\sum_{\ell=1}^{\infty}\frac{(-)^\ell}{\ell!}\frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left[\frac{1}{r}S_{bL-1}\left(t-\frac{r}{c}\right)\right], \tag{2.11b}$$

$$V_{ij}^{\rm ext}=G\sum_{\ell=2}^{\infty}\frac{(-)^\ell}{\ell!}\partial_{L-2}\left[\frac{1}{r}M^{(2)}_{ijL-2}\left(t-\frac{r}{c}\right)\right]+G\sum_{\ell=2}^{\infty}\frac{(-)^\ell}{\ell!}\frac{2\ell}{\ell+1}\partial_{aL-2}\left[\frac{1}{r}\varepsilon_{ab(i}S^{(1)}_{j)bL-2}\left(t-\frac{r}{c}\right)\right]. \tag{2.11c}$$

The notation ∂_L is shorthand for a product of partial derivatives, $\partial_L=\partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}$ where $\partial_i=\partial/\partial x^i$. In a similar way $\partial_{L-1}=\partial_{i_1}\cdots\partial_{i_{\ell-1}}$, $\partial_{aL-1}=\partial_a\partial_{L-1}$, and so on. The superscript (n) indicates n time derivatives, and the indices in parentheses are symmetrized. The potentials (2.11) satisfy the source-free d'Alembertian equation and the (exact) differential identities

$$\partial_t V^{\rm ext}+\partial_i V_i^{\rm ext}=0, \tag{2.12a}$$

$$\partial_t V_i^{\rm ext}+\partial_j V_{ij}^{\rm ext}=0. \tag{2.12b}$$

Furthermore, $V_{ij}^{\rm ext}$ is trace-free: $V_{ii}^{\rm ext}=0$. Having defined these potentials one introduces a more complicated potential $W_{ij}^{\rm ext}$ by the formula

$$W_{ij}^{\rm ext}=V_{ij}^{\rm ext}+\mathcal{F}_{B=0}\Box_R^{-1}\left[r^B\left(-\partial_i V^{\rm ext}\partial_j V^{\rm ext}+\frac{1}{2}\delta_{ij}\partial_k V^{\rm ext}\partial_k V^{\rm ext}\right)\right]. \tag{2.13}$$

The second term appearing here involves the retarded integral (2.3) but regularized by means of the analytic continuation process defined in Ref. [27]. One multiplies the source by r^B where $r=|\mathbf{x}|$ and B is a complex number. Then applying the operator $\Box_R^{-1}$ defines a function of B which can be analytically continued all over the complex plane deprived of the integers, but admitting there a Laurent expansion with only some simple poles [27]. The looked-for solution is equal to the finite part at $B=0$ (in short $\mathcal{F}_{B=0}$) or constant term $\sim B^0$ in the Laurent expansion when $B\to 0$. This regularization process is made indispensable by the fact that we are looking for solutions of the wave equation in the form of multipole expansions, which are valid only in the exterior of the source and are singular when considered formally inside the source.

With the external potentials $V^{\rm ext}$, $V_i^{\rm ext}$, and $W_{ij}^{\rm ext}$ one has a result similar to Eq. (2.6), namely, the expression of the external field $h^{\mu\nu}_{\rm can}$ (where the notation ''can'' means that we are considering specifically the ''canonical'' construction of the external field as defined in Sec. 4.3 of [27]) up to the intermediate accuracy $O(6,5,6)$:

$$h^{00}_{\rm can}=-\frac{4}{c^2}V^{\rm ext}+\frac{4}{c^4}[W^{\rm ext}_{ii}-2(V^{\rm ext})^2]+O(6), \tag{2.14a}$$

$$h^{0i}_{\rm can}=-\frac{4}{c^3}V^{\rm ext}_i+O(5), \tag{2.14b}$$

$$h^{ij}_{\rm can}=-\frac{4}{c^4}W^{\rm ext}_{ij}+O(6). \tag{2.14c}$$

Because this expression has the same form as Eq. (2.6), the nonlinearities in the exterior field will have in turn the same form as Eq. (2.9). Up to order $O(8,7,8)$ we obtain the ''canonical'' field

$$h^{\mu\nu}_{\rm can}=Gh^{\mu\nu}_{\rm can(1)}+G^2q^{\mu\nu}_{\rm can(2)}+\mathcal{F}_{B=0}\Box_R^{-1}[r^B\overline{\Lambda}^{\mu\nu}(V^{\rm ext},W^{\rm ext})]+O(8,7,8), \tag{2.15}$$

satisfying the (exact) harmonic gauge condition

$$\partial_\nu h^{\mu\nu}_{\rm can}=0. \tag{2.16}$$

In the last term of (2.15) the symbol $\mathcal{F}_{B=0}$ and the regularization factor r^B have the same meaning as in (2.13). The effective nonlinear source $\overline{\Lambda}^{\mu\nu}$ is given by Eqs. (2.9) but expressed with the external potentials $V^{\rm ext}$, $V_i^{\rm ext}$, and $W_{ij}^{\rm ext}$ instead of the inner potentials V, V_i, and W_{ij} appearing in (2.7). The first term $Gh^{\mu\nu}_{\rm can(1)}$ in Eq. (2.15) is linear in the external potentials:

$$Gh^{00}_{\rm can(1)}=-\frac{4}{c^2}V^{\rm ext}, \tag{2.17a}$$

$$Gh^{0i}_{\rm can(1)}=-\frac{4}{c^3}V^{\rm ext}_i, \tag{2.17b}$$

$$Gh^{ij}_{\rm can(1)}=-\frac{4}{c^4}V^{\rm ext}_{ij}. \tag{2.17c}$$

This term is the solution of the linearized (vacuum) equations on which is based the post-Minkowskian algorithm for the construction of the canonical metric in Ref. [27]. The powers of c^{-1} in Eqs. (2.17) are such that $h^{\mu\nu}_{\rm can(1)}=O(2,3,4)$. Finally, the term $G^2q^{\mu\nu}_{\rm can(2)}$ in (2.15) is a particular solution of the source-free wave equation which has to be added in order that the harmonic gauge condition (2.16) be satisfied. It was proved in Appendix A of paper I that this term is of order

$$G^2q^{\mu\nu}_{\rm can(2)}=O(7,7,7), \tag{2.18}$$

and thus can be safely neglected if we are interested in the mass multipole moments to 2PN order only. In the present work, investigating the 2.5PN order, the term (2.18) cannot *a priori* be neglected. However, we shall see that the 2.5PN order is needed only in the sum of the 00 component and the spatial trace of this term, i.e., $G^2(q^{00}_{\rm can(2)}+q^{ii}_{\rm can(2)})$. Relying on our previous papers we know that this sum is made of some retarded waves of the type $\partial_L[r^{-1}X(t-r/c)]$ with scalar or dipolar multipolarity only ($\ell=0$ or 1). Furthermore, we know that the dependence on c^{-1} of such a wave is $O(5+\ell_1+\ell_2-\ell)$ where ℓ_1 and ℓ_2 are the multipolarities of the two interacting moments composing the wave [we use the same notation as, e.g., in Appendix A of paper I; see also after Eq. (5.5) below]. With $\ell=0$ or 1 a term of order $O(7)$ necessarily has $\ell_1+\ell_2=2$ or 3. The term in question is made of the product of the mass M with the quadrupoles M_{ij} and S_{ij}, or of the product of the mass dipole M_i with the quadrupole M_{ij}. The first possibility is excluded because M_{ij} and S_{ij} are STF (thus no scalar or dipole wave with no free index can be formed), and the second possibility does not exist in a mass-centered frame where $M_i=0$. So we have proved that in a mass-centered frame we have

$$G^2(q^{00}_{\rm can(2)}+q^{ii}_{\rm can(2)})=O(8), \tag{2.19}$$

which is all that will be needed in the following.

III. THE MATCHING BETWEEN THE INNER AND OUTER FIELDS

A. Relations between inner and outer potentials

Our matching requirement is that there exists a change of coordinates valid in the external near zone and transforming the inner gravitational field $h^{\alpha\beta}(x)$ given by Eq. (2.7) into the outer field $h^{\alpha\beta}_{\rm can}(x_{\rm can})$ given by Eq. (2.15). Let this change of coordinates be

$$x^\mu_{\rm can}(x)=x^\mu+\varphi^\mu(x) \tag{3.1}$$

where x^μ are the inner coordinates used in Sec. II A and $x^\mu_{\rm can}$ the outer coordinates used in Sec. II B. The vector φ^μ has been shown in paper I to be of order

$$\varphi^\mu=O(3,4). \tag{3.2}$$

Because the two coordinate systems x^μ and $x^\mu_{\rm can}$ are harmonic [at least approximately; see Eqs. (2.10) and (2.16)], and because $h^{\alpha\beta}=O(2,3,4)$, in addition to Eq. (3.2), we have

$$\Box\varphi^{\mu}=O(7,8). \tag{3.3}$$

The matching equations consistent with the order $O(8,7,8)$ read from paper I as

$$h^{00}_{\rm can}(x)=h^{00}(x)+\partial\varphi^{00}+2h^{0\mu}\partial_\mu\varphi^0-\partial_\mu(h^{00}\varphi^\mu)+\partial_i\varphi^0\partial_i\varphi^0+O(8), \tag{3.4a}$$

$$h^{0i}_{\rm can}(x)=h^{0i}(x)+\partial\varphi^{0i}+O(7), \tag{3.4b}$$

$$h^{ij}_{\rm can}(x)=h^{ij}(x)+\partial\varphi^{ij}+O(8), \tag{3.4c}$$

where both the inner and outer fields are expressed in the same (inner) coordinate system x^μ. We denote by $\partial\varphi^{\mu\nu}$ the linear part of the coordinate transformation:

$$\partial\varphi^{\mu\nu}=\partial^\mu\varphi^\nu+\partial^\nu\varphi^\mu-\eta^{\mu\nu}\partial_\lambda\varphi^\lambda. \tag{3.5}$$

The nonlinear part of the transformation enters to this order only the 00 component (3.4a). The matching equations were used in paper I first to obtain the relations between the external potentials and the *multipole* expansions of the corresponding inner potentials. The insertion of the intermediate expressions of $h^{\mu\nu}$ and $h^{\mu\nu}_{\rm can}$ valid up to $O(6,5,6)$ and given by (2.6) and (2.14) yields

$$V^{\rm ext}=\mathcal{M}(V)+c\,\partial_t\varphi^0+O(4), \tag{3.6a}$$

$$V^{\rm ext}_i=\mathcal{M}(V_i)-\frac{c^3}{4}\partial_i\varphi^0+O(2), \tag{3.6b}$$

$$W^{\rm ext}_{ij}=\mathcal{M}(W_{ij})-\frac{c^4}{4}[\partial_i\varphi^j+\partial_j\varphi^i-\delta_{ij}(\partial_0\varphi^0+\partial_k\varphi^k)]+O(2). \tag{3.6c}$$

The script letter $\mathcal{M}$ refers to the multipole expansion. Note that the first equation is valid to post-Newtonian order [see the remainder $O(4)$] while the two others are valid to Newtonian order only [remainder $O(2)$]. The multipole expansions of V, V_i, and W_{ij} are given by

$$\mathcal{M}(V)=G\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{V}^L\left(t-\frac{r}{c}\right)\right], \tag{3.7a}$$

$$\mathcal{M}(V_i)=G\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{V}^L_i\left(t-\frac{r}{c}\right)\right], \tag{3.7b}$$

$$\mathcal{M}(W_{ij})=G\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{W}^L_{ij}\left(t-\frac{r}{c}\right)\right]+\mathcal{F}_{B=0}\Box^{-1}_R\left\{r^B\left[-\partial_i\mathcal{M}(V)\partial_j\mathcal{M}(V)+\frac{1}{2}\delta_{ij}\partial_k\mathcal{M}(V)\partial_k\mathcal{M}(V)\right]\right\}, \tag{3.7c}$$

where the (reducible) multipole moments $\mathcal{V}^L$, $\mathcal{V}^L_i$, and $\mathcal{W}^L_{ij}$ are

$$\mathcal{V}^L(t)=\int d^3\mathbf{x}\hat{x}_L\int_{-1}^1 dz\,\delta_\ell(z)\sigma(\mathbf{x},t+z|\mathbf{x}|/c), \tag{3.8a}$$

$$\mathcal{V}^L_i(t)=\int d^3\mathbf{x}\hat{x}_L\int_{-1}^1 dz\,\delta_\ell(z)\sigma_i(\mathbf{x},t+z|\mathbf{x}|/c), \tag{3.8b}$$

$$\mathcal{W}^L_{ij}(t)=\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}_L\int_{-1}^1 dz\,\delta_\ell(z)\left[\sigma_{ij}+\frac{1}{4\pi G}\left(\partial_iV\partial_jV-\frac{1}{2}\delta_{ij}\partial_kV\partial_kV\right)\right](\mathbf{x},t+z|\mathbf{x}|/c). \tag{3.8c}$$

The notation $\hat{x}_L$ is for the trace-free projection of the product of ℓ spatial vectors $x_L\equiv x_{i_1}\cdots x_{i_\ell}$. The function $\delta_\ell(z)$ is given by

$$\delta_\ell(z)=\frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1-z^2)^\ell,\qquad \int_{-1}^1 dz\,\delta_\ell(z)=1. \tag{3.9}$$

Some explanations of the expressions (3.7)–(3.9) are in order. First, notice that the expressions of the multipole expansions of the potentials V and V_i whose sources have a compact support are quite standard. They can be found in this form in Appendix B of Ref. [29] (but were derived earlier in an alternative form [41]). Notably, the presence of the function $\delta_\ell(z)$ is due to the time delays of the propagation of the waves with finite velocity c inside the source. By contrast, the expression of the multipole expansion of the potential W_{ij} whose source extends everywhere in space is more complicated. The second term in Eq. (3.7c) ensures that $\mathcal{M}(W_{ij})$ satisfies the correct equation [deduced from (2.2c)] outside the source, namely, $\Box\mathcal{M}(W_{ij})=-\partial_i\mathcal{M}(V)\partial_j\mathcal{M}(V)+\frac{1}{2}\delta_{ij}\partial_k\mathcal{M}(V)\partial_k\mathcal{M}(V)$, where the right side agrees numerically with $-\partial_iV\partial_jV+\frac{1}{2}\delta_{ij}\partial_kV\partial_kV$ outside the source. This second term involves the multipole expansion $\mathcal{M}(V)$ of the inner potential and not V itself. This is in conformity with the use of the regularized operator $\mathcal{F}\Box^{-1}_R$ which is defined only when acting on multipole expansions [such as in Eq. (2.13)]. On the other hand, the integrand of (3.8c) involves the noncompact supported source of W_{ij}, where appears the potential V itself, i.e., not in multipole expanded form. Thus the integrand of (3.8c) is valid everywhere inside and outside the source. Very far from the source it diverges because of the presence of the product $\hat{x}_L$ of ℓ spatial vectors, behaving like $|\mathbf{x}|^\ell$ at spatial infinity.

The well-defined meaning of the integral results from the presence of the regularization factor $|\mathbf{x}|^B$ and the finite part symbol. We notice that no *ad hoc* prescription is necessary in order to obtain the multipole moments in a well-defined form even in the case of a noncompact supported source. This is proved in paper I.

B. The matching equation

The relations (3.6) linking $V^{\rm ext}$, $V_i^{\rm ext}$, and $W_{ij}^{\rm ext}$ to the multipole expansions $\mathcal{M}(V)$, $\mathcal{M}(V_i)$, and $\mathcal{M}(W_{ij})$ serve us in reexpressing the nonlinearities in the external metric in terms of the potentials belonging to the inner metric. The result of paper I is

$$\overline{\Lambda}^{\mu\nu}(V^{\rm ext},W^{\rm ext})=\overline{\Lambda}^{\mu\nu}(\mathcal{M}(V),\mathcal{M}(W))+\Box\Omega^{\mu\nu}+O(8,7,8), \tag{3.10}$$

where the components of the tensor $\Omega^{\mu\nu}$ are given by

$$\Omega^{00}=-\frac{8}{c^3}\left[\mathcal{M}(V)\partial_t\varphi^0+\mathcal{M}(V_i)\partial_i\varphi^0-\frac{c}{2}\partial_\mu(\mathcal{M}(V)\varphi^\mu)\right]+\partial_i\varphi^0\partial_i\varphi^0, \tag{3.11a}$$

$$\Omega^{0i}=\Omega^{ij}=0. \tag{3.11b}$$

The only nonzero component of this tensor is the 00 component which is of order

$$\Omega^{00}=O(6). \tag{3.12}$$

This is because $\varphi^0=O(3)$, $\varphi^i=O(4)$. Both sides of Eq. (3.10) are multiplied by r^B, and one applies afterwards the retarded integral $\Box_R^{-1}$ and takes the finite part at $B=0$. We find

$$\begin{aligned}\mathcal{F}_{B=0}\Box_R^{-1}[r^B\overline{\Lambda}^{\mu\nu}(V^{\rm ext},W^{\rm ext})]&=\mathcal{F}_{B=0}\Box_R^{-1}[r^B\overline{\Lambda}^{\mu\nu}(\mathcal{M}(V),\mathcal{M}(W))]\\&\quad+\Omega^{\mu\nu}+X^{\mu\nu}+O(8,7,8),\end{aligned} \tag{3.13}$$

where one must be careful that an extra term $X^{\mu\nu}$ with respect to paper I arises whose components are given by

$$X^{00}=\mathcal{F}_{B=0}\Box_R^{-1}[r^B\Box\Omega^{00}]-\Omega^{00}, \tag{3.14a}$$

$$X^{0i}=X^{ij}=0. \tag{3.14b}$$

The only nonzero component of this term is the 00 component, which satisfies

$$\Box X^{00}=0;\quad X^{00}=O(7). \tag{3.15}$$

The fact that X^{00} is a (retarded) solution of the source-free d'Alembert equation is clear from its definition (3.14a) and the main property of the operator $\mathcal{F}\Box_R^{-1}$ which is the inverse of $\Box$ (when acting on multipolar sources). The fact that X^{00} is of order $O(7)$, that is, exactly one order in c^{-1} smaller than the order of the corresponding Ω^{00}, is not immediately obvious but will be proved below. The quantity X^{00} was rightly neglected in paper I but has to be considered here since it will contribute to the mass moments at 2.5PN order.

We are now in a position to write down a matching equation valid up to the neglect of $O(8,7,8)$ terms. The equation is obtained by insertion into Eqs. (3.4) of both the inner field $h^{\mu\nu}$ given by (2.7) and the outer field $h^{\mu\nu}_{\rm can}$ given by (2.15). Use is made of the link derived in (3.13) between the external and internal nonlinearities. We find that $\Omega^{\mu\nu}$ cancels to the required order the nonlinear part of the coordinate transformation, so that only the linear part $\partial\varphi^{\mu\nu}$ given by Eq. (3.5) remains. The resulting matching equation [extending the less accurate Eq. (3.35) of paper I] reads as

$$\begin{aligned}Gh^{\mu\nu}_{\rm can(1)}+G^2q^{\mu\nu}_{\rm can(2)}&=\Box_R^{-1}\left[\frac{16\pi G}{c^4}\overline{\lambda}(V,W)T^{\mu\nu}+\overline{\Lambda}^{\mu\nu}(V,W)\right]\\&\quad-\mathcal{F}_{B=0}\Box_R^{-1}[r^B\overline{\Lambda}^{\mu\nu}(\mathcal{M}(V),\mathcal{M}(W))]\\&\quad-X^{\mu\nu}+\partial\varphi^{\mu\nu}+O(8,7,8).\end{aligned} \tag{3.16}$$

There are two new terms with respect to (3.35) in paper I: $G^2q^{\mu\nu}_{\rm can(2)}$ in the left side, and $-X^{\mu\nu}$ in the right side which are both terms of 2.5PN order. Now, by exactly the same reasoning as in paper I one can transform the difference between the two retarded integrals in the right side into an explicit multipole expansion parametrized by some moments $\overline{\mathcal{T}}^{\mu\nu}_L(t)$: namely,

$$Gh^{\mu\nu}_{\rm can(1)}+G^2q^{\mu\nu}_{\rm can(2)}=-\frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\overline{\mathcal{T}}^{\mu\nu}_L(t-r/c)\right]-X^{\mu\nu}+\partial\varphi^{\mu\nu}+O(8,7,8). \tag{3.17}$$

These moments are given by

$$\overline{\mathcal{T}}^{\mu\nu}_L(t)=\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}_L\int_{-1}^{1}dz\,\delta_\ell(z)\overline{\tau}^{\mu\nu}(\mathbf{x},t+z|\mathbf{x}|/c) \tag{3.18}$$

where $\overline{\tau}^{\mu\nu}$ denotes the total stress-energy tensor of the material and gravitational fields (valid up to the considered precision),

$$\overline{\tau}^{\mu\nu}(V,W)=\overline{\lambda}(V,W)T^{\mu\nu}+\frac{c^4}{16\pi G}\overline{\Lambda}^{\mu\nu}(V,W), \tag{3.19}$$

which is conserved in the sense

$$\partial_\nu\overline{\tau}^{\mu\nu}=O(3,4). \tag{3.20}$$

It is important to note that the effective stress-energy tensor $\overline{\tau}^{\mu\nu}$ is a functional of the potentials V, V_i, and W_{ij} valid everywhere inside and outside the source. There is no contribution of the multipole expansions of the potentials in the final result (see paper I).

The left side of the matching equation (3.17) is a functional of the original multipole moments M_L and S_L parametrizing the exterior metric. On the other hand, the right side is a functional of the actual densities of mass, current, and stress of the material fields in the source. To find the explicit expressions of M_L and S_L in terms of these source densities we decompose the right side into irreducible multipole moments. Inspection of the reasoning done in Sec. IV A of paper I shows that this reasoning is still valid in the present, more accurate, case. As a result we find

$$Gh^{\mu\nu}_{\mathrm{can}(1)}[M_L,S_L]+G^2q^{\mu\nu}_{\mathrm{can}(2)}=Gh^{\mu\nu}_{\mathrm{can}(1)}[I_L,J_L]-X^{\mu\nu}+\partial\xi^{\mu\nu}+O(8,7,8), \tag{3.21}$$

where the last term is a linear coordinate transformation associated with the vector $\xi^\mu=\varphi^\mu+\omega^\mu$ where ω^μ is the same as in Eq. (4.4) of paper I. The linearized metric $Gh^{\mu\nu}_{\mathrm{can}(1)}$ in the right side takes the same expression as in the left side but is parametrized instead of M_L and S_L by the (STF) *source* multipole moments I_L and J_L given by

$$\begin{aligned}I_L(t)=\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\int_{-1}^{1}dz\Big[\delta_\ell(z)\hat{x}_L\overline{\Sigma}\\ -\frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\,\delta_{\ell+1}(z)\hat{x}_{iL}\partial_t\overline{\Sigma}_i\\ +\frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\,\delta_{\ell+2}(z)\hat{x}_{ijL}\partial_t^2\overline{\Sigma}_{ij}\Big]\\ \times(\mathbf{x},t+z|\mathbf{x}|/c),\end{aligned} \tag{3.22a}$$

$$\begin{aligned}J_L(t)=\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\int_{-1}^{1}dz\Big[\delta_\ell(z)\varepsilon_{ab<i_\ell}\hat{x}_{L-1>a}\overline{\Sigma}_b\\ -\frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\,\delta_{\ell+1}(z)\varepsilon_{ab<i_\ell}\hat{x}_{L-1>ac}\partial_t\overline{\Sigma}_{bc}\Big]\\ \times(\mathbf{x},t+z|\mathbf{x}|/c).\end{aligned} \tag{3.22b}$$

To obtain these expressions from Eq. (3.17) one uses some techniques similar to the ones employed by Damour and Iyer [52] in the case of linearized gravity. We have posed

$$\overline{\Sigma}=\frac{\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2}, \tag{3.23a}$$

$$\overline{\Sigma}_i=\frac{\overline{\tau}^{0i}}{c}, \tag{3.23b}$$

$$\overline{\Sigma}_{ij}=\overline{\tau}^{ij}. \tag{3.23c}$$

The equation (3.21) can be solved uniquely for the multipole moments M_L and S_L. To do so it suffices to notice that for any gauge term $\partial\xi^{\mu\nu}\equiv\partial^\mu\xi^\nu+\partial^\nu\xi^\mu-\eta^{\mu\nu}\partial_\lambda\xi^\lambda$ the identity

$$\frac{1}{2}\partial^2_{ij}[\partial\xi^{00}+\partial\xi^{kk}]+\partial_0[\partial_i\partial\xi^{0j}+\partial_j\partial\xi^{0i}]+\partial_0^2\Big[\partial\xi^{ij}+\frac{1}{2}\delta^{ij}(\partial\xi^{00}-\partial\xi^{kk})\Big]\equiv0 \tag{3.24}$$

holds. This is nothing but the vanishing of the $0i0j$ component of the linearized Riemann tensor when computed with $g^{\mathrm{gauge}}_{\mu\nu}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$. Applying (3.24) to the gauge term of (3.21) and using the form of the multipole moment decomposition of $h^{\mu\nu}_{\mathrm{can}(1)}$ one finds

$$M_L=I_L+\delta I_L+O(6), \tag{3.25a}$$

$$S_L=J_L+O(4), \tag{3.25b}$$

where δI_L in the mass moments is of 2.5PN order and comes from the decomposition of X^{00} (which, we recall, is a retarded solution of the wave equation) into multipole moments according to

$$X^{00}=\frac{4G}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\Big[\frac{1}{r}\,\delta I_L(t-r/c)\Big]. \tag{3.26}$$

There is no contribution coming from $G^2q^{\mu\nu}_{\mathrm{can}(2)}$ in the left side of (3.21) thanks to the result proved in Eq. (2.19) in the case where we are using a mass-centered frame. The result (3.25a) generalizes to 2.5PN order the result (4.7a) of paper I. In a future work we shall investigate more systematically the relations linking the exterior moments M_L and S_L to the source moments I_L and J_L as they have been defined here.

IV. EXPLICIT EXPRESSIONS OF THE MULTIPOLE MOMENTS

Two things must be done in order to obtain the expressions of the moments M_L and S_L of Eqs. (3.25). First, one must expand when $c\to+\infty$ the source moments I_L and J_L of Eqs. (3.22) up to consistent order. Secondly, one must evaluate the 2.5PN modification δI_L entering the mass moments (3.25a).

A. The source multipole moments

The post-Newtonian expansion of the source moments (3.22) is straighforwardly performed using a formula which was given in Eq. (B.14) of Ref. [29]: namely,

$$\begin{aligned}\int_{-1}^{1}dz\,\delta_\ell(z)\overline{\Sigma}(\mathbf{x},t+z|\mathbf{x}|/c)\\ =\overline{\Sigma}(\mathbf{x},t)+\frac{\mathbf{x}^2}{2c^2(2\ell+3)}\partial_t^2\overline{\Sigma}(\mathbf{x},t)\\ +\frac{\mathbf{x}^4}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\overline{\Sigma}(\mathbf{x},t)+O(6).\end{aligned} \tag{4.1}$$

Explicit expressions of the densities $\overline{\Sigma}$, $\overline{\Sigma}_i$, and $\overline{\Sigma}_{ij}$ have also to be inserted into I_L and J_L; these are easily evaluated with the help of Eqs. (2.8), (2.9), (3.19), and (3.23) with the result (identical to paper I)

$$\overline{\Sigma}=\left[1+\frac{4V}{c^2}-\frac{8}{c^4}(W_{ii}-V^2)\right]\sigma-\frac{1}{\pi Gc^2}\partial_iV\partial_iV+\frac{1}{\pi Gc^4}\left\{-V\partial_t^2V-2V_i\partial_t\partial_iV-W_{ij}\partial_{ij}^2V-\frac{1}{2}(\partial_tV)^2+2\partial_iV_j\partial_jV_i+2\partial_iV\partial_iW_{jj}-\frac{7}{2}V\partial_iV\partial_iV\right\}, \tag{4.2a}$$

$$\overline{\Sigma}_i=\left[1+\frac{4V}{c^2}\right]\sigma_i+\frac{1}{\pi Gc^2}\left\{\partial_kV(\partial_iV_k-\partial_kV_i)+\frac{3}{4}\partial_tV\partial_iV\right\}+O(4), \tag{4.2b}$$

$$\overline{\Sigma}_{ij}=\sigma_{ij}+\frac{1}{4\pi G}\left\{\partial_iV\partial_jV-\frac{1}{2}\delta_{ij}\partial_kV\partial_kV\right\}+O(2). \tag{4.2c}$$

Note the important fact that the remainder in Eq. (4.1) is $O(6)$ and not $O(5)$, and thus does not contribute to the 2.5PN order [similarly, the other remainders in (4.2) will not contribute]. With Eqs. (4.1) and (4.2) we recover the same expressions as in Eqs. (4.12) and (4.13) of paper I for the source moments in raw form:

$$I_L(t)=\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\left\{\left[1+\frac{4}{c^2}V-\frac{8}{c^4}(W_{ii}-V^2)\right]\hat{x}_L\sigma-\frac{1}{\pi Gc^2}\hat{x}_L\partial_iV\partial_iV+\frac{1}{\pi Gc^4}\hat{x}_L\left[-V\partial_t^2V-2V_i\partial_t\partial_iV-W_{ij}\partial_{ij}^2V-\frac{1}{2}(\partial_tV)^2+2\partial_iV_j\partial_jV_i+2\partial_iV\partial_iW_{jj}-\frac{7}{2}V\partial_iV\partial_iV\right]+\frac{|\mathbf{x}|^2\hat{x}_L}{2c^2(2\ell+3)}\partial_t^2\left[\left(1+\frac{4V}{c^2}\right)\sigma-\frac{1}{\pi Gc^2}\partial_iV\partial_iV\right]+\frac{|\mathbf{x}|^4\hat{x}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\sigma-\frac{2(2\ell+1)|\mathbf{x}|^2\hat{x}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\sigma_i-\frac{4(2\ell+1)\hat{x}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\left[\left(1+\frac{4V}{c^2}\right)\sigma_i+\frac{1}{\pi Gc^2}\left\{\partial_kV(\partial_iV_k-\partial_kV_i)+\frac{3}{4}\partial_tV\partial_iV\right\}\right]+\frac{2(2\ell+1)\hat{x}_{ijL}}{c^4(\ell+1)(\ell+2)(2\ell+5)}\partial_t^2\left[\sigma_{ij}+\frac{1}{4\pi G}\partial_iV\partial_jV\right]\right\}+O(6), \tag{4.3a}$$

$$J_L(t)=\mathcal{F}_{B=0}\varepsilon_{ab<i_\ell}\int d^3\mathbf{x}|\mathbf{x}|^B\left\{\hat{x}_{L-1>a}\left(1+\frac{4}{c^2}V\right)\sigma_b+\frac{|\mathbf{x}|^2\hat{x}_{L-1>a}}{2c^2(2\ell+3)}\partial_t^2\sigma_b+\frac{1}{\pi Gc^2}\hat{x}_{L-1>a}\left[\partial_kV(\partial_bV_k-\partial_kV_b)+\frac{3}{4}\partial_tV\partial_bV\right]-\frac{(2\ell+1)\hat{x}_{L-1>ac}}{c^2(\ell+2)(2\ell+3)}\partial_t\left[\sigma_{bc}+\frac{1}{4\pi G}\partial_bV\partial_cV\right]\right\}+O(4). \tag{4.3b}$$

The remainders $O(6)$ and $O(4)$ are negligible. The retarded potentials V, V_i, and W_{ij} are then replaced by their post-Newtonian expansions when $c\to+\infty$. It is easily seen that the accuracy of the expansions of V and W_{ij} given in paper I is not sufficient and has to be pushed one order farther. The relevant expansions are

$$V=U+\frac{1}{2c^2}\partial_t^2X-\frac{2G}{3c^3}K^{(3)}+O(4), \tag{4.4a}$$

$$V_i=U_i+O(2), \tag{4.4b}$$

$$W_{ij}=P_{ij}-\frac{G}{2c}\left[Q_{ij}^{(3)}+\frac{1}{3}\delta_{ij}K^{(3)}\right]+O(2), \tag{4.4c}$$

where the Newtonian-like potentials U, X, U_i, and P_{ij} are defined as in paper I by

$$U(\mathbf{x},t)=G\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma(\mathbf{x}',t), \tag{4.5a}$$

$$X(\mathbf{x},t)=G\int d^3\mathbf{x}'|\mathbf{x}-\mathbf{x}'|\sigma(\mathbf{x}',t), \tag{4.5b}$$

$$U_i(\mathbf{x},t)=G\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sigma_i(\mathbf{x}',t), \tag{4.5c}$$

$$P_{ij}(\mathbf{x},t)=G\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ij}+\frac{1}{4\pi G}\left(\partial_iU\partial_jU-\frac{1}{2}\delta_{ij}\partial_kU\partial_kU\right)\right](\mathbf{x}',t), \tag{4.5d}$$

and where the new terms involve the trace-free quadrupole moment Q_{ij} and moment of inertia K associated with the mass distribution σ: namely,

$$Q_{ij}(t)=\int d^3\mathbf{x}\sigma(\mathbf{x},t)\hat{x}_{ij}, \tag{4.6a}$$

$$K(t)=\int d^3\mathbf{x}\sigma(\mathbf{x},t)\mathbf{x}^2. \quad (4.6b)$$

When substituting the expansions (4.4) into the source moments (4.3) all the terms coming from the Newtonian-like potentials U, X, U_i, and P_{ij} lead to the same expressions as in paper I, while the terms coming from the moments Q_{ij} and K lead to some correction terms (in I_L only). Let us write

$$I_L=\tilde{I}_L+\delta\tilde{I}_L+O(6), \quad (4.7a)$$

$$J_L=\tilde{J}_L+O(4), \quad (4.7b)$$

where $\tilde{I}_L$ and $\tilde{J}_L$ are the 2PN-accurate moments which were obtained in paper I, and where $\delta\tilde{I}_L$ denotes a 2.5PN correction term [which is distinct from δI_L found in Eq. (3.25a)]. After the transformation of $\tilde{I}_L$ as in Sec. IV B of paper I one can write $\tilde{I}_L$ and $\tilde{J}_L$ in the form

$$\begin{aligned}\tilde{I}_L(t)=\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\Bigg\{&\hat{x}_L\left[\sigma+\frac{4}{c^4}(\sigma_{ii}U-\sigma P_{ii})\right]+\frac{|\mathbf{x}|^2\hat{x}_L}{2c^2(2\ell+3)}\partial_t^2\sigma-\frac{4(2\ell+1)\hat{x}_{iL}}{c^2(\ell+1)(2\ell+3)}\partial_t\left[\left(1+\frac{4U}{c^2}\right)\sigma_i\right.\\&\left.+\frac{1}{\pi Gc^2}\left(\partial_kU[\partial_iU_k-\partial_kU_i]+\frac{3}{4}\partial_tU\partial_iU\right)\right]+\frac{|\mathbf{x}|^4\hat{x}_L}{8c^4(2\ell+3)(2\ell+5)}\partial_t^4\sigma-\frac{2(2\ell+1)|\mathbf{x}|^2\hat{x}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)}\partial_t^3\sigma_i\\&+\frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\hat{x}_{ijL}\partial_t^2\left[\sigma_{ij}+\frac{1}{4\pi G}\partial_iU\partial_jU\right]+\frac{1}{\pi Gc^4}\hat{x}_L\Big[-P_{ij}\partial_{ij}^2U-2U_i\partial_t\partial_iU\\&+2\partial_iU_j\partial_jU_i-\frac{3}{2}(\partial_tU)^2-U\partial_t^2U\Big]\Bigg\}, \end{aligned} \quad (4.8a)$$

$$\begin{aligned}\tilde{J}_L(t)=\mathcal{F}_{B=0}\varepsilon_{ab<i_\ell}\int d^3\mathbf{x}|\mathbf{x}|^B\Bigg\{&\hat{x}_{L-1>a}\left(1+\frac{4}{c^2}U\right)\sigma_b+\frac{|\mathbf{x}|^2\hat{x}_{L-1>a}}{2c^2(2\ell+3)}\partial_t^2\sigma_b+\frac{1}{\pi Gc^2}\hat{x}_{L-1>a}\left[\partial_kU(\partial_bU_k-\partial_kU_b)+\frac{3}{4}\partial_tU\partial_bU\right]\\&-\frac{(2\ell+1)\hat{x}_{L-1>ac}}{c^2(\ell+2)(2\ell+3)}\partial_t\left[\sigma_{bc}+\frac{1}{4\pi G}\partial_bU\partial_cU\right]\Bigg\}.\end{aligned} \quad (4.8b)$$

The moments $\tilde{I}_L$ and $\tilde{J}_L$ constituted the central result of paper I and they were the basis of the application to inspiraling compact binaries in paper II. On the other hand, the 2.5PN correction term $\delta\tilde{I}_L$ is obtained by simple inspection of Eq. (4.3a). A simplifying fact in obtaining $\delta\tilde{I}_L$ is that the moments Q_{ij} and K are only functions of time so that their spatial gradients vanish. We obtain

$$\delta\tilde{I}_L=\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\left\{\frac{2G}{3c^5}K^{(3)}\sigma\hat{x}_L+\frac{1}{2\pi c^5}Q_{ij}^{(3)}\hat{x}_L\partial_{ij}^2U\right\}. \quad (4.9)$$

The second term is an integral having *a priori* a noncompact support. However, it can be transformed into a manifestly compact-support form by means of the formula

$$\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\frac{\hat{x}_L}{|\mathbf{x}-\mathbf{y}|}=-\frac{2\pi}{2\ell+3}|\mathbf{y}|^2\hat{y}_L. \quad (4.10)$$

This formula is proved by noticing that the integral defined by $I_B=\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}_L|\mathbf{x}-\mathbf{y}|^{-1}$ is proportional to a prefactor $|\mathbf{y}|^B$ and satisfies $\Delta_\mathbf{y}I_B=-4\pi|\mathbf{y}|^B\hat{y}_L$. These two facts imply that $I_B=\Delta_\mathbf{y}^{-1}(-4\pi|\mathbf{y}|^B\hat{y}_L)$ where $\Delta_\mathbf{y}^{-1}$ is defined as in Eq. (3.9) of Ref. [27], and thus that $I_B=-4\pi|\mathbf{y}|^{B+2}\hat{y}_L/(B+2)(B+2\ell+3)$, which yields (4.10) after taking the finite part. The more complicated formula (4.23) in paper I can interestingly be compared with (4.10). Thanks to (4.10) we can write

$$\delta\tilde{I}_L=\frac{G}{c^5}\int d^3\mathbf{x}\left\{\frac{2}{3}K^{(3)}\sigma\hat{x}_L-\frac{1}{2\ell+3}Q_{ij}^{(3)}\sigma\partial_{ij}^2[|\mathbf{x}|^2\hat{x}_L]\right\}. \quad (4.11)$$

Expanding the spatial derivatives in the second term yields finally

$$\delta\tilde{I}_L=\frac{G}{c^5}\left\{\frac{2}{3}K^{(3)}Q_L-\frac{4\ell}{2\ell+3}Q^{(3)}_{k<i_\ell}Q_{L-1>k}-\frac{\ell(\ell-1)}{2\ell-1}Q^{(3)}_{<i_\ell i_{\ell-1}}K_{L-2>}\right\} \quad (4.12)$$

where we have posed

$$Q_L(t)=\int d^3\mathbf{x}\sigma\hat{x}_L, \quad (4.13a)$$

$$K_{L-2}(t)=\int d^3\mathbf{x}\sigma\mathbf{x}^2\hat{x}_{L-2}. \quad (4.13b)$$

These definitions are in conformity with the earlier definitions (4.6). (The brackets $< >$ denote the STF projection.)

B. The 2.5PN modification of the mass moments

In addition to the previous contribution $\delta\tilde{I}_L$ which is part of the source multipole moments I_L, we have seen in Eq. (3.25a) that there exists also a 2.5PN contribution δI_L entering M_L. Evidently the contribution δI_L is as important as $\delta\widetilde{I}_L$ in that it contributes also to the asymptotic wave form depending on the moments M_L. [Note that the terminology referring to I_L and J_L as *the* multipole moments of the source by contrast to M_L and S_L which are viewed as some intermediate moments devoid of direct physical significance is somewhat arbitrary. All that matters in the end is to express (by any convenient means) the asymptotic wave form in terms of physical quantities belonging to the source.]

The 2.5PN term δI_L results from the multipole decomposition of the quantity X^{00} which is itself determined from the other quantity Ω^{00}; see Eqs. (3.26), (3.14a), and (3.11a). The first step in the computation of δI_L is to find the vector φ^μ of the coordinate transformation between the inner and external metrics. Since φ^μ is of order $O(3,4)$ and is d'Alembertian-free to order $O(7,8)$ [see Eqs. (3.2) and (3.3)], there exist four sets of STF tensors $W_L(t)$, $X_L(t)$, $Y_L(t)$, and $Z_L(t)$ such that

$$\varphi^0=-\frac{4G}{c^3}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}W_L(t-r/c)\right]+O(7), \tag{4.14a}$$

$$\begin{aligned}\varphi^i=&\frac{4G}{c^4}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_{iL}\left[\frac{1}{r}X_L(t-r/c)\right]\\&+\frac{4G}{c^4}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[\frac{1}{r}Y_{iL-1}(t-r/c)\right]\right.\\&\left.+\frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left[\frac{1}{r}Z_{bL-1}(t-r/c)\right]\right\}+O(8).\end{aligned} \tag{4.14b}$$

The powers of $1/c$ in front of these terms are such that $W_L,\ldots,Z_L$ have a nonzero limit when $c\to+\infty$. We compute these tensors to the lowest order in $1/c$. To do this let us recall the relations between $V_i^{\rm ext}$ and $W_{ij}^{\rm ext}$ and the multipole expansions $\mathcal{M}(V_i)$ and $\mathcal{M}(W_{ij})$ as obtained in Eqs. (3.6b) and (3.6c). We have

$$V_i^{\rm ext}=G\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{V}_i^L(t-r/c)\right]-\frac{c^3}{4}\partial_i\varphi^0+O(2), \tag{4.15a}$$

$$\begin{aligned}V_{ij}^{\rm ext}=&G\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{W}_{ij}^L(t-r/c)\right]-\frac{c^4}{4}[\partial_i\varphi^j+\partial_j\varphi^i\\&-\delta_{ij}(\partial_0\varphi^0+\partial_k\varphi^k)]+O(2),\end{aligned} \tag{4.15b}$$

where we have transformed the relation for $W_{ij}^{\rm ext}$ into a simpler relation for $V_{ij}^{\rm ext}$, and where the moments take to lowest order the form [see Eqs. (3.8b) and (3.8c)]:

$$\mathcal{V}_i^L(t)=\int d^3\mathbf{x}\hat{x}_L\sigma_i(\mathbf{x},t)+O(2), \tag{4.16a}$$

$$\mathcal{W}_{ij}^L(t)=\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}_L\overline{\Sigma}_{ij}(\mathbf{x},t)+O(2). \tag{4.16b}$$

The (noncompact-supported) stress density $\overline{\Sigma}_{ij}$ is defined by Eq. (4.2c). Having written (4.15) and (4.16) and knowing the explicit multipole decompositions of $V_i^{\rm ext}$ and $V_{ij}^{\rm ext}$ given by Eqs. (2.11b) and (2.11c), it is a simple matter to compute the tensors $W_L,\ldots,Z_L$ by decomposition of the integrands $\hat{x}_L\sigma_i$ and $\hat{x}_L\overline{\Sigma}_{ij}$ entering (4.16) into irreducible tensorial pieces with respect to their $\ell+1$ and $\ell+2$ indices. We do not detail this computation but simply give the result, which is

$$W_L=\frac{2\ell+1}{(\ell+1)(2\ell+3)}\int d^3\mathbf{x}\hat{x}_{iL}\sigma_i+O(2), \tag{4.17a}$$

$$X_L=\frac{2\ell+1}{2(\ell+1)(\ell+2)(2\ell+5)}\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\hat{x}_{ijL}\overline{\Sigma}_{ij}+O(2), \tag{4.17b}$$

$$Y_L=\frac{3\ell(2\ell-1)}{(\ell+1)(2\ell+3)}\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\left(\hat{x}_{i<L-1}\overline{\Sigma}_{i_\ell>i}-\frac{1}{3}\hat{x}_L\overline{\Sigma}_{ii}\right)+O(2), \tag{4.17c}$$

$$Z_L=-\frac{2\ell+1}{(\ell+2)(2\ell+3)}\mathcal{F}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B\varepsilon_{ab<i_\ell}\hat{x}_{L-1>bc}\overline{\Sigma}_{ac}+O(2). \tag{4.17d}$$

The tensor W_L is manifestly of compact-supported form. There is agreement for this tensor with the previous result obtained in Eqs. (2.22a) and (2.19c) of Ref. [29]. As they are written, the other tensors X_L, Y_L, and Z_L do not have a compact-supported form. However, Y_L can be rewritten equivalently in such a form:

$$Y_L=\int d^3\mathbf{x}\left\{\frac{3(2\ell+1)}{(\ell+1)(2\ell+3)}\hat{x}_{iL}\partial_t\sigma_i-\hat{x}_L\left(\sigma_{ii}-\frac{1}{2}\sigma U\right)\right\}+O(2). \tag{4.17e}$$

The transformation of Y_L into the form (4.17e) is done using the results (4.2) and (4.18) of paper I.

The quantities Ω^{00} and X^{00} can now be evaluated. As is clear from its structure and the form of φ^μ and $\mathcal{M}(V)$, $\mathcal{M}(V_i)$, the quantity Ω^{00} is made up of a sum of quadratic products of retarded waves. We shall write

$$\Omega^{00}=\frac{1}{c^6}\sum_{p,q}\hat{\partial}_P\left[\frac{1}{r}F(t-r/c)\right]\hat{\partial}_Q\left[\frac{1}{r}G(t-r/c)\right]+O(10) \tag{4.18}$$

where F and G are some functions of the retarded time symbolizing some derivatives of the functions $\mathcal{V}^L$ and $\mathcal{V}_i^L$ and W_L, X_L, Y_L, and Z_L (all indices suppressed). We assume (as can always be done) that the derivative operators are trace-free: $\hat{\partial}_P\equiv\partial_{<i_1}\partial_{i_2}\cdots\partial_{i_p>}$ and $\hat{\partial}_Q\equiv\partial_{<j_1}\partial_{j_2}\cdots\partial_{j_q>}$. The power of $1/c$ in front indicates the true order of magnitude of Ω^{00} when $c\to+\infty$ [see (3.12)], and the remainder $O(10)$ comes from the uncontrolled remainder terms in (4.14). To evaluate X^{00} we need to know the action of the operator $\mathcal{F}_{B=0}\Box_R^{-1} r^B \Box - \mathbb{1}$, where $\mathbb{1}$ denotes the unit operator, on the generic term composing Ω^{00}. Actually we shall be interested only in that part of X^{00} which is strictly larger than the remainder $O(8)$ we neglect in (3.16). In that case, a useful formula shows that all the terms in Ω^{00} which are composed of the product of two waves with multipolarities $p\geqslant 1$ *and* $q\geqslant 1$ yield negligible terms in X^{00}. This formula, which is proved in Appendix A, reads

$$(\mathcal{F}_{B=0}\Box_R^{-1} r^B \Box - \mathbb{1})\left\{\hat{\partial}_P\left[\frac{1}{r}F(t-r/c)\right]\hat{\partial}_Q\left[\frac{1}{r}G(t-r/c)\right]\right\}$$
$$= \frac{1}{c}\hat{\partial}_{PQ}\left[\frac{1}{r}(\delta_{p,0}F^{(1)}G+\delta_{0,q}FG^{(1)})\right]+O(3) \qquad (4.19)$$

($\delta_{p,q}$ is the Kronecker symbol). When both $p\geqslant 1$ and $q\geqslant 1$ the right side of (4.19) is of order $O(3)$ relatively to the left side and the corresponding term in X^{00} is of negligible order $O(9)$. We can thus limit our consideration to the terms in Ω^{00} involving at least one monopolar wave $p=0$ or $q=0$. We insert into Eq. (3.11a) the multipole expansions (3.7a) and (3.7b) together with those of $\varphi^0$and φ^i given by (4.14). By straightforward application of (4.19) to each of the resulting terms one finds

$$X^{00}=\frac{16G^2}{c^7}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{1}{r}(W^{(2)}\mathcal{V}_L - W^{(1)}\mathcal{V}_L^{(1)}\right.$$
$$\left. + \ell Y^{(1)}_{<i_\ell}\mathcal{V}_{L-1>})\right]+O(9), \qquad (4.20)$$

where the functions $W(t)$ and $Y_i(t)$ are given by (4.17a) in which $\ell=0$ and by (4.17c) in which $\ell=1$, and where we have used the law of conservation of mass implying $\mathcal{V}^{(1)}=O(2)$ and our assumption of mass-centered frame implying $\mathcal{V}_i^{(1)}=O(2)$. [See the definitions of the functions $\mathcal{V}^L$ and $\mathcal{V}_i^L$ in Eqs. (3.8a) and (3.8b); in (4.20) we denote $\mathcal{V}_L\equiv\mathcal{V}^L$ and $\mathcal{V}_{L-1}\equiv\mathcal{V}^{L-1}$ for the function (3.8a) although this notation is slightly ambiguous with (3.8b).] To lowest order the function $\mathcal{V}_L(t)$ reduces to

$$\mathcal{V}_L = Q_L + O(2), \qquad (4.21)$$

where $Q_L(t)$ is the moment defined in Eq. (4.13a). On the other hand, $W(t)$ satisfies

$$W=\frac{1}{3}\int d^3\mathbf{x}\, x_i\sigma_i + O(2) = \frac{1}{6}K^{(1)}+O(2), \qquad (4.22)$$

where $K(t)$ is the moment of inertia (4.6b). Similarly, one finds

$$Y_i = \frac{1}{5}G_i^{(1)} + O(2) \qquad (4.23a)$$

where the vector $G_i(t)$ reads

$$G_i = \int d^3\mathbf{x}\left(\sigma_i\mathbf{x}^2 - \frac{1}{2}\sigma_j x_i x_j\right). \qquad (4.23b)$$

With this notation we end up with the 2.5PN correction term δI_L [compare (3.26) and (4.20)],

$$\delta I_L = \frac{G}{c^5}\left\{\frac{2}{3}K^{(3)}Q_L - \frac{2}{3}K^{(2)}Q_L^{(1)} + \frac{4\ell}{5}G^{(2)}_{<i_\ell}Q_{L-1>}\right\}$$
$$+O(7). \qquad (4.24)$$

Summarizing the results so far, we have explicitly computed the mass multipole moment M_L given by Eq. (3.25a). It contains a 2.5PN contribution issuing from the source moment I_L and computed in Eq. (4.12), and also the direct 2.5PN modification computed in Eq. (4.24). We can write

$$M_L = \widetilde{I}_L + \Delta I_L + O(6) \qquad (4.25)$$

where $\widetilde{I}_L$ is given by Eq. (4.8a) (this was the result of paper I), and where $\Delta I_L = \delta\widetilde{I}_L + \delta I_L$ is given by

$$\Delta I_L = \frac{G}{c^5}\left\{\frac{4}{3}K^{(3)}Q_L - \frac{2}{3}K^{(2)}Q_L^{(1)}\right.$$
$$-\frac{\ell(\ell-1)}{2\ell-1}K_{<L-2}Q^{(3)}_{i_{\ell-1}i_\ell>} - \frac{4\ell}{2\ell+3}Q^{(3)}_{k<i_\ell}Q_{L-1>k}$$
$$\left.+\frac{4\ell}{5}G^{(2)}_{<i_\ell}Q_{L-1>}\right\}+O(7). \qquad (4.26)$$

We recall that the tensors Q_L, K_{L-2}, and G_i are defined in Eqs. (4.13) and (4.23b). The low orders in ℓ read

$$\Delta I = \frac{4G}{3c^5}MK^{(3)}+O(7), \qquad (4.27a)$$

$$\Delta I_i = \frac{4G}{5c^5}MG_i^{(2)}+O(7), \qquad (4.27b)$$

$$\Delta I_{ij} = \frac{G}{c^5}\left\{\frac{4}{3}K^{(3)}Q_{ij} - \frac{2}{3}K^{(2)}Q_{ij}^{(1)} - \frac{2}{3}KQ_{ij}^{(3)} - \frac{8}{7}Q^{(3)}_{k<i}Q_{j>k}\right\}$$
$$+O(7), \qquad (4.27c)$$

where M is the total mass such that $Q=M+O(2)$, and where we have used a frame such that $Q_i=O(2)$. The quadrupolar correction term (4.27c) will contribute to the asymptotic wave form at the 2.5PN order. The dipolar correction term (4.27b) will be used to determine the center of mass of the system at this order.

V. THE 2.5PN-ACCURATE GRAVITATIONAL LUMINOSITY

The mass and current multipole moments M_L and S_L are determined up to the neglect of $O(6)$ and $O(4)$ terms, respectively, and can be used to compute the gravitational luminosity (or energy loss rate) of the system at 2.5PN order. To compute the wave form at the same order would necessitate a more accurate determination of the current moments, up to the neglect of $O(5)$ terms. We shall leave this computation for future work.

Let $X^\mu=(cT,\mathbf{X})$ be a coordinate system valid in the neighborhood of future null infinity and such that the metric admits a Bondi-type expansion when $R\equiv|\mathbf{X}|\to+\infty$ with $T_R\equiv T-R/c$ staying constant. See [42] for the proof (within the present formalism) of the existence and construction of such a coordinate system. The relation between T_R and the retarded time of the harmonic coordinates $x^\mu_{\rm can}$ is

$$T_R=t_{\rm can}-\frac{r_{\rm can}}{c}-\frac{2GM}{c^3}\ln\left(\frac{r_{\rm can}}{cb}\right)+O(1/r_{\rm can})+O(5),\qquad(5.1)$$

where b is some arbitrary constant time scale. [Actually, to be consistent with the 2.5PN precision one should consider also the next-order post-Newtonian term in Eq. (5.1); however, this term will not be needed in the following.] It is sufficient to control the transverse and trace-free projection of the leading-order term $\sim R^{-1}$ in the spatial metric. A multipole decomposition yields a parametrization into two and only two sets of STF moments U_L and V_L which depend on T_R and can be referred to as the ''radiative'' or ''observable'' mass and current moments. These are chosen so that they reduce in the limit $c\to+\infty$ to the ℓth time derivatives of the ordinary Newtonian mass and current moments of the source. The total luminosity $\mathcal{L}=\mathcal{L}(T_R)$ of the gravitational wave emission when expressed in terms of U_L and V_L reads [24]

$$\mathcal{L}=\sum_{\ell=2}^{+\infty}\frac{G}{c^{2\ell+1}}\left\{\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!}U_L^{(1)}U_L^{(1)}+\frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!c^2}V_L^{(1)}V_L^{(1)}\right\}.\qquad(5.2)$$

Considering $\mathcal{L}$ to 2.5PN order one retains in (5.2) all the terms up to the neglect of a remainder $O(6)$, and finds

$$\mathcal{L}=\frac{G}{c^5}\left\{\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)}+\frac{1}{c^2}\left[\frac{1}{189}U_{ijk}^{(1)}U_{ijk}^{(1)}+\frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right]+\frac{1}{c^4}\left[\frac{1}{9072}U_{ijkm}^{(1)}U_{ijkm}^{(1)}+\frac{1}{84}V_{ijk}^{(1)}V_{ijk}^{(1)}\right]+O(6)\right\}.\qquad(5.3)$$

Because the powers of $1/c$ go by steps of two in $\mathcal{L}$, this expression is in fact the same as already used in paper I.

The only problem is to find the relations between the radiative moments U_L, V_L and the moments M_L, S_L we have previously determined. We rely on previous papers ([29,30] and paper I) having written the general form of these relations as

$$U_L(T_R)=M_L^{(\ell)}(T_R)+\sum_{n\geq 2}\frac{G^{n-1}}{c^{3(n-1)+\Sigma\underline{\ell}_i-\ell}}X_{nL}(T_R),\qquad(5.4a)$$

$$\varepsilon_{ai_\ell i_{\ell-1}}V_{aL-2}(T_R)=\varepsilon_{ai_\ell i_{\ell-1}}S_{aL-2}^{(\ell-1)}(T_R)+\sum_{n\geq 2}\frac{G^{n-1}}{c^{3(n-1)+\Sigma\underline{\ell}_i-\ell}}Y_{nL}(T_R),\qquad(5.4b)$$

where the functions X_{nL} and Y_{nL} represent some nonlinear (and in general nonlocal) functionals of the moments M_L and S_L. The powers of $1/c$ in Eqs. (5.4) come from the dimensionality of the functionals X_{nL} and Y_{nL}, which is chosen to be that of a product of n multipole moments and their time derivatives. We can write symbolically

$$X_{nL},Y_{nL}\sim M_{L_1}^{(a_1)}M_{L_2}^{(a_2)}\cdots S_{L_n}^{(a_n)}.\qquad(5.5)$$

The notation in Eqs. (5.4) and (5.5) is the same as in paper I; in particular, $\Sigma\ell_i$ denotes the total number of indices on the n moments in the term in question, and $\Sigma\underline{\ell}_i$ denotes the sum $\Sigma\ell_i+s$ where s is the number of current moments. Here we shall need only the fact that $\Sigma\underline{\ell}_i$ is larger than the multipolarity ℓ by an even positive integer $2k$ which represents the number of contracted indices between the moments composing the term (with the current moments carrying their associated Levi-Cività symbol): i.e.,

$$\sum\underline{\ell}_i=\ell+2k.\qquad(5.6)$$

With the latter equation it is simple to control the type of nonlinearities which are present in the radiative moments to 2.5PN order. Since the reasoning has already been done in paper I to 2PN order we consider only the case which is further needed, that of a term of pure 2.5PN order in the mass-type quadrupole moment U_{ij} (having $\ell=2$). By Eqs. (5.4a) and (5.6) this case corresponds to $3(n-1)+2k=5$. The only solution is $n=2$ (quadratic nonlinearity) and $k=1$ (one contraction of indices between the moments). With two moments, one contraction and $\ell=2$ one has $\underline{\ell}_1+\underline{\ell}_2=4$. Furthermore, one of the two moments is nonstatic, hence $\underline{\ell}_2\geq 2$, say, so we obtain only two possibilities, $(\underline{\ell}_1,\underline{\ell}_2)=(1,3)$ or (2,2). The first possibility is excluded because the moment having $\underline{\ell}_1=1$ is necessarily the constant mass dipole M_i which has been set to zero. There remains the second possibility $\underline{\ell}_1=\underline{\ell}_2=2$ which corresponds either to the interaction between two mass-type quadrupole moments M_{ij} or to the interaction of the (constant) current-type dipole S_i with M_{ij}.

We combine these facts with the study done in Ref. [30] of the occurrence of ''hereditary'' terms in the asymptotic metric at the quadratic approximation $n=2$. Two and only two types of hereditary terms were found: the ''tail'' terms coming from the interaction between the monopole M and nonstatic multipoles, and the nonlinear ''memory'' term which is made of the interaction between two nonstatic multipoles. The tail terms are of order c^{-3} and have been included in paper I, but the memory term arises at the order c^{-5} (in the radiative quadrupole U_{ij}). The latter term can be straightforwardly computed from Eqs. (2.42a), (2.21), and (2.11a) in Ref. [30]. An equivalent result can be found in Ref. [43]. For discussions on the memory term, see Refs. [44,45]. It is clear by the previous reasoning that the memory term represents the hereditary part of the 2.5PN contribution in the radiative moment U_{ij}, corresponding to the interaction of two moments M_{ij}. Associated with this term there are also some instantaneous terms having the same structure and exhausting (*a priori*) the possibilities of sharing time derivatives between the two moments.

Gathering these results with the results of Ref. [30] and paper I we obtain the expression of the radiative quadrupole U_{ij} to 2.5PN order as

$$U_{ij}(T_R)=M^{(2)}_{ij}(T_R)+\frac{2GM}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{\tau}{2b}\right)+\frac{11}{12}\right]M^{(4)}_{ij}\times(T_R-\tau)+\frac{G}{c^5}\Bigg\{-\frac{2}{7}\int_{-\infty}^{T_R}du M^{(3)}_{k<i}(u)M^{(3)}_{j>k}(u)+\alpha M^{(3)}_{k<i}M^{(2)}_{j>k}+\beta M^{(4)}_{k<i}M^{(1)}_{j>k}+\gamma M^{(5)}_{k<i}M_{j>k}+\lambda S_k M^{(4)}_{m<i}\varepsilon_{j>km}\Bigg\}+O(6). \quad (5.7)$$

The constant b in the term of order c^{-3} (tail integral) is the same as in Eq. (5.1). The memory term is the integral in the braces of order c^{-5}. The qualitatively different nature of these two integrals can be clearly understood when taking the limit $T_R\to+\infty$ corresponding to very late times after the system has ceased to emit radiation. In this limit the third and higher time derivatives of M_{ij} are expected to tend to zero, so the tail term tends to zero, while by contrast the memory term tends to the finite limit $-(2G/7c^5)\int_{-\infty}^{+\infty}du M^{(3)}_{k<i}M^{(3)}_{j>k}$ (see Refs. [44,45]). The coefficients α, β, γ, and λ are some purely numerical coefficients in front of instantaneous terms (which depend on T_R only). These coefficients can be obtained by a long computation using the algorithm of Ref. [27]; however, we shall not need them in the application below (they will be computed in a future work). The higher-order radiative moments take similar expressions but are needed only to a lower precision. The relevant expressions for U_{ijk} and V_{ij} have been written in paper I, and read

$$U_{ijk}(T_R)=M^{(3)}_{ijk}(T_R)+\frac{2GM}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{\tau}{2b}\right)+\frac{97}{60}\right]\times M^{(5)}_{ijk}(T_R-\tau)+O(5), \quad (5.8a)$$

$$V_{ij}(T_R)=S^{(2)}_{ij}(T_R)+\frac{2GM}{c^3}\int_0^{+\infty}d\tau\left[\ln\left(\frac{\tau}{2b}\right)+\frac{7}{6}\right]\times S^{(4)}_{ij}(T_R-\tau)+O(5), \quad (5.8b)$$

while the other needed moments are given by

$$U_{ijkm}(T_R)=M^{(4)}_{ijkm}(T_R)+O(3), \quad (5.9a)$$

$$V_{ijk}(T_R)=S^{(3)}_{ijk}(T_R)+O(3). \quad (5.9b)$$

The expressions (5.7)–(5.9) of the radiative moments are to be inserted into the gravitational luminosity (5.3). This leads to a natural (though not unique) decomposition of $\mathcal{L}$ into instantaneous and tail contributions,

$$\mathcal{L}=\mathcal{L}_{\rm inst}+\mathcal{L}_{\rm tail}. \quad (5.10)$$

The contribution $\mathcal{L}_{\rm inst}$ depends only on the instant T_R and is given by

$$\mathcal{L}_{\rm inst}=\frac{G}{c^5}\Bigg\{\frac{1}{5}M^{(3)}_{ij}M^{(3)}_{ij}+\frac{1}{c^2}\left[\frac{1}{189}M^{(4)}_{ijk}M^{(4)}_{ijk}+\frac{16}{45}S^{(3)}_{ij}S^{(3)}_{ij}\right]+\frac{1}{c^4}\left[\frac{1}{9072}M^{(5)}_{ijkm}M^{(5)}_{ijkm}+\frac{1}{84}S^{(4)}_{ijk}S^{(4)}_{ijk}\right]+\frac{2G}{5c^5}M^{(3)}_{ij}[(\alpha-2/7)M^{(3)}_{ik}M^{(3)}_{jk}+(\alpha+\beta)M^{(4)}_{ik}M^{(2)}_{jk}+(\beta+\gamma)M^{(5)}_{ik}M^{(1)}_{jk}+\gamma M^{(6)}_{ik}M_{jk}+\lambda S_k M^{(5)}_{mi}\varepsilon_{jkm}]+O(6)\Bigg\}. \quad (5.11)$$

Note that this involves the term coming from the nonlinear memory which is instantaneous in the energy loss (the memory effect exists only in the wave form). The tail contribution depends on all instants $T_R-\tau$ anterior to T_R and reads

$$\mathcal{L}_{\rm tail}=\frac{4G^2M}{c^5}\Bigg\{\frac{1}{5c^3}M^{(3)}_{ij}(T_R)\int_0^{+\infty}d\tau M^{(5)}_{ij}(T_R-\tau)\ln\left(\frac{\tau}{2b_1}\right)+\frac{1}{189c^5}M^{(4)}_{ijk}(T_R)\int_0^{+\infty}d\tau M^{(6)}_{ijk}(T_R-\tau)\ln\left(\frac{\tau}{2b_2}\right)+\frac{16}{45c^5}S^{(3)}_{ij}(T_R)\int_0^{+\infty}d\tau S^{(5)}_{ij}(T_R-\tau)\ln\left(\frac{\tau}{2b_3}\right)+O(6)\Bigg\}, \quad (5.12)$$

where we have set for simplicity

$$b_1=be^{-11/12},\quad b_2=be^{-97/60},\quad b_3=be^{-7/6}. \quad (5.13)$$

The luminosity (5.10)–(5.13) in which the moments have been determined in Sec. IV is our final result for the general case of a (semirelativistic) isolated system.

VI. APPLICATION TO INSPIRALING COMPACT BINARIES

The authors of Ref. [21] (paper II) applied the results of paper I to an inspiraling compact binary system modeled by two point masses moving on a circular orbit. Here we do the same for the results derived previously and obtain the energy loss rate and associated laws of variation of the frequency and phase of the binary to 2.5PN order.

A. The equations of motion

The equations of motion of two point masses at the 2.5PN approximation are needed for this application. These equations have been obtained in the same coordinates as used here by Damour and Deruelle [37] studying the dynamics of the binary pulsar. For inspiraling compact binaries one needs only to specialize these equations to the case of an orbit which is circular (apart from the gradual inspiral). A relevant summary of the Damour-Deruelle equations of motion is presented for the reader's convenience in Appendix B. Here we quote the results valid for circular orbits, following mostly the notation of paper II. Mass parameters are denoted by

$$m\equiv m_1+m_2,\quad X_1\equiv\frac{m_1}{m},X_2\equiv\frac{m_2}{m},\quad \nu\equiv X_1X_2 \quad (6.1)$$

(the total mass is henceforth denoted by m to conform with paper II). The individual positions of the two bodies in harmonic coordinates are y_1^i and y_2^i. Their relative separation and relative velocity are

$$x^i = y_1^i - y_2^i, \quad v^i = \frac{dx^i}{dt}. \tag{6.2}$$

A small ordering post-Newtonian parameter is defined to be

$$\gamma = \frac{Gm}{rc^2} \tag{6.3}$$

with $r=|\mathbf{x}|$. Next we use the fact that the origin of the coordinate system is located at the center of mass of the binary. This means that $M_i=0$ where M_i is the dipole mass moment of the external field. By Eqs. (4.25) and (4.27b) this means $\tilde{I}_i + \Delta I_i = O(6)$ where $\tilde{I}_i$ is the dipole moment which was computed in paper II (*before* expressing it in the relative frame) and where $\Delta I_i = (4G/5c^5) m G_i^{(2)} + O(7)$ where $G_i = \int d^3\mathbf{x}(\sigma_i \mathbf{x}^2 - \frac{1}{2}\sigma_j x_i x_j)$. For circular orbits one finds $G_i = m\nu(X_2 - X_1) r^2 v^i + O(2)$ and thus $\Delta I_i = (4G/5c^5) Gm^3 \nu (X_1 - X_2)(v^i/r) + O(7)$. This readily shows how the relations (3.7) of paper II are to be extended to 2.5PN order. We find

$$y_1^i = [X_2 + 3\nu\gamma^2 (X_1 - X_2)] x^i - \frac{4}{5} \frac{G^2 m^2 \nu}{rc^5} (X_1 - X_2) v^i + O(6), \tag{6.4a}$$

$$y_2^i = [-X_1 + 3\nu\gamma^2 (X_1 - X_2)] x^i - \frac{4}{5} \frac{G^2 m^2 \nu}{rc^5} (X_1 - X_2) v^i + O(6). \tag{6.4b}$$

This result is in agreement with the 2.5PN-accurate center of mass theorem of Refs. [37]. The assumption that the orbit is circular apart from the adiabatic inspiral due to reaction effects of order $O(5)$ implies that the scalar product of x^i and v^i is of small $O(5)$ order: $\mathbf{x}\cdot\mathbf{v} \equiv (xv) = O(5)$. By Eqs. (6.4) this implies also $(nv_1) = O(5)$ and $(nv_2) = O(5)$ where $n^i \equiv x^i/r$. These facts drastically simplify the equations of motion of the binary given by Eqs. (B1)–(B2) in Appendix B. The result when expressed in the relative frame simply reads

$$\frac{dv^i}{dt} = -\omega_{2\mathrm{PN}}^2 x^i - \frac{32}{5c^5} \frac{G^3 m^3 \nu}{r^4} v^i + O(6), \tag{6.5}$$

where we have introduced the angular frequency $\omega_{2\mathrm{PN}}$ defined by

$$\omega_{2\mathrm{PN}}^2 \equiv \frac{Gm}{r^3} \left[1 - (3-\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2 \right) \gamma^2 \right]. \tag{6.6}$$

This frequency represents the orbital frequency of the *exact* circular periodic orbit at the 2PN order [see Eq. (3.11) in paper II]. The relation between the norm of the relative velocity $v=|\mathbf{v}|$ and $\omega_{2\mathrm{PN}}$ is obtained by multiplying both sides of (6.5) by x^i. Using $(xv)=O(5)$ and $d(xv)/dt = O(10)$ [because $d(xv)/dt$ is of the same order as the square of reaction effects] we find

$$v = r\omega_{2\mathrm{PN}} + O(6). \tag{6.7}$$

Finally, we write the result for the orbital energy $E \equiv E^{2.5\mathrm{PN}}$ entering the left side of the energy balance equation $dE/dt = -\mathcal{L}^N + O(6)$ derived in Eq. (B10) of Appendix B. This energy is computed from Eqs. (B6) and (B11), in which one uses the circular orbit assumption, together with (6.7). The result is

$$E = -\frac{c^2}{2} m\nu\gamma \left\{ 1 - \frac{1}{4}(7-\nu)\gamma - \frac{1}{8}(7 - 49\nu - \nu^2)\gamma^2 \right\} + O(6). \tag{6.8}$$

There are no terms of order $\gamma^{5/2} = O(5)$ for circular orbits because the term $O(5)$ in Eq. (B11) is proportional to (nv) and thus vanishes in this case.

B. The energy loss rate

The gravitational luminosity $\mathcal{L}$ of a general source was split into two contributions, an instantaneous one $\mathcal{L}_{\mathrm{inst}}$ given by Eq. (5.11) and a tail one $\mathcal{L}_{\mathrm{tail}}$ given by Eq. (5.12). We shall basically show that only $\mathcal{L}_{\mathrm{tail}}$ contributes to the 2.5PN order in the case (but only in this case) of a binary system moving on a *circular* orbit.

Let us consider first the contribution $\mathcal{L}_{\mathrm{inst}}$. The only moment it contains which is required with full 2.5PN accuracy is the mass quadrupole moment

$$M_{ij} = \tilde{I}_{ij} + \Delta I_{ij} + O(6) \tag{6.9}$$

where $\tilde{I}_{ij}$ results from Eq. (4.8a) and ΔI_{ij} is given by Eq. (4.27c). For a circular orbit, the moment of inertia K is constant [indeed $K = m\nu r^2 + O(2)$]: hence, ΔI_{ij} reduces to two terms only:

$$\Delta I_{ij} = \frac{G}{c^5} \left\{ -\frac{2}{3} K Q_{ij}^{(3)} - \frac{8}{7} Q_{k<i}^{(3)} Q_{j>k} \right\} + O(7). \tag{6.10}$$

We prove that the contribution in $\mathcal{L}_{\mathrm{inst}}$ which is due to ΔI_{ij} is in fact zero. Indeed, this contribution is made of the contracted product between $Q_{ij}^{(3)}$ and $\Delta I_{ij}^{(3)}$ [recall that $M_{ij} = Q_{ij} + O(2)$], and hence of the contracted products $Q_{ij}^{(3)} Q_{ij}^{(6)}$ and $Q_{ij}^{(3)} Q_{ik}^{(6-q)} Q_{jk}^{(q)}$. But in the circular case an odd number of time derivatives of Q_{ij} yields a term proportional to $x^{<i}v^{j>}$ while an even number yields either $x^{<i}x^{j>}$ or $v^{<i}v^{j>}$. Thus a contracted product of the type $Q_{ij}^{(n)} Q_{ij}^{(p)}$ where $n+p$ is *odd* necessarily involves one scalar product (xv) and is thus zero; similarly, a product of the type $Q_{ij}^{(n)} Q_{ik}^{(p)} Q_{jk}^{(q)}$ where $n+p+q$ is odd is also zero. The same is true of a product like $\varepsilon_{ijk} Q_{jm}^{(r)} Q_{km}^{(s)}$ where $r+s$ is *even*. These simple facts show that (6.10) cannot contribute to the energy loss rate. Furthermore, we find that all the terms in $\mathcal{L}_{\mathrm{inst}}$ which involve the contracted products of *three* moments [i.e., all the terms of order c^{-5} in Eq. (5.11)] are also zero. Hence we can write, in the circular orbit case,

$$\mathcal{L}_{\rm inst}=\frac{G}{c^5}\left\{\frac{1}{5}\tilde{I}^{(3)}_{ij}\tilde{I}^{(3)}_{ij}+\frac{1}{c^2}\left[\frac{1}{189}\tilde{I}^{(4)}_{ijk}\tilde{I}^{(4)}_{ijk}+\frac{16}{45}\tilde{J}^{(3)}_{ij}\tilde{J}^{(3)}_{ij}\right]+\frac{1}{c^4}\left[\frac{1}{9072}\tilde{I}^{(5)}_{ijkm}\tilde{I}^{(5)}_{ijkm}+\frac{1}{84}\tilde{J}^{(4)}_{ijk}\tilde{J}^{(4)}_{ijk}\right]+O(6)\right\}. \quad (6.11)$$

Now recall that the moments $\tilde{I}_L$ and $\tilde{J}_L$ are the ones which were used as the starting point in the computation of paper II. In using paper II one must be careful that in this paper the equations of motion have a precision limited to 2PN instead of 2.5PN. We first note that the 2.5PN terms in the mass-centered frame relations (6.4) will cancel out when expressing the quadrupole mass moment in terms of the relative variables. Furthermore, during the computation of this moment in paper II the equations of motion were used only to reduce some terms which were already of 1PN order. Thus all the expressions of the moments $\tilde{I}_L$ and $\tilde{J}_L$ computed in paper II (but not the expressions of their time derivatives) can be used in the present paper without modification. Notably, the mass quadrupole $\tilde{I}_{ij}$ takes the expression (3.74) of paper II, modulo negligible $O(6)$ terms:

$$\tilde{I}_{ij}=\mathrm{STF}_{ij}\,\nu m\left\{x^{ij}\left[1-\frac{\gamma}{42}(1+39\nu)-\frac{\gamma^2}{1512}(461+18395\nu+241\nu^2)\right]+\frac{r^2}{c^2}v^{ij}\left[\frac{11}{21}(1-3\nu)+\frac{\gamma}{378}(1607-1681\nu+229\nu^2)\right]\right\}+O(6). \quad (6.12)$$

However, we need the third time derivative of (6.12) which does involve, because of the more precise equations of motion, some new terms with respect to paper II. We find

$$\tilde{I}^{(3)}_{ij}=\mathrm{STF}_{ij}\,\nu m\left\{-8\frac{Gm}{r^3}x^iv^j\left[1-\frac{\gamma}{42}(149-69\nu)+\frac{\gamma^2}{1512}(7043-7837\nu+3703\nu^2)\right]+\frac{64}{5}\frac{G^3m^3\nu}{r^4c^5}\left[\frac{Gm}{r^3}x^{ij}-3v^{ij}\right]\right\}+O(6). \quad (6.13)$$

But for the same reason as before these new terms will not contribute to the energy loss for circular orbits. Thus we conclude that $\mathcal{L}_{\rm inst}$ is exactly given by the expression found in Eq. (4.12) of paper II modulo negligible terms of relative order $O(6)\equiv O(\gamma^3)$. We can thus write

$$\mathcal{L}_{\rm inst}=\frac{32c^5}{5G}\nu^2\gamma^5\left\{1-\left(\frac{2927}{336}+\frac{5}{4}\nu\right)\gamma+\left(\frac{293383}{9072}+\frac{380}{9}\nu\right)\gamma^2+O(\gamma^3)\right\}. \quad (6.14)$$

Let us now turn our attention to the tail part of the gravitational luminosity as defined by Eq. (5.12). The correct formula to compute the tail integrals in $\mathcal{L}_{\rm tail}$ is

$$\int_0^{+\infty}d\tau\ln\left(\frac{\tau}{2b}\right)\cos(\Omega\tau)=-\frac{\pi}{2\Omega}, \quad (6.15)$$

where Ω denotes the (real) angular frequency of the radiation. It was proved in Ref. [46] that this formula is to be applied as it stands (i.e., even though the integral is not absolutely convergent) to a fixed (nondecaying) orbit whose frequency is equal to the current value of the orbital frequency $\omega\equiv\omega_{\rm 2PN}(T_R)$. The numerical errors made in applying this formula were shown to be of relative order $O(\ln c/c^5)$, which is always negligible for our purpose here (because of the explicit powers of c^{-1} in front of the tail integrals). Thus we replace in the integrand of (5.12) the moments M_L and S_L by their expressions valid for fixed circular orbits and up to the appropriate precision. The necessary formulas (taken from paper II) are

$$M^{(3)}_{ij}=-8\,\mathrm{STF}_{ij}\frac{Gm^2\nu}{r^3}x^iv^j\left[1-\frac{\gamma}{42}(149-69\nu)\right]+O(4), \quad (6.16a)$$

$$M^{(5)}_{ij}=32\,\mathrm{STF}_{ij}\frac{G^2m^3\nu}{r^6}x^iv^j\left[1-\frac{\gamma}{42}(275-111\nu)\right]+O(4), \quad (6.16b)$$

$$M^{(4)}_{ijk}=3(X_2-X_1)\mathrm{STF}_{ijk}\frac{Gm^2\nu}{r^3}\left\{7\frac{Gm}{r^3}x^{ijk}-20x^iv^{jk}\right\}+O(2), \quad (6.16c)$$

$$M^{(6)}_{ijk}=3(X_2-X_1)\mathrm{STF}_{ijk}\frac{G^2m^6\nu}{r^6}\left\{-61\frac{Gm}{r^3}x^{ijk}+182x^iv^{jk}\right\}+O(2), \quad (6.16d)$$

$$S^{(3)}_{ij}=-(X_2-X_1)\mathrm{STF}_{ij}\frac{Gm^2\nu}{r^3}\varepsilon^{abi}v^jx^av^b+O(2), \quad (6.16e)$$

$$S^{(5)}_{ij}=(X_2-X_1)\mathrm{STF}_{ij}\frac{G^2m^3\nu}{r^6}\varepsilon^{abi}v^jx^av^b+O(2). \quad (6.16f)$$

The quadrupole moment includes 1PN terms while the other moments are Newtonian. The computation of $\mathcal{L}_{\rm tail}$ involves many scalar products between x^i or v^i given at the current time T_R, i.e., $x^i\equiv x^i(T_R)$ or $v^i\equiv v^i(T_R)$, with the same vectors but evaluated at an arbitrary earlier time $T_R-\tau$, say, $x'^i\equiv x^i(T_R-\tau)$ and $v'^i\equiv v^i(T_R-\tau)$. Relevant formulas for these scalar products are

$$(xx')=r^2\cos(\omega\tau), \quad (6.17a)$$

$$(vv')=r^2\omega^2\cos(\omega\tau), \quad (6.17b)$$

$$(xv')=-(vx')=r^2\omega\sin(\omega\tau), \quad (6.17c)$$

where ω is the current value of the frequency. The reduction of $\mathcal{L}_{\rm tail}$ is quite straightforward. We need to use $\omega^2=(Gm/r^3)[1-(3-\nu)\gamma+O(\gamma^2)]$ (the 1PN correction in ω^2 is sufficient) and the formula (6.15) where $\Omega=\omega$, 2ω, or 3ω. One uses also $(X_2-X_1)^2=1-4\nu$. The result reads

$$\mathcal{L}_{\rm tail}=\frac{32c^5}{5G}\nu^2\gamma^5\left\{4\pi\gamma^{3/2}-\left(\frac{25663}{672}+\frac{109}{8}\nu\right)\pi\gamma^{5/2}+O(\gamma^3)\right\}. \tag{6.18}$$

The complete 2.5PN-accurate gravitational luminosity generated by an inspiraling compact binary moving on a quasicircular orbit is therefore obtained as the sum of Eqs. (6.14) and (6.18). We obtain

$$\mathcal{L}=\frac{32c^5}{5G}\nu^2\gamma^5\left\{1-\left(\frac{2927}{336}+\frac{5}{4}\nu\right)\gamma+4\pi\gamma^{3/2}+\left(\frac{293383}{9072}+\frac{380}{9}\nu\right)\gamma^2-\left(\frac{25663}{672}+\frac{109}{8}\nu\right)\pi\gamma^{5/2}+O(\gamma^3)\right\}, \tag{6.19}$$

where we recall that the post-Newtonian ordering parameter is $\gamma=Gm/rc^2$ with r being the harmonic-coordinate orbital separation. This expression was already obtained to 1PN order in Refs. [23,47,48], to 1.5PN order in Refs. [30,13,49,46], and to 2PN order in Refs. [20–22,12]. The 2.5PN order added in this paper is like the 1.5PN order due to the presence of the radiation tails in the wave zone, as indicated by the irrational number π in factor coming from the formula (6.15). Hence there is up to 2.5PN order a clean separation between the integer post-Newtonian approximations which come from instantaneous relativistic effects in the source multipole moments and the half-integer approximations which are due to hereditary effects in the wave zone. However, the next post-Newtonian approximation (3PN) is expected to involve both types of effects.

C. The orbital phase

Once the energy loss has been derived in a particular coordinate system it can be re-expressed in a coordinate-independent way by using the directly observable frequency $\omega\equiv\omega_{2\rm PN}$ instead of the parameter γ. Defining $x=(Gm\omega/c^3)^{2/3}$, we obtain the inverse of Eq. (6.6) as

$$\gamma=x\left[1+\left(1-\frac{\nu}{3}\right)x+\left(1-\frac{65\nu}{12}\right)x^2+O(x^3)\right], \tag{6.20}$$

which is substituted into Eq. (6.19) with the result

$$\mathcal{L}=\frac{32c^5}{5G}\nu^2x^5\left\{1-\left(\frac{1247}{336}+\frac{35}{12}\nu\right)x+4\pi x^{3/2}+\left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)x^2-\left(\frac{8191}{672}+\frac{535}{24}\nu\right)\pi x^{5/2}+O(x^3)\right\}. \tag{6.21}$$

An important check is obtained in the test-body limit $\nu\to 0$ of this result, which is found to agree with the 2.5PN truncation of the result of perturbation theory known through the equivalent of 4PN order [15–17] [see Eq. (43) of Ref. [17]].

As for the orbital energy $E\equiv E^{2.5\rm PN}$ which has been given in Eq. (6.8) following the equations of motion summarized in Appendix B, it reads in terms of x as

$$E=-\frac{c^2}{2}m\nu x\left\{1-\frac{1}{12}(9+\nu)x-\frac{1}{8}\left(27-19\nu+\frac{\nu^2}{3}\right)x^2+O(x^3)\right\}. \tag{6.22}$$

The laws of variation of the frequency ω and phase ϕ of the binary during the orbital decay will be computed using the energy balance equation

$$\frac{dE}{dT_R}=-\mathcal{L}, \tag{6.23}$$

where T_R is the retarded time in the wave zone. Note that we meet here the remarks made in the Introduction. Up to now the analysis has been rigorous within the framework of post-Newtonian theory; namely, $\mathcal{L}$ has been computed from a well-defined formalism based on convergent integrals for the multipole moments (see Sec. IV). The reduction of $\mathcal{L}$ to binary systems using δ functions to describe the compact objects can probably be justified at the 2.5PN order by using the results of Ref. [11] (see paper II). Furthermore, E is computed directly from the Damour-Deruelle equations of motion [37]. We now *postulate* the validity of the energy balance equation (6.23) where both E and $\mathcal{L}$ take their full 2.5PN accuracy. This equation has been proved here [in Eq. (B9) of Appendix B] only when $\mathcal{L}$ takes its Newtonian value $\mathcal{L}^N$. It was proved before to Newtonian order in Refs. [32–37]. On the other hand, the balance equation is also known to hold at 1PN order [38,39], and even for tails at 1.5PN order [40,30]. To prove it at 2.5PN order such as in (6.23) would mean knowing all the radiation reaction effects in the binary's equations of motion up to 5PN order. To palliate this it has been customary in this field (see Refs. [7–9,13,14,18,19,12,21]) to compute the binary's orbital decay assuming that the energy balance equation is valid to high order (the angular momentum balance equation is unnecessary for circular orbits).

Adopting the same approach to the problem, we introduce as in paper II the adimensional time

$$\theta=\frac{c^3\nu}{5Gm}T_R. \tag{6.24}$$

Equation (6.23) is then transformed into the ordinary differential equation

$$\frac{dx}{d\theta}=64x^5\left\{1-\left(\frac{743}{336}+\frac{11}{4}\nu\right)x+4\pi x^{3/2}+\left(\frac{34103}{18144}+\frac{13661}{2016}\nu+\frac{59}{18}\nu^2\right)x^2-\left(\frac{4159}{672}+\frac{173}{8}\nu\right)\pi x^{5/2}+O(x^3)\right\}. \tag{6.25}$$

On the other hand, the differential equation for the instantaneous phase ϕ of the binary is $d\phi = \omega dT_R = (5/\nu) x^{3/2} d\theta$ or, equivalently,

$$\frac{d\phi}{dx} = \frac{5}{64\nu} x^{-7/2} \left\{ 1 + \left(\frac{743}{336} + \frac{11}{4}\nu \right) x - 4\pi x^{3/2} + \left(\frac{3\,058\,673}{1\,016\,064} + \frac{5429}{1008}\nu + \frac{617}{144}\nu^2 \right) x^2 - \left(\frac{7729}{672} + \frac{3}{8}\nu \right) \pi x^{5/2} + O(x^3) \right\}. \tag{6.26}$$

The frequency and phase follow from the integration of these equations. The frequency (or x parameter) reads

$$x = \frac{1}{4} \Theta^{-1/4} \left\{ 1 + \left(\frac{743}{4032} + \frac{11}{48}\nu \right) \Theta^{-1/4} - \frac{\pi}{5} \Theta^{-3/8} + \left(\frac{19\,583}{254\,016} + \frac{24\,401}{193\,536}\nu + \frac{31}{288}\nu^2 \right) \Theta^{-1/2} + \left(-\frac{11\,891}{53\,760} + \frac{29}{1920}\nu \right) \pi \Theta^{-5/8} + O(\Theta^{-3/4}) \right\}, \tag{6.27}$$

where $\Theta = \theta_c - \theta$ denotes the (adimensional) time left till the final coalescence, and the phase is

$$\phi = -\frac{1}{32\nu} \left\{ x^{-5/2} + \left(\frac{3715}{1008} + \frac{55}{12}\nu \right) x^{-3/2} - 10\pi x^{-1} + \left(\frac{15\,293\,365}{1\,016\,064} + \frac{27\,145}{1\,008}\nu + \frac{3085}{144}\nu^2 \right) x^{-1/2} + \left(\frac{38\,645}{1\,344} + \frac{15}{16}\nu \right) \pi \ln\left(\frac{x}{x_0} \right) + O(x^{1/2}) \right\}, \tag{6.28}$$

where the constant x_0 is determined by initial conditions. By substituting Eq. (6.27) into (6.28) one obtains the phase as a function of time:

$$\phi = -\frac{1}{\nu} \left\{ \Theta^{5/8} + \left(\frac{3715}{8064} + \frac{55}{96}\nu \right) \Theta^{3/8} - \frac{3\pi}{4} \Theta^{1/4} + \left(\frac{9\,275\,495}{14\,450\,688} + \frac{284\,875}{258\,048}\nu + \frac{1\,855}{2\,048}\nu^2 \right) \Theta^{1/8} - \left(\frac{38\,645}{172\,032} + \frac{15}{2\,048}\nu \right) \pi \ln\left(\frac{\Theta}{\Theta_0} \right) + O(\Theta^{-1/8}) \right\}, \tag{6.29}$$

where Θ_0 denotes some constant. Note the presence of the logarithm of the frequency or coalescing time in the phase at the 2.5PN approximation. The expressions (6.25)–(6.29) are valid only in the post-Newtonian regime.

Let us evaluate the contribution of each post-Newtonian term in the phase (6.28) to the accumulated number $\mathcal{N}$ of gravitational-wave cycles between some initial and final frequencies ω_i and ω_f. Note that such a computation can only be indicative of the relative orders of magnitude of the different terms in the phase. A full analysis would require the knowledge of the power spectral density of the noise in a detector, and a complete simulation of the parameter estimation using matched filtering [7–9,50,51]. The contribution to $\mathcal{N}$ due to the 2.5PN term in (6.28) is

$$\mathcal{N}_{2.5\text{PN}} = -\frac{1}{48\nu} \left(\frac{38\,645}{1344} + \frac{15}{16}\nu \right) \ln\left(\frac{\omega_f}{\omega_i} \right). \tag{6.30}$$

In the case of the inspiral of two neutron stars of mass $1.4M_\odot$, and with the values $\omega_i/\pi = 10$ Hz (set by seismic noise) and $\omega_f/\pi = 1000$ Hz (set by photon shot noise), we find, from Ref. [12] and Eq. (6.30)

	Newtonian	1PN	1.5PN	2PN	2.5PN
$\mathcal{N}$	16 050	439	−208	9	−11

[Note that as long as the final frequency ω_f is determined only by the detector's characteristics (and not by the location of the innermost stable orbit), the term $\mathcal{N}_{2.5\text{PN}}$ depends on the masses only through the mass ratio ν.]

In this (indicative) example we see that the contribution of the 2.5PN order more than compensates the contribution of the previous 2PN order. (But of course both contributions have to be included in the filters since they have different functional dependences on the frequency.) However, note that finite mass effects (proportional to ν) are numerically small at the 2.5PN order, contrary to the 2PN order where they are quite significant [21,12]. In the previous example, they contribute only to -0.1 cycle as compared to $\mathcal{N}_{2.5\text{PN}} = -11$.

APPENDIX A: THE PROOF OF A FORMULA

To prove the formula (4.19) we begin with the identity

$$r^B \Box f = \Box(r^B f) - B(B+1) r^{B-2} f - 2B r^{B-1} \partial_r f \tag{A1}$$

which permits the transformation of the left side of (4.19), namely,

$${}_P X_Q \equiv (\mathcal{F}_{B=0} \Box_R^{-1} r^B \Box - \mathbb{1}) \left\{ \hat{\partial}_P \left[\frac{1}{r} F(t - r/c) \right] \times \hat{\partial}_Q \left[\frac{1}{r} G(t - r/c) \right] \right\}, \tag{A2}$$

into

$${}_P X_Q = \mathcal{F}_{B=0} \Box_R^{-1} \left(-B r^{B-1} [2\partial_r + r^{-1}] \times \left\{ \hat{\partial}_P \left[\frac{1}{r} F(t - r/c) \right] \hat{\partial}_Q \left[\frac{1}{r} G(t - r/c) \right] \right\} \right). \tag{A3}$$

Because of the presence of the explicit factor B in the integrand, the equation (A3) appears to be the *residue* of the

Laurent expansion of the retarded integral near $B=0$ (we have discarded a term proportional to B^2 which is known to give zero contribution). Residues of retarded integrals have been investigated in Eq. (4.26) of Ref. [40] and yield retarded solutions of the wave equation. By a simple dimensional analysis we find that ${}_PX_Q$ admits the structure

$$ {}_PX_Q \sim \sum_{k=0}^{[(p+q)/2]} \frac{1}{c^{2k+1}} \hat{\partial}_L \left[\frac{1}{r} F^{(a)}(t-r/c) G^{(b)}(t-r/c) \right]. \tag{A4} $$

Here, $[(p+q)/2]$ is the integer part of $(p+q)/2$, $\hat{\partial}_L$ denotes a trace-free derivative operator composed of ℓ spatial derivatives where $\ell = p+q-2k$, and the number of time derivatives on F and G is $a+b=2k+1$. Omitted in (A4) are pure dimensionless coefficients and numerous Kronecker symbols. The explicit dependence on $1/c$ present in Eq. (A4) shows that if we are interested only in the leading-order term when $c \to +\infty$ we can limit ourselves to the computation of the terms having the *maximum* number of space derivatives, i.e., $\ell_{\max}=p+q$ (corresponding to $k=0$). The errors made in keeping only such terms are of relative order $O(3)$. We use to compute (A3) the formula

$$ \hat{\partial}_P \left[\frac{1}{r} F(t-r/c) \right] = \hat{n}_P \sum_{i=0}^{p} a_i^p \frac{F^{(p-i)}(t-r/c)}{c^{p-i} r^{1+i}}, \tag{A5} $$

where the numerical coefficient is

$$ a_i^p = \frac{(-)^p (p+i)!}{2^i i! (p-i)!} \tag{A6} $$

[see Eq. (A35a) in [27]]. This yields a double sum of terms having in front the product of the trace-free tensors $\hat{n}_P$ and $\hat{n}_Q$. This product can be decomposed into trace-free tensors of multipolarities $\ell = p+q-2k$ with $k=0,\dots,[(p+q)/2]$; however, if we are looking only for the term having $\ell_{\max}=p+q$ one can simply replace the product $\hat{n}_P \hat{n}_Q$ by the trace-free tensor $\hat{n}_{PQ}$ (with 1 as a coefficient in front). Hence

$$ {}_PX_Q = \sum_{i,j} a_i^p a_j^q \mathcal{F}_{B=0} \Box_R^{-1} \Bigg\{ -B r^{B-1} \hat{n}_{PQ} [2\partial_r + r^{-1}] \times \frac{F^{(p-i)}(t-r/c) G^{(q-j)}(t-r/c)}{c^{p+q-i-j} r^{2+i+j}} \Bigg\} + O(3), \tag{A7} $$

where the remainder $O(3)$ comes from the errors made in the replacement $\hat{n}_P \hat{n}_Q \to \hat{n}_{PQ}$. The integrand in (A7) is easily transformed into a sum of terms of the form $r^{B-K} \hat{n}_{PQ} H(t-r/c)$ times the factor B, where K is an integer. Using the fact that the residues of such terms exist only when $K \geqslant p+q+3$ [see Eq. (4.26) in [40]], one finds that the summation indices i and j in (A7) must satisfy $i+j=p+q$ or $p+q-1$. This greatly simplifies the computation of (A7) which is done using the expression (A6) of the coefficients a_i^p and Eq. (4.26) in [40]. The result is

$$ {}_PX_Q = \hat{\partial}_{PQ} \left[\frac{1}{rc} (\delta_{p,0} F^{(1)} G + \delta_{0,q} F G^{(1)}) \right] + O(3), \tag{A8} $$

where $\delta_{p,0}$ denotes the usual Kronecker symbol.

APPENDIX B: THE DAMOUR-DERUELLE EQUATIONS OF MOTION

We present a summary of results concerning the equations of motion of two point masses moving under their mutual gravitational influence up to the 2.5PN order. These equations were obtained by Damour and Deruelle [37] (see also the presentation [11] to which we are close here). The equations of motion of body 1 (say) take the Newtonian-like form

$$ \frac{dy_1^i}{dt} = v_1^i, \tag{B1a} $$

$$ \frac{dv_1^i}{dt} = A_1^i + \frac{1}{c^2} B_1^i + \frac{1}{c^4} C_1^i + \frac{1}{c^5} D_1^i + O(6), \tag{B1b} $$

where y_1^i and v_1^i denote the instantaneous position and coordinate velocity of body 1 (in the harmonic coordinate system), A_1^i is the usual Newtonian acceleration of the body, and B_1^i, C_1^i, and D_1^i represent the relativistic corrections of order 1PN, 2PN, and 2.5PN, respectively. The equations (12.7)–(12.12) in Ref. [11] give these terms as

$$ A_1^i = -\frac{Gm_2}{r^2} n^i, \tag{B2a} $$

$$ B_1^i = \frac{Gm_2}{r^2} \Bigg\{ n^i \left[-v_1^2 - 2v_2^2 + 4(v_1 v_2) + \frac{3}{2}(nv_2)^2 + 5\frac{Gm_1}{r} + 4\frac{Gm_2}{r} \right] + (v_1^i - v_2^i)[4(nv_1) - 3(nv_2)] \Bigg\}, \tag{B2b} $$

$$C_1^i=\frac{Gm_2}{r^2}\Bigg\{n^i\Bigg[-2v_2^4+4v_2^2(v_1v_2)-2(v_1v_2)^2+\frac{3}{2}v_1^2(nv_2)^2+\frac{9}{2}v_2^2(nv_2)^2-6(v_1v_2)(nv_2)^2-\frac{15}{8}(nv_2)^4+\frac{Gm_1}{r}\Bigg(-\frac{15}{4}v_1^2$$
$$+\frac{5}{4}v_2^2-\frac{5}{2}(v_1v_2)+\frac{39}{2}(nv_1)^2-39(nv_1)(nv_2)+\frac{17}{2}(nv_2)^2\Bigg)+\frac{Gm_2}{r}(4v_2^2-8(v_1v_2)+2(nv_1)^2-4(nv_1)(nv_2)$$
$$-6(nv_2)^2)\Bigg]+(v_1^i-v_2^i)\Bigg[v_1^2(nv_2)+4v_2^2(nv_1)-5v_2^2(nv_2)-4(v_1v_2)(nv_1)+4(v_1v_2)(nv_2)-6(nv_1)(nv_2)^2+\frac{9}{2}(nv_2)^3$$
$$+\frac{Gm_1}{r}\Bigg(-\frac{63}{4}(nv_1)+\frac{55}{4}(nv_2)\Bigg)+\frac{Gm_2}{r}(-2(nv_1)-2(nv_2))\Bigg]\Bigg\}+\frac{G^3m_2}{r^4}n^i\Bigg\{-\frac{57}{4}m_1^2-9m_2^2-\frac{69}{2}m_1m_2\Bigg\}, \tag{B2c}$$

$$D_1^i=\frac{4}{5}\frac{G^2m_1m_2}{r^3}\Bigg\{v^i\Bigg[-v^2+2\frac{Gm_1}{r}-8\frac{Gm_2}{r}\Bigg]+n^i(nv)\Bigg[3v^2-6\frac{Gm_1}{r}+\frac{52}{3}\frac{Gm_2}{r}\Bigg]\Bigg\}, \tag{B2d}$$

where m_1 and m_2 are the two masses, v_1^i and v_2^i the two velocities, $r=|\mathbf{y}_1-\mathbf{y}_2|$ the separation, and $n^i=(y_1^i-y_2^i)/r$. Scalar products are denoted by, e.g., $\mathbf{v}_1\cdot\mathbf{v}_2=(v_1v_2)$. In the last equation $v^i=v_1^i-v_2^i$. The equations of the second body are obtained by exchanging the labels 1 and 2, being careful of the fact that n^i and v^i change sign in the exchange. The acceleration term D_1^i is responsible for the dominant damping or radiation reaction effects in the dynamics of the binary.

The equations of motion when truncated to 2PN order (neglecting the damping term D_1^i) admit a Lagrangian formulation and the associated conservations laws [37]. The 2PN Lagrangian in harmonic coordinates depends not only on the positions and velocities of the two bodies, but also on their accelerations $a_{1,2}^i=dv_{1,2}^i/dt$: $L^{2\mathrm{PN}}=L^{2\mathrm{PN}}(y,v,a)$. It is given by

$$L^{2\mathrm{PN}}(y,v,a)=\sum\Bigg\{\frac{1}{2}m_1v_1^2+\frac{1}{2}\frac{Gm_1m_2}{r}+\frac{1}{8}m_1v_1^4+\frac{Gm_1m_2}{r}\Bigg[\frac{3}{2}v_1^2-\frac{7}{4}(v_1v_2)-\frac{1}{4}(nv_1)(nv_2)-\frac{1}{2}\frac{Gm_1}{r}\Bigg]+\frac{1}{16}m_1v_1^6$$
$$+\frac{Gm_1m_2}{r}\Bigg[\frac{7}{8}v_1^4+\frac{15}{16}v_1^2v_2^2-2v_1^2(v_1v_2)+\frac{1}{8}(v_1v_2)^2-\frac{7}{8}(nv_1)^2v_2^2+\frac{3}{4}(nv_1)(nv_2)(v_1v_2)+\frac{3}{16}(nv_1)^2(nv_2)^2\Bigg]$$
$$+\frac{G^2m_1^2m_2}{r^2}\Bigg[\frac{1}{4}v_1^2+\frac{7}{4}v_2^2-\frac{7}{4}(v_1v_2)+\frac{7}{2}(nv_1)^2+\frac{1}{2}(nv_2)^2-\frac{7}{2}(nv_1)(nv_2)\Bigg]+Gm_1m_2\Bigg[(na_1)\Bigg(\frac{7}{8}v_2^2$$
$$-\frac{1}{8}(nv_2)^2\Bigg)-\frac{7}{4}(v_2a_1)(nv_2)\Bigg]\Bigg\}+\frac{G^3m_1m_2}{r^3}\Bigg[\frac{1}{2}m_1^2+\frac{1}{2}m_2^2+\frac{19}{4}m_1m_2\Bigg]. \tag{B3}$$

The summation symbol runs over the two bodies 1 and 2. The equations obtained by variation of this Lagrangian, and in which the accelerations are replaced *after* variation by their (Newtonian) values, are equivalent to the 2PN equations of motion [Eqs. (B1) and (B2) where $D_1^i=0$] modulo negligible $O(6)$ terms. From this Lagrangian one constructs the 2PN integral of energy,

$$\tilde{E}^{2\mathrm{PN}}=\sum\Bigg\{v_1^i\Bigg(\frac{\partial L^{2\mathrm{PN}}}{\partial v_1^i}-\frac{d}{dt}\frac{\partial L^{2\mathrm{PN}}}{\partial a_1^i}\Bigg)+a_1^i\frac{\partial L^{2\mathrm{PN}}}{\partial a_1^i}\Bigg\}-L^{2\mathrm{PN}}, \tag{B4}$$

and one replaces in it the accelerations by their (Newtonian) values, resulting in

$$\tilde{E}^{2\mathrm{PN}}(y,v,a)=E^{2\mathrm{PN}}(y,v)+O(6). \tag{B5}$$

The explicit expression of the energy $E^{2\mathrm{PN}}$ as a function of the positions and velocities is

$$E^{2\mathrm{PN}}(y,v)=\sum\Bigg\{\frac{1}{2}m_1v_1^2-\frac{1}{2}\frac{Gm_1m_2}{r}+\frac{3}{8c^2}m_1v_1^4+\frac{Gm_1m_2}{c^2r}\Bigg[\frac{3}{2}v_1^2-\frac{7}{4}(v_1v_2)-\frac{1}{4}(nv_1)(nv_2)+\frac{1}{2}\frac{Gm_1}{r}\Bigg]+\frac{5}{16c^4}m_1v_1^6$$
$$+\frac{Gm_1m_2}{c^4r}\Bigg[\frac{21}{8}v_1^4+\frac{31}{16}v_1^2v_2^2-\frac{55}{8}v_1^2(v_1v_2)+\frac{17}{8}(v_1v_2)^2-\frac{13}{8}(nv_1)^2v_2^2-\frac{9}{8}(nv_1)(nv_2)v_2^2+\frac{3}{4}(nv_1)(nv_2)(v_1v_2)$$
$$+\frac{13}{8}(v_1v_2)(nv_2)^2+\frac{3}{16}(nv_1)^2(nv_2)^2+\frac{3}{8}(nv_1)(nv_2)^3\Bigg]+\frac{G^2m_1^2m_2}{c^4r^2}\Bigg[-\frac{3}{2}v_1^2+\frac{7}{4}v_2^2+\frac{29}{4}(nv_1)^2+\frac{1}{2}(nv_2)^2$$
$$-\frac{13}{4}(nv_1)(nv_2)\Bigg]\Bigg\}-\frac{G^3m_1m_2}{c^4r^3}\Bigg[\frac{1}{2}m_1^2+\frac{1}{2}m_2^2+\frac{19}{4}m_1m_2\Bigg]. \tag{B6}$$

The fact that $\tilde{E}^{2\mathrm{PN}}$ is the integral of energy of the 2PN equations of motion implies that $E^{2\mathrm{PN}}$ as a function of the positions and velocities satisfies the identity

$$\sum \left\{ v_1^i \frac{\partial E^{2\mathrm{PN}}}{\partial y_1^i} + \left[A_1^i + \frac{1}{c^2} B_1^i + \frac{1}{c^4} C_1^i \right] \frac{\partial E^{2\mathrm{PN}}}{\partial v_1^i} \right\} \equiv O(6), \tag{B7}$$

from which we find the law of variation of $E^{2\mathrm{PN}}$ under the complete 2.5PN dynamics of the binary: namely,

$$\frac{dE^{2\mathrm{PN}}}{dt} = \sum \frac{1}{c^5} D_1^i \frac{\partial E^{2\mathrm{PN}}}{\partial v_1^i} + O(6). \tag{B8}$$

This equation makes it clear how the damping acceleration term D_1^i drives the variation of the energy [see Eq. (15.3) in Ref. [11]]. Since the right side of (B8) is a small post-Newtonian term it can be evaluated by inserting in place of $E^{2\mathrm{PN}}$ its Newtonian value given by the first two terms in (B6). Thus

$$\frac{dE^{2\mathrm{PN}}}{dt} = \sum \frac{1}{c^5} m_1 v_1^i D_1^i + O(6). \tag{B9}$$

Using the expression (B2d) of the damping term we arrive after a short calculation at the balancelike equation

$$\frac{dE^{2.5\mathrm{PN}}}{dt} = -\mathcal{L}^{\mathrm{N}} + O(6), \tag{B10}$$

which involves in the left side the 2PN energy (B6) augmented by a pure 2.5PN term, namely,

$$E^{2.5\mathrm{PN}} = E^{2\mathrm{PN}} + \frac{8}{5c^5} \frac{G^2 m^3 \nu^2}{r^2} (nv) v^2, \tag{B11}$$

and in the right side the expression

$$\mathcal{L}^{\mathrm{N}} = \frac{8}{15c^5} \frac{G^3 m^4 \nu^2}{r^4} \big(12 v^2 - 11 (nv)^2\big). \tag{B12a}$$

It is simple to rewrite the latter expression in the well-known form

$$\mathcal{L}^{\mathrm{N}} = \frac{G}{5c^5} Q_{ij}^{(3)} Q_{ij}^{(3)} + O(7). \tag{B12b}$$

As this is the standard (Newtonian) quadrupole formula we conclude that Eq. (B10) proves the energy balance between the loss of orbital energy $E^{2.5\mathrm{PN}}$ of the binary and the Newtonian energy flux carried out by the gravitational waves. Note that it is important that the energy which enters the left side of the balance equation is given as an *instantaneous* functional of the two world lines of the binary (see Damour [11] for a discussion).

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