Generalized Killing equations and Taub-NUT spinning space

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The generalized Killing equations for the configuration space of spinning particles (spinning space) are analyzed. Simple solutions of the homogeneous part of these equations are expressed in terms of Killing-Yano tensors. The general results are applied to the case of the four-dimensional Euclidean Taub-NUT manifold. [S0556-2821(96)05512-9]

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I. SPINNING PARTICLES AND CONSTANTS OF MOTION

The pseudoclassical limit of the Dirac theory of a spin 1/2 fermion in curved space-time is described by the supersymmetric extension of the usual relativistic point particle [1]. The configuration space of spinning particles (spinning space) is an extension of an ordinary Riemannian manifold, parametrized by local coordinates $\{x^\mu\}$, to a graded manifold parametrized by local coordinates $\{x^\mu,\psi^\mu\}$, with the first set of variables being Grassmann even (commuting) and the second set Grassmann odd (anticommuting). The equation of motion of a spinning particle on a geodesic is derived from the action

$$S=\int d\tau\left(\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu+\frac{i}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{D\tau}\right). \tag{1}$$

The corresponding world-line Hamiltonian is given by

$$H=\tfrac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu, \tag{2}$$

where $\Pi_\mu=g_{\mu\nu}\dot{x}^\nu$ is the covariant momentum.

For any constant of motion $\mathcal{J}(x,\Pi,\psi)$, the brackets with H vanish, $\{H,\mathcal{J}\}=0$, with the Poisson-Dirac brackets for functions of the covariant phase-space variables (x,Π,ψ) defined by

$$\{F,G\}=\mathcal{D}_\mu F\frac{\partial G}{\partial\Pi_\mu}-\frac{\partial F}{\partial\Pi_\mu}\mathcal{D}_\mu G-\mathcal{R}_{\mu\nu}\frac{\partial F}{\partial\Pi_\mu}\frac{\partial G}{\partial\Pi_\nu}+i(-1)^{a_F}\frac{\partial F}{\partial\psi^\mu}\frac{\partial G}{\partial\psi_\mu}, \tag{3}$$

where the notation used is

$$\mathcal{D}_\mu F=\partial_\mu F+\Gamma^\lambda_{\mu\nu}\Pi_\lambda\frac{\partial F}{\partial\Pi_\nu}-\Gamma^\lambda_{\mu\nu}\psi^\nu\frac{\partial F}{\partial\psi^\lambda},$$
$$\mathcal{R}_{\mu\nu}=\frac{i}{2}\psi^\rho\psi^\sigma R_{\rho\sigma\mu\nu}, \tag{4}$$

and a_F is the Grassmann parity of F: $a_F=(0,1)$ for $F=$(even,odd).

If we expand $\mathcal{J}(x,\Pi,\psi)$ in a power series in the canonical momentum

$$\mathcal{J}=\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{J}^{(n)\mu_1\cdots\mu_n}(x,\psi)\Pi_{\mu_1}\cdots\Pi_{\mu_n}, \tag{5}$$

then the brackets $\{H,\mathcal{J}\}$ vanish for arbitrary Π_μ if, and only if, the components of $\mathcal{J}$ satisfy the generalized Killing equations [1]:

$$\mathcal{J}^{(n)}_{(\mu_1\cdots\mu_n;\mu_{n+1})}+\frac{\partial\mathcal{J}^{(n)}_{(\mu_1\cdots\mu_n}}{\partial\psi^\sigma}\Gamma^\sigma_{\mu_{n+1})\lambda}\psi^\lambda=\frac{i}{2}\psi^\rho\psi^\sigma R_{\rho\sigma\nu(\mu_{n+1}}\mathcal{J}^{(n+1)\nu}{}_{\mu_1\ldots\mu_n)}, \tag{6}$$

where the parentheses denote full symmetrization over the indices enclosed.

The solutions of the generalized Killing equations (6) can be divided into two clases [2,3]: *generic* ones, which exist for any spinning particle model (1) and *nongeneric* ones, which depend on the specific background space considered. To the first class belong proper-time translations and supersymmetry, generated by the Hamiltonian and supercharge:

$$Q_0=\Pi_\mu\psi^\mu. \tag{7}$$

In addition, there is also a ''chiral'' symmetry generated by the chiral charge

$$\Gamma_*=\frac{i^{[d/2]}}{d!}\sqrt{g}\epsilon_{\mu_1\cdots\mu_d}\psi^{\mu_1}\cdots\psi^{\mu_d} \tag{8}$$

and a dual supersymmetry whose generator is

$$Q^*=i\{\Gamma_*,Q_0\}=\frac{i^{[d/2]}}{(d-1)!}\sqrt{g}\epsilon_{\mu_1\cdots\mu_d}\Pi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_d}, \tag{9}$$

where d is the dimension of space-time.

The *nongeneric* conserved quantities depend on the explicit form of the metric $g_{\mu\nu}(x)$. It was a great success of Gibbons *et al.* [3] to have been able to prove that the Killing-Yano tensors can be understood as objects generating *non-*

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generic supersymmetries. A tensor $f_{\mu_1 \ldots \mu_r}$ is called a Killing-Yano tensor of valence r if it is totally antisymmetric and satisfies the equation

$$f_{\mu_1 \cdots \mu_{r-1}(\mu_r;\lambda)} = 0. \tag{10}$$

In order to solve the system of coupled differential equations (6) one starts with a $\tilde{\mathcal{J}}^{(n)}_{\mu_1 \cdots \mu_n}$ solution of the homogeneous equation:

$$\tilde{\mathcal{J}}^{(n)}_{(\mu_1 \cdots \mu_n;\mu_{n+1})} + \frac{\partial \tilde{\mathcal{J}}^{(n)}_{(\mu_1 \cdots \mu_n}}{\partial \psi^\sigma} \Gamma^\sigma_{\mu_{n+1})\lambda} \psi^\lambda = 0. \tag{11}$$

This solution is introduced on the right-hand side (RHS) of the generalized Killing equation (6) for $\mathcal{J}^{(n-1)}_{\mu_1 \cdots \mu_{n-1}}$ and the iteration is carried on to $n=0$.

In fact, for the bosonic sector, neglecting the Grassmann variables $\{\psi^\mu\}$, all the generalized Killing equations (6) are homogeneous and decoupled. The first equation shows that $\mathcal{J}_0$ is a trivial constant, the next one is the equation for the Killing vectors and so on. In general, the homogeneous equation for a given n defines a Killing tensor of valence n and $\mathcal{J}^{(n)}_{\mu_1 \cdots \mu_n} \Pi^{\mu_1} \cdots \Pi^{\mu_n}$ is a first integral of the geodesic equation [4].

For the spinning particles, even if one starts with a Killing tensor of valence n, solution of Eq. (11) in which all spin degrees of freedom are neglected, the components $\mathcal{J}^{(m)}_{\mu_1 \cdots \mu_m}$ $(m<n)$ will receive a nontrivial spin contribution.

In what follows we should like to stress that the very starting homogeneous equation (11) can have solutions depending on the Grassmann coordinates $\{\psi^\mu\}$. That is the case of the manifolds admitting Killing-Yano tensors. For example, for the first equation (11), i.e., $n=0$,

$$\tilde{\mathcal{J}}^{(0)} = \frac{i}{4} f_{\mu\nu} \psi^\mu \psi^\nu \tag{12}$$

is a solution if $f_{\mu\nu}$ is a Killing-Yano tensor covariantly constant. Moreover $\tilde{\mathcal{J}}^{(0)}$ is a separately conserved quantity.

Going to the next equation (11) with $n=1$, a natural solution is

$$\tilde{\mathcal{J}}^{(1)}_\mu = R_\mu f_{\lambda\sigma} \psi^\lambda \psi^\sigma \tag{13}$$

where R_μ is a Killing vector $(R_{(\mu;\nu)}=0)$ and again $f_{\lambda\sigma}$ is a Killing-Yano tensor covariantly constant. Introducing this solution in the RHS of Eq. (6) with $n=0$, after some calculations, we get, for $\mathcal{J}^{(0)}$,

$$\mathcal{J}^{(0)} = \frac{i}{2} R_{[\mu;\nu]} f_{\lambda\sigma} \psi^\mu \psi^\nu \psi^\lambda \psi^\sigma, \tag{14}$$

where the square brackets denote antisymmetrization with norm one. Finally, from Eq. (5) with the aid of Eqs. (13) and (14), we get a new constant of motion which is peculiar to the spinning case:

$$\mathcal{J} = f_{\mu\nu} \psi^\mu \psi^\nu \left(R_\lambda \Pi^\lambda + \frac{i}{2} R_{[\lambda;\sigma]} \psi^\lambda \psi^\sigma \right). \tag{15}$$

Another ψ-dependent solution of the $n=1$, Eq. (11) can be generated from a Killing-Yano tensor of valence r:

$$\tilde{\mathcal{J}}^{(1)}_{\mu_1} = f_{\mu_1 \mu_2 \cdots \mu_r} \psi^{\mu_2} \ldots \psi^{\mu_r}. \tag{16}$$

Following the above prescription we get, for $\mathcal{J}^{(0)}$,

$$\mathcal{J}^{(0)} = \frac{i}{r+1} (-1)^{r+1} f_{[\mu_1 \cdots \mu_r;\mu_{r+1}]} \psi^{\mu_1} \cdots \psi^{\mu_{r+1}} \tag{17}$$

and the constant of motion corresponding to these solutions of the Killing equations is

$$Q_f = f_{\mu_1 \cdots \mu_r} \Pi^{\mu_1} \psi^{\mu_2} \cdots \psi^{\mu_r} + \frac{i}{r+1} (-1)^{r+1} f_{[\mu_1 \cdots \mu_r;\mu_{r+1}]} \psi^{\mu_1} \cdots \psi^{\mu_{r+1}}. \tag{18}$$

Therefore, the existence of a Killing-Yano tensor of valence r is equivalent to the existence of a supersymmetry for the spinning space with supercharge Q_f which anticommutes with Q_0. A similar result was obtained in Ref. [5] in which the role of the generalized Killing-Yano tensors, with the framework extended to include electromagnetic interactions is discussed.

Finally, we should like to mention the special case of a covariantly constant tensor $\mathcal{J}^{(n)}_{\mu_1 \cdots \mu_n}$, symmetric in the first r indices and antisymmetric in the remaining ones. Using such kind of tensor, the Killing equations are decoupled even in the spinning case, the quantity $\mathcal{J}^{(n)}_{\mu_1 \cdots \mu_n} \Pi^{\mu_1} \cdots \Pi^{\mu_r} \psi^{\mu_{r+1}} \cdots \psi^{\mu_n}$ being conserved along the geodesics.

In the main, with some ability, it is possible to investigate higher orders of Eq. (11), but it seems that one cannot go much far with simple, transparent expressions. Instead of that, we shall apply the above constructions to a concrete case, namely, the four-dimensional Euclidean Taub-Newman-Unti-Tamburino (NUT) manifold.

II. TAUB-NUT SPINNING SPACE

Much attention has been paid to the Euclidean Taub-NUT metric, since in the long distance limit the relative motion of two monopoles is described approximately by its geodesics [6,7]. As it is well known, the geodesic motion of the Taub-NUT metric admits the Kepler-type symmetry [8–11]. On the other hand, the Kaluza-Klein monopole of Gross and Perry [12] and of Sorkin [13] was obtain by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory.

In a special choice of coordinates, the Euclidean Taub-NUT metric takes the form

$$ds^2 = V(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) + 16 m^2 V^{-1}(r)(d\chi + \cos\theta d\varphi)^2, \tag{19}$$

with $V(r) = 1 + 4m/r$. There are four Killing vectors [8–11]

$$D_A = R^\mu_A \partial_\mu, \quad A = 0, \ldots, 3 \tag{20}$$

corresponding to the invariance of the metric (19) under spatial rotations ($A=1,2,3$) and χ translations ($A=0$). In the purely bosonic case these invariances would correspond to conservation of angular momentum and ''relative electric charge'' [8–10]:

$$\vec{j}=\vec{r}\times\vec{p}+q\frac{\vec{r}}{r}, \tag{21}$$

$$q=16m^2V(r)(\dot{\chi}+\cos\theta\dot{\varphi}), \tag{22}$$

where $\vec{p}=[1/V(r)]\dot{\vec{r}}$ is the ''mechanical momentum'' which is only part of the momentum canonically conjugate to $\vec{r}$.

Finally, there is a conserved vector analogous to the Runge-Lenz vector of the Kepler-type problema,

$$\vec{K}=\frac{1}{2}\vec{K}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=\frac{1}{2}\left[\vec{p}\times\vec{j}+\left(\frac{q^2}{4m}-4mE\right)\frac{\vec{n}}{r}\right], \tag{23}$$

where the conserved energy E, from Eq. (2), is

$$E=\frac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu=\frac{1}{2}V^{-1}(r)\left[\dot{\vec{r}}^{\,2}+\left(\frac{q}{4m}\right)^2\right]. \tag{24}$$

In the Taub-NUT geometry there are known to exist four Killing-Yano tensors [9]. The first three Killing-Yano tensors $f_{i\mu\nu}$ are covariantly constants (with vanishing field strength):

$$f_i=8m(d\chi+\cos\theta d\varphi)\wedge dx_i-\epsilon_{ijk}\left(1+\frac{4m}{r}\right)dx_j\wedge dx_k. \tag{25}$$

The fourth Killing-Yano tensor is

$$f_Y=8m(d\chi+\cos\theta d\varphi)\wedge dr+4r(r+2m)\left(1+\frac{r}{4m}\right)\sin\theta d\theta\wedge d\varphi, \tag{26}$$

having only one nonvanishing component of the field strength:

$$f_{Yr\theta;\varphi}=2\left(1+\frac{r}{4m}\right)r\sin\theta. \tag{27}$$

The corresponding supercharges (18) constructed from the Killing-Yano tensors (25) and (26) are Q_i and Q_Y. The supercharges Q_i together with Q_0 from Eq. (7) realize the $N=4$ supersymmetry algebra [14]:

$$\{Q_A,Q_B\}=-2i\delta_{AB}H, \quad A,B=0,\ldots,3, \tag{28}$$

making manifest the link between the existence of the Killing-Yano tensors and the hyper-Kähler geometry of the Taub-NUT manifold.

Starting with these results from the bosonic sector of the Taub-NUT space one can proceed with the spin contributions. The first generalized Killing equation (6) shows that with each Killing vector R_A^μ (20), there is an associated Killing scalar B_A [15]. A simple expression for the Killing scalar was given in Ref. [14]:

$$B_A=\frac{i}{2}R_{A[\mu;\nu]}\psi^\mu\psi^\nu. \tag{29}$$

Therefore the total angular momentum and ''relative electric charge'' become, in the spinning case,

$$\vec{J}=\vec{B}+\vec{j}, \tag{30}$$

$$J_0=B_0+q, \tag{31}$$

where $\vec{J}=(J_1,J_2,J_3)$ and $\vec{B}=(B_1,B_2,B_3)$.

The above constants of motion are superinvariant:

$$\{J_A,Q_0\}=0, \quad A=0,\ldots,3. \tag{32}$$

The Lie algebra defined by the Killing vectors is realized by the constants of motion (30) and (31) through the Poisson-Dirac brackets (3).

Similarly, introducing the Killing tensors $\vec{K}_{\mu\nu}$ (23) into the RHS of the second generalized Killing equation (6), we get the corresponding Killing vectors $\vec{\mathcal{R}}_\mu$ having a spin-dependent part $\vec{S}_\mu$ [16]:

$$\vec{\mathcal{R}}_\mu=\vec{R}_\mu+\vec{S}_\mu, \tag{33}$$

where $\vec{R}_\mu$ are the standard Killing vectors. The ψ-dependent parts of the Killing vectors $\vec{S}_\mu$ contribute to the Runge-Lenz vector for the spinning space:

$$\vec{\mathcal{K}}=\tfrac{1}{2}\vec{K}_{\mu\nu}\cdot\dot{x}^\mu\dot{x}^\nu+\vec{S}_\mu\cdot\dot{x}^\mu. \tag{34}$$

In terms of the supercharges Q_i and Q_Y, the components of the Runge-Lenz vector $\vec{\mathcal{K}}$ are given by [14]

$$\mathcal{K}_i=\frac{i}{2}\{Q_Y,Q_i\}, \quad i=1,2,3. \tag{35}$$

The nonvanishing Poisson brackets are (after some algebra)

$$\{J_i,J_j\}=\epsilon_{ijk}J_k, \tag{36}$$

$$\{J_i,\mathcal{K}_j\}=\epsilon_{ijk}\mathcal{K}_k, \tag{37}$$

$$\{\mathcal{K}_i,\mathcal{K}_j\}=\frac{1}{4}\left(\frac{J_0^2}{16m^2}-2E\right)\epsilon_{ijk}J_k, \tag{38}$$

similar to the results from the bosonic sector [9].

Taking into account the existence of the Killing-Yano covariantly constant tensors $f_{i\mu\nu}$ (25), three constants of motion can be obtained using the prescription (12),

$$S_i=\frac{i}{4}f_{i\mu\nu}\psi^\mu\psi^\nu, \quad i=1,2,3, \tag{39}$$

which realize an SO(3) Lie algebra similar to that of the angular momentum (36):

$$\{S_i,S_j\}=\epsilon_{ijk}S_k. \tag{40}$$

These components of the spin are separately conserved and can be combined with the angular momentum $\vec{J}$ to define a new improved form of the angular momentum $I_i = J_i - S_i$ with the property that it preserves the algebra

$$\{I_i, I_j\} = \epsilon_{ijk} I_k \tag{41}$$

and that it commutes with the SO(3) algebra generated by the spin S_i:

$$\{I_i, S_j\} = 0. \tag{42}$$

Let us note also the following Dirac brackets of S_i with supercharges:

$$\{S_i, Q_0\} = -\frac{Q_i}{2}, \quad \{S_i, Q_j\} = \frac{1}{2}(\delta_{ij} Q_0 + \epsilon_{ijk} Q_k). \tag{43}$$

We can combine these two SO(3) algebras (40) and (41) to obtain the generators of a conserved SO(4) symmetry among the constants of motion, a standard basis which is spanned by $M_i^{\pm} = I_i \pm S_i$ [14]. We should like to remark that there is no spin component such as in Eq. (39) to be used for a improved ''relative electric charge'' J_0. The reason is that the fourth Killing-Yano tensor f_Y (26) is not covariantly constant. Of course, we can add to J_0 a combination of the three separately conserved quantities S_i but this is not a natural ''improved relative electric charge.'' Moreover, it is impossible to modify a particular solution J_0 (31) of the generalized Killing equation (6) for $n=0$ and Killing vectors in the RHS by adding solutions of the homogeneous part of this equation in order to recover the conserved quantity (22) from the standard Taub-NUT case as it was suggested in Ref. [17].

In fact, from Eq. (31), in the spinning case, q is not separately conserved and we have [15]

$$\dot{q}(J_0 - q) = \dot{q} B_0 = -B_0 \dot{B}_0 = \frac{256 m^4}{(4m+r)^5} \dot{r} \Gamma_* . \tag{44}$$

Therefore, q is not separately conserved in the spinning space excepting the case $\Gamma_* = 0$. Γ_* can be zero if there is a relation between Grassmann variables ψ^μ. Such a relation can be realized imposing, for example, $Q_0 = 0$. This constraint can be correlated with the absence of an intrinsic electric dipole moment of physical fermions (leptons and quarks) [1]. The conservation of Q_0 guarantees that this condition can be satisfied at all times, irrespective of the presence of external fields.

However, in general, the motion of a spinning particle governed by the action (1), does not fix the value of the supercharge Q_0. We have the freedom to choose its value and any choice gives a consistent model [1]. On the other hand, in Refs. [15,18], a simple exact solution of the generalized Killing equations, corresponding to trajectories lying on a cone, is given. Again, this particular solution requires that $\Gamma_* = 0$ and the constant of motion J_0 reduces to the standard ''relative electric charge'' q.

Finally, let us consider a solution of the homogeneous equation (11) for $n=1$ of the type (13). Using the Killing vectors (20) and the Killing-Yano tensors (25), we can form the combinations

$$\tilde{\mathcal{J}}^{(1)}_{Aj\mu} = R_{A\mu} f_{j\lambda\sigma} \psi^\lambda \psi^\sigma, \quad A = 0, \ldots, 3; \quad j = 1,2,3. \tag{45}$$

After some algebra we get the new constants of motion of the form (15):

$$\mathcal{J}_{Aj} = f_{j_\lambda \sigma} \psi^\lambda \psi^\sigma \left(R_{A\mu} \Pi^\mu + \frac{i}{2} R_{A[\alpha;\beta]} \psi^\alpha \psi^\beta \right)$$
$$= -4 i S_j J_A, \quad A = 0, \ldots, 3; \quad j = 1,2,3 \tag{46}$$

where we used Eqs. (15), (29)–(31), and (39). Strictly speaking, the constants $\mathcal{J}_{Aj}$ are not completely new, being expressed in terms of the constants J_A and S_j. However, the combinations (46) arise in a natural way as solutions of the generalized Killing equations and appear only in the spinning case. Moreover, we can form a sort of Runge-Lenz vector involving only Grassmann components,

$$L_i = \frac{1}{m} \epsilon_{ijk} S_j J_k, \quad i,j,k = 1,2,3, \tag{47}$$

with the commutation relations such as in Eqs. (37) and (38):

$$\{L_i, J_j\} = \epsilon_{ijk} L_k, \tag{48}$$

$$\{L_i, L_j\} = (\vec{S}\vec{J} - \vec{S}^2) \frac{1}{m^2} \epsilon_{ijk} J_k . \tag{49}$$

Note also the following Dirac brackets of L_i with supercharges:

$$\{L_i, Q_0\} = -\frac{1}{2m} \epsilon_{ijk} Q_j J_k, \tag{50}$$

$$\{L_i, Q_j\} = \frac{1}{2m} (\epsilon_{ijk} Q_0 J_k - \delta_{ij} Q_k M_k^- + Q_i M_j^-). \tag{51}$$

III. CONCLUDING REMARKS

The constants of motion of a scalar particle in a curved space-time are determined by the symmetries of the manifold, and are expressible in terms of the Killing vectors and tensors, i.e., if a space-time admits a Killing tensor $K_{\mu_1 \cdots \mu_r}$ of valence r, then the quantity $K_{\mu_1 \cdots \mu_r} \Pi^{\mu_1} \cdots \Pi^{\mu_r}$ is conserved along the geodesic. On the other hand, the Killing-Yano tensors can be understood as objects generating *nongeneric* supersymmetries [3]. They have been also used to investigate the motion of spinning particles including electromagnetic interactions [5] and torsion [19]. The aim of this paper was to point out the role of the Killing-Yano tensors to generate solution of the homogeneous parts of the generalized Killing equations. This solution must be included in the complete solution of the system of coupled Killing equations. The general procedure was applied to the particular case of the Taub-NUT spinning space. The extension of these results for the motion of spinning particles in spaces with torsion and/or in the presence of an electromagnetic field will be discussed elsewhere.

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