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Large-scale magnetic fields from hydromagnetic turbulence in the very early universe

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We investigate hydromagnetic turbulence of primordial magnetic fields using magnetohydrodynamics (MHD) in an expanding universe. We present the basic, covariant MHD equations, find solutions for MHD waves in the early universe, and investigate the equations numerically for random magnetic fields in two spatial dimensions. We find the formation of magnetic structures at larger and larger scales as time goes on. In three dimensions we use a cascade (shell) model that has been rather successful in the study of certain aspects of hydrodynamic turbulence. Using such a model we find that after $\sim 10^9$ times the initial time the scale of the magnetic field fluctuation (in the comoving frame) has increased by 4–5 orders of magnitude as a consequence of an inverse cascade effect (i.e., transfer of energy from smaller to larger scales). Thus *at large scales* primordial magnetic fields are considerably stronger than expected from considerations which do not take into account the effects of MHD turbulence. [S0556-2821(96)02712-9]

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I. INTRODUCTION

It has been suggested that primordial magnetic fields might arise during the early cosmic phase transitions [1], and recently it has been shown that magnetic fields are indeed a stable feature of the electroweak phase transition [2]. In a first order phase transition magnetic fields could also be generated when bubbles of the new vacuum collide, whence a ring of magnetic field may arise in the intersecting region [3]. It has also been suggested that at very high temperatures the ground state of a non-Abelian particle theory is a ''ferromagnetic'' vacuum with a permanent nonzero magnetic field [4]. The plasma of the early universe has a high conductivity so that a primordial magnetic field would be imprinted on the comoving plasma and would dissipate very slowly [5]. Such a field could then contribute to the seed field needed to understand the presently observed galactic magnetic fields [6], which have been measured both in the Milky Way and in other spiral galaxies, including their halos. Typically the observed present day magnetic field is of the order of $\sim 10^{-6}$ G.

Locally the primordial field could be very large; it is limited only by the magnetic energy density and effects it induces in the electron statistics which both affect primordial nucleosynthesis [7]. The actual limits depend on whether or not one assumes a homogenous field, but a typical upper limit at nucleosynthesis is $B \leqslant 10^{12}$ G. Because flux conservation implies that $B_{\rm rms} \sim R^{-2}$, where R is the scale factor, the field could have been much stronger at earlier times. On dimensional grounds, a typical value of the magnetic field fluctuation should be $B_{\rm rms} \sim T^2$, so that at the time of the electroweak phase transition one could locally obtain fields as high as 10^{24} G. Depending on how such a strong, random magnetic field scales at large distances, it could [8] be the seed field needed to explain the magnetic fields observed on the scale of galaxies and larger.

However, even assuming that a primordial magnetic field is created at some very early epoch, a number of issues remain to be worked out before one can say anything definite about the role of primordial fields in generating galactic magnetic fields. At earliest times magnetic fields are generated by particle physics processes with length scales typical of particle physics. If the inflation hypothesis proves correct, then after inflation rather long correlation lengths are possible [9]. The question is if it is at all possible for the small-scale fluctuations to grow to large scales, and what exactly is the scaling behavior of $B_{\rm rms}$ or the correlator $\langle B(r+x) B(x) \rangle$. Even in an inflationary scenario it would be of interest to see if the relatively large scale can grow even further. To study these problems one needs to consider the detailed evolution of the magnetic field to account for such issues as what happens when uncorrelated field regions come into contact with each other during the course of the expansion of the universe. In general, turbulence is an essential feature of such phenomena. These questions can only be answered by considering magnetohydrodynamics (MHD) in an expanding universe [10]. It is the main purpose of this paper to investigate the subsequent development of the primordial field. Expressed in a general way, our conclusion turns out to be that MHD turbulence is operative, and hence the scale of magnetic fields is considerably larger than one would expect

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if MHD turbulence was ignored. This means that the previous estimates of the strength of the primordial magnetic field ''today'' need to be reconsidered.

We begin by posing the basic equations and consider certain simplified models. A full (1+3)-dimensional numerical simulation would be desirable, but is beyond the scope of the present paper. In Sec. II we derive the relativistic MHD equations for relativistic plasma, which is appropriate for the very early universe. In Sec. III we discuss the appearance of waves in relativistic MHD. To elucidate the various MHD effects pertaining to the early universe, we also present numerical solutions to the MHD equations for a two-dimensional slice. In Sec. IV we study a cascade model that reflects important properties of fully three-dimensional turbulence. The cascade model has been rather successful in ordinary hydrodynamics. We find that in the early universe magnetic energy is transferred from small scales to large scales. We also compute the correlation function $\langle B(r+x)B(x)\rangle$ in the cascade model. In Sec. V we offer an interpretation of our results.

II. RELATIVISTIC MHD IN THE EXPANDING UNIVERSE

We begin by presenting a derivation of the fully general relativistic MHD equations (see also Ref. [11], where further references can be found), which we rewrite in a form suitable for our numerical work. We consider the early universe as consisting of ideal fluid with an equation of state of the form $p=\frac{1}{3}\rho$, where p is pressure, ρ the energy density, and the speed of light is set to unity. We further assume that the fluid supports a (random) magnetic field. The energy-momentum tensor is then given by

$$T^{\mu\nu}=(p+\rho)U^{\mu}U^{\nu}+pg^{\mu\nu}+\frac{1}{4\pi}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma}-\frac{1}{4}g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right), \qquad (1)$$

where U^{μ} is the four-velocity of the plasma, normalized as $U^{\mu}U_{\mu}=-1$, and $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ is the electromagnetic field tensor. Note that, as long as diffusion can be neglected, the presence of the magnetic field does not change the equation of state.

The magnetic energy is assumed to be much smaller than the radiation energy, so that it can be neglected as far as the expansion of the universe is concerned. We therefore assume a flat, isotropic, and homogeneous universe with a Robertson-Walker metric $ds^2=-dt^2+R^2(t)d\mathbf{x}^2$. Although the magnetic field generates local bulk motion, this may still be consistent with isotropy and homogeneity at sufficiently large scales, e.g., if the magnetic field is random, i.e., statistically homogeneous and isotropic on scales much larger than the intrinsic correlation scale of the field. Even very large magnetic fields, together with the ensuing very fast bulk motion, might not contradict isotropy and homogeneity. The equations of motion for the fluid arise from energy-momentum conservation

$$T^{\mu\nu}{}_{;\nu}\equiv\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\nu}}\sqrt{-g}T^{\mu\nu}+\Gamma^{\mu}_{\nu\lambda}T^{\nu\lambda}=0. \qquad (2)$$

The Maxwell equations read

$$F^{\mu\nu}{}_{;\nu}=J^{\mu}, \quad F_{[\mu\nu,\lambda]}=0. \qquad (3)$$

We define $F_{\mu\nu}$ in terms of the electric and magnetic fields

$$F_{i0}=RE^{i}, \quad F_{ij}=\epsilon_{ijk}R^2B^{k}, \qquad (4)$$

where latin letters go from 1 to 3. With this definition the expression for the total energy has no R factors and takes therefore the familiar form

$$T^{00}=(p+\rho)\gamma^2-p+\tfrac{1}{2}(\mathbf{B}^2+\mathbf{E}^2), \qquad (5)$$

where $\gamma=U^0$.

In order to solve (2) and (3) we rewrite the equations of motion explicitly in 3+1 dimensions. We start by writing (2) as

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\nu}}[\sqrt{g}(p+\rho)U^{\mu}U^{\nu}]+\Gamma^{\mu}_{\sigma\lambda}(p+\rho)U^{\sigma}U^{\lambda}+g^{\mu\nu}\frac{\partial p}{\partial x^{\nu}}=F^{\mu\nu}g_{\nu\sigma}J^{\sigma}, \qquad (6)$$

where $\sqrt{-g}=R^3$, and the nonvanishing Christoffel symbols are $\Gamma^{0}_{ij}=R\dot{R}\delta_{ij}$ and $\Gamma^{i}_{0j}=(\dot{R}/R)\delta^{i}_{j}=\Gamma^{i}_{j0}$. It is useful to define $U^{i}=\gamma R^{-1}v^{i}$, because then the normalization $U^{\mu}U_{\mu}=-1$ gives the familiar form for the Lorentz factor $\gamma=(1-\mathbf{v}^2)^{-1/2}$.

For $\mu=i$ we obtain

$$\frac{\partial R^4\mathbf{S}}{\partial t}=\frac{1}{R}[-(\boldsymbol{\nabla}\cdot\mathbf{v})(R^4\mathbf{S})-(\mathbf{v}\cdot\boldsymbol{\nabla})(R^4\mathbf{S})-\boldsymbol{\nabla}(R^4p)+(R^3\mathbf{J})\times(R^2\mathbf{B})], \qquad (7)$$

where $\mathbf{S}=(p+\rho)\gamma^2\mathbf{v}$. It should be noticed that in this equation all quantities are scaled by the appropriate powers of R. Thus, e.g., $R^4\mathbf{S}$ is expected to be independent of R, because $p+\rho$ scales like $1/R^4$, and $\mathbf{v}$ is expected to be independent of R. Also, $\boldsymbol{\nabla}$ occurs always multiplied by $1/R$, or, alternatively, the operator $\partial/\partial t$ is replaced by itself multiplied by R, which means that time is replaced by conformal time $\tilde{t}=\int dt/R$. To emphasize this, it is convenient to introduce new scaled ''tilde'' variables:

$$\tilde{\mathbf{S}}=R^4\mathbf{S}, \quad \tilde{p}=R^4p, \quad \tilde{\rho}=R^4\rho, \quad \tilde{\mathbf{B}}=R^2\mathbf{B}, \quad \tilde{\mathbf{J}}=R^3\mathbf{J}, \quad \text{and} \quad \tilde{\mathbf{E}}=R^2\mathbf{E}. \qquad (8)$$

It should be noticed that $\mathbf{v}$ is not scaled. Equation (7) can then be written

$$\frac{\partial\tilde{\mathbf{S}}}{\partial\tilde{t}}=-(\boldsymbol{\nabla}\cdot\mathbf{v})\tilde{\mathbf{S}}-(\mathbf{v}\cdot\boldsymbol{\nabla})\tilde{\mathbf{S}}-\boldsymbol{\nabla}\tilde{p}+\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}. \qquad (9)$$

For $\mu=0$ we obtain, using scaled quantities,

$$\left(1-\frac{1}{4\gamma^2}\right)\frac{\partial\ln\tilde{\rho}}{\partial\tilde{t}}+\frac{\partial\ln\gamma^2}{\partial\tilde{t}}+\mathbf{v}\cdot\boldsymbol{\nabla}\ln(\tilde{\rho}\gamma^2)+\boldsymbol{\nabla}\cdot\mathbf{v}=\frac{\tilde{\mathbf{J}}\cdot\tilde{\mathbf{E}}}{(\tilde{p}+\tilde{\rho})\gamma^2}. \qquad (10)$$

In order to solve this equation numerically with an explicit code we need to eliminate the time derivative $\partial \ln\gamma^2/\partial t$. To this end we first solve the normalization condition for γ^2:

$$\gamma^2=\frac{1}{2}+\left(\frac{1}{4}+\frac{\tilde{\mathbf{S}}^2}{(\tilde{p}+\tilde{\rho})^2}\right)^{1/2}. \tag{11}$$

We then differentiate

$$\frac{\partial \ln\gamma^2}{\partial\tilde{t}}=\frac{1}{\gamma^2(2\gamma^2-1)}\frac{\partial\tilde{\mathbf{S}}^2/\partial\tilde{t}}{(\tilde{p}+\tilde{\rho})^2}-\frac{\gamma^2-1}{\gamma^2-\frac{1}{2}}\frac{\partial\ln\tilde{\rho}}{\partial\tilde{t}}. \tag{12}$$

Combining (10) and (12) we obtain a final equation suitable for numerical work:

$$\frac{2\gamma^2+1}{4\gamma^2(2\gamma^2-1)}\frac{\partial\ln\tilde{\rho}}{\partial\tilde{t}}=-\frac{\partial\tilde{\mathbf{S}}^2/\partial\tilde{t}}{\left(\frac{4}{3}\tilde{\rho}\gamma\right)^2(2\gamma^2-1)}-\mathbf{v}\cdot\boldsymbol{\nabla}\ln(\tilde{\rho}\gamma^2)-\boldsymbol{\nabla}\cdot\mathbf{v}+\frac{\tilde{\mathbf{J}}\cdot\tilde{\mathbf{E}}}{\frac{4}{3}\tilde{\rho}\gamma^2}, \tag{13}$$

where we have used $\rho+p=\frac{4}{3}\rho$ for later convenience. In this equation only scale invariant quantities enter.

The Maxwell equations can be written explicitly as

$$\frac{\partial\tilde{\mathbf{B}}}{\partial\tilde{t}}=-\nabla\times\tilde{\mathbf{E}},\quad \boldsymbol{\nabla}\cdot\tilde{\mathbf{B}}=0, \tag{14}$$

and

$$\tilde{\mathbf{J}}=\boldsymbol{\nabla}\times\tilde{\mathbf{B}}-\frac{\partial\tilde{\mathbf{E}}}{\partial\tilde{t}},\quad \boldsymbol{\nabla}\cdot\tilde{\mathbf{E}}=\tilde{\rho}_e \tag{15}$$

where ρ_e is the charge density and $\tilde{\rho}_e=R^3\rho_e$. Further,

$$\tilde{\mathbf{E}}=-\mathbf{v}\times\tilde{\mathbf{B}}, \tag{16}$$

which is valid in the limit of high conductivity [11]. Again, these equations have the natural scaling properties with respect to powers of R. We emphasize that in the relativistic regime the displacement current $-\partial\tilde{\mathbf{E}}/\partial t$ cannot be neglected. However, in all cases considered we were able to solve for $-\dot{\tilde{\mathbf{E}}}=\dot{\mathbf{v}}\times\tilde{\mathbf{B}}+\mathbf{v}\times\dot{\tilde{\mathbf{B}}}$ iteratively by evaluating $\dot{\mathbf{v}}$ and $\dot{\tilde{\mathbf{B}}}$ from the previous iteration.

The equation of energy conservation is $T^{0\nu}{}_{;\nu}=0$, or

$$\frac{1}{R^3}\frac{\partial}{\partial t}(R^3T^{00})+\frac{\partial}{\partial x^j}T^{0j}+R\dot{R}T^{jj}=0, \tag{17}$$

but since $T^{\nu}{}_{\nu}=0$, we have $T^{jj}=T^j{}_j/R^2=-T^0{}_0/R^2=T^{00}/R^2$, and therefore the energy equation is

$$\frac{\partial}{\partial t}R^4T^{00}=-\frac{\partial}{\partial x^j}R^4T^{0j}, \tag{18}$$

or integrated over the whole space

$$\frac{dR^4E_{\rm tot}}{dt}=0, \tag{19}$$

where

$$E_{\rm tot}=\int T^{00}d^3x\equiv\langle T^{00}\rangle. \tag{20}$$

Hence $R^4E_{\rm tot}$ is conserved.

The conclusion from the above expressions is thus that *the MHD equations in an expanding universe with zero curvature are the same as the relativistic MHD equations in a nonexpanding universe, provided the dynamical quantities are replaced by the scaled "tilde" variables, and provided conformal time $\tilde{t}$ is used.* The effect of this is, as usual, that the expansion slows down the rate of dynamical evolution.

It should be noted that the velocity $\mathbf{v}$ is the bulk velocity. Thus, in general, we expect that $\mathbf{v}$ is nonrelativistic. This is physically reasonable since, although the gas particles move with velocity near unity, we expect no strong collective effects which could give rise to a relativistic bulk velocity. The equations for nonrelativistic bulk motions of a relativistic gas are given in the Appendix.

In the early universe conductivity is large, and hence the diffusion length is also large. The conductivity of the isotropic relativistic electron gas, which interacts with heavy (nonrelativistic) ions, is related to the Coulomb scattering cross section and reads [12]

$$\sigma=\frac{\omega_p^2}{4\pi\sigma_{\rm coll}n_e}\simeq\frac{T}{3\pi\alpha}, \tag{21}$$

where ω_p is the plasma frequency, $\sigma_{\rm coll}$ is the collision cross section, and α is the fine structure constant. This result is valid for fields smaller than the critical field $B_c=m_e^2/e=4.41\times10^{13}$ G, above which the electrons cannot be treated as free, and the conductivity (21) should be multiplied by a factor B/B_c. On dimensional grounds, conductivity of the fully relativistic standard model gas will also scale as $\sigma\sim T$. The expansion rate of the radiation dominated universe is given by

$$H\equiv\frac{\dot{R}}{R}=\frac{1}{2t}=\sqrt{\frac{8\pi^3g_*}{90}}\frac{T^2}{M_P}, \tag{22}$$

where g_* is the number of the effective degrees of freedom, and $M_P=1.2\times10^{19}$ GeV is the Planck mass. Equation (22) also provides the time-temperature relationship [13], and the inverse the length scale of the universe. A measure of the importance of the diffusion is the magnetic Reynolds number, which may be defined as Re$=Lv\sigma$, where L and v are, respectively, the typical length scale and velocity in the system under consideration. A Reynolds number less than 1 means that diffusion dominates. In the early universe, say at the electroweak phase transition $T_{\rm EW}\simeq100$ GeV where in the standard model $g_*=106.75$, the Reynolds number is huge, typically

$$\mathrm{Re}_U\sim v\sigma H^{-1}\sim\frac{M_P}{T}\sim10^{17}, \tag{23}$$

where v has been arbitrarily chosen to be 10^{-2}. In this sense the very early universe is almost a perfect conductor. Also, the extremely large value of the Reynold's number indicates a turbulent situation, which we shall find by other methods later.

III. ASPECTS OF RELATIVISTIC MHD

A. Magnetohydrodynamic waves

Let us begin by first presenting some general considerations. In the framework of relativistic MHD in an expanding universe, we can still discuss waves. Although the equations exhibited in the previous section are considerably more complicated than their nonrelativistic counterparts, the MHD waves are linear perturbations of the standard cosmological background. Thus the bulk velocity $\mathbf{v}$ must necessarily be small relative to the velocity of light. It therefore follows that the displacement current can be ignored. The background is homogeneous, and we assume the relativistic relation $p=\rho/3$ for the background as well as for the fluctuations. The continuity equation, i.e., (10), gives to the lowest order the well known result $\rho=\text{const}/R^4$. To the next order we get

$$\frac{\partial R^4 \delta\rho}{\partial t}+\frac{4}{3}R^4\rho\left(\frac{1}{R}\boldsymbol{\nabla}\cdot\mathbf{v}\right)=0, \tag{24}$$

where $\delta\rho$ is the fluctuation in ρ. Also, from (7) we get, to lowest order in the fluctuations,

$$\frac{\partial R^4 \delta\mathbf{S}}{\partial t}=-\frac{1}{R}\boldsymbol{\nabla}R^4\left(\frac{1}{3}\delta\rho+\mathbf{B}\delta\mathbf{B}\right)+\frac{1}{R}(R^2\mathbf{B})\boldsymbol{\nabla}(R^2\delta\mathbf{B}). \tag{25}$$

Here $\delta\mathbf{S}=\frac{4}{3}\rho\mathbf{v}$, and $\delta\mathbf{B}$ is the fluctuation of the background field $\mathbf{B}$, which is assumed to behave like $\sim 1/R^2$. Of course, $\delta\mathbf{B}$ is expected to have a similar scaling behavior as a function of time, but it also has a spatial dependence. Finally, we have the fluctuation equation

$$\frac{\partial R^2 \delta\mathbf{B}}{\partial t}=\frac{1}{R}\nabla\times(\mathbf{v}\times R^2\mathbf{B}), \tag{26}$$

which follows from (14), since the displacement current can be ignored for small bulk velocities.

We now seek a wave solution which, because of the structure of (24)–(26), must contain the scale factor R to the power -2:

$$\delta\mathbf{B}=\frac{\mathbf{b}_0}{R^2}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega\tilde{t})], \tag{27}$$

and

$$\mathbf{v}=\mathbf{v}_0\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega\tilde{t})], \tag{28}$$

$$\delta\rho=\frac{\text{const}}{R^4}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega\tilde{t})], \tag{29}$$

where $\mathbf{b}_0$ and $\mathbf{v}_0$ are constants. These expressions satisfy the basic fluctuation equations (24)–(26) with

$$\tilde{t}=\int_{t_0}^{t}\frac{dt'}{R(t')}, \tag{30}$$

where t_0 is the initial time, in accordance with the results obtained in the previous section.

We therefore see that with the scaling properties mentioned above the equations are similar to the nonrelativistic case [provided we use the time $\tilde{t}$ in (30)]. Thus we find the group velocity $\partial\omega/\partial k=B/\sqrt{p+\rho}$. Because the scaling properties of $\mathbf{B}$ and $\sqrt{p+\rho}$ with respect to the expansion of the universe are the same, it follows that the group velocity is independent of R. As far as the phase velocities are concerned, the same is true. Assuming the background field to be in the x direction, then $\delta\mathbf{B}$ and $\mathbf{v}$ are in the z direction, as in the case of nonrelativistic waves [14]. One then finds that the velocities are given by

$$\frac{1}{2}\left(\sqrt{\frac{1}{3}+\frac{3\mathbf{B}^2}{4\rho}+\frac{2B_x}{\sqrt{\rho}}}\pm\sqrt{\frac{1}{3}+\frac{3\mathbf{B}^2}{4\rho}-\frac{2B_x}{\sqrt{\rho}}}\right). \tag{31}$$

Of course, these velocities are given in terms of the conformal time $\tilde{t}$. It should be noted that the assumption of small bulk velocities can only be maintained if $|\mathbf{B}|\ll\sqrt{\rho}$. If this condition is not satisfied, we cannot expect the nonlinear effects to be small.

B. Two-dimensional slice

Ideally, we would like to solve the MHD equations in three (plus one) dimensions. However, as indicated in Sec. II, this is a major computational task. We restrict ourselves therefore to a two-dimensional slice only. The main conclusion will be that much of the qualitative behavior of nonrelativistic MHD carries over to the case of relativistic MHD.

We solve (9) and (13)–(16) numerically using sixth order centered differences to compute the spatial derivatives and a third order Runge-Kutta scheme for the time step. We adopt random initial conditions for $\mathbf{B}$. First we select $\mathbf{B}$ at each grid point independently from a Gaussian distribution, but in order to guarantee that $\boldsymbol{\nabla}\cdot\tilde{\mathbf{B}}=0$ for all times, we advance the z component of the vector potential by means of the equation $\partial\tilde{A}_z/\partial\tilde{t}=\mathbf{e}_z\cdot(\mathbf{v}\times\tilde{\mathbf{B}})$, where $\tilde{\mathbf{B}}=\boldsymbol{\nabla}\times(\tilde{A}_z\mathbf{e}_z)$. The initial A_z is computed by solving $\nabla^2 A_z=-J_z$. Initially ($t=t_0$, i.e., $\tilde{t}=0$) we put ρ to unity. Periodic boundary conditions are adopted in the x and y directions. The initial velocity is chosen such that the velocity vanishes everywhere. Thus the effects that we subsequently see are entirely generated by the random initial field $\mathbf{B}$. We emphasize that our calculations are exploratory, and hence we do not use any particular model (e.g., based on the electroweak theory) in selecting the initial random magnetic field.

Our new equations (9) and (13)–(16) are scale invariant, so it is sufficient to solve them on a computational domain with size $L=1$. The results for a different domain size L' are the same, but taken at a different time $\tilde{t}'=(L'/L)\tilde{t}$.

As in all turbulence calculations there has to be some diffusion to prevent the accumulation of energy at the smallest scale. In order to restrict the effects of dissipation only to the largest possible wave numbers we use hyperdiffusion, i.e., instead of the usual diffusion operator ∇^2, we use an operator of the form $-\nabla^4$ for the evolution of all variables. This technique is well known in turbulence research (see, e.g., [15]). Also, since this procedure is merely of computational relevance, we did not use the relativistic expressions.

The minimum diffusion coefficient ν we can afford is given by the empirical constraint that the ''mesh Reynolds number'' $\mathrm{Re}_{\mathrm{mesh}}=U\delta x/\nu$ must not exceed the value 5–10. Here, δx is the mesh size and U is a typical velocity that includes the velocity of waves and bulk motions. As was pointed out before, in the early universe the Reynolds number is very large, which means that the magnetic diffusivity $\eta=4\pi/\sigma$ should be much smaller than the adopted value of ν. In other words, in order to have realistic values of ν, δx has to be extremely small. However, the maximum number of mesh points, $N=L/\delta x$, is limited by computer memory and time. Our present, rather exploratory, calculations were carried out on a workstation, and so we restricted ourselves to $N_{\mathrm{max}}=128$. Even on larger computers we would never reach realistic values. This demonstrates the difficulty of a realistic simulation. It is obvious that numerical simulations with a low Reynolds number cannot provide a realistic picture of the early universe MHD. However, we believe they are useful in illustrating the qualitative features of the problem.

The evolution of the magnetic field is compared in Fig. 1 for lower and higher resolution. As time goes on, the coalescence of magnetic structures leads to the gradual formation of larger and larger scales. In the higher resolution case there are more small-scale structures, but also here the development of large-scale fields is evident. In turbulence research this phenomenon is known as an inverse cascade. Such cascade processes are linked to certain conservation properties that the basic equations obey. For further details, see Ref. [16]. We mention here only a few important aspects. An inverse cascade exists both in two-dimensional and in three-dimensional MHD turbulence. The only difference is that in the two-dimensional case it is an inverse cascade of the magnetic potential, whereas in the three-dimensional case it is an inverse cascade of the magnetic helicity density $\mathbf{A}\cdot\mathbf{B}$. In fact, the conserved quantities in the two cases are $\int d^2x\mathbf{A}^2$ and $\int d^3x\mathbf{A}\cdot\mathbf{B}$, respectively. For comparison we also mention that the difference between two- and three-dimensional hydrodynamic (nonmagnetic) turbulence is more drastic. In two-dimensional hydrodynamics there is an inverse energy cascade associated with the conservation of enstrophy (mean squared vorticity), which has no counterpart in three-dimensional hydrodynamics.

The significance of an inverse cascade is that it leads to a transfer of magnetic energy to larger and larger scales. This process is due to the nonlinear terms giving rise to mode interactions. Energy spreads over different scales until some balance is achieved where the kinetic and magnetic energy spectra have a certain slope. In the ordinary MHD turbulence a relevant energy spectrum could be the Iroshnikov-Kraichnan spectrum [17], where the spectral energy varies as $k^{-3/2}$, or a Kolmogorov type spectrum like $k^{-5/3}$. These different spectra describe equilibrium situations, but in any case it is clear that the spectrum will be very different from white noise, which has a k^{+2} power spectrum (see Sec. IV B below). The possibility of energy transfer from small to large scales via an inverse cascade could be of major importance in cosmology. It could provide a seed field at the parsec or kiloparsec scale, albeit at small amplitude.

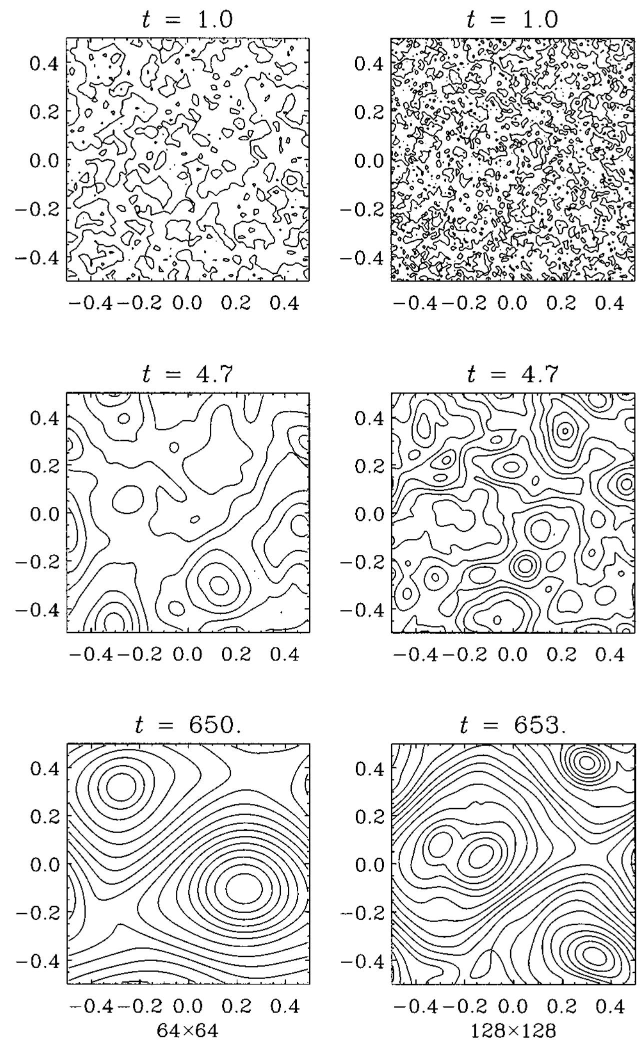

FIG. 1. Left column: magnetic field lines at different times at low resolution (64×64 mesh points). Right column: magnetic field lines at different times at higher resolution (128×128 mesh points).

IV. A CASCADE MODEL

A. Description

The ultimate goal is to solve the basic MHD equations in three dimensions at high resolution, using random initial conditions. Although we would be unable to cover a realistically large range of length scales, it is important to know whether dynamo action could be possible in a relativistic flow. This is a major task, which would go beyond the scope of this paper. To see the difficulties involved in such a program, the reader should recall the difficulties in making long-term weather predictions based on the Navier-Stokes equations. Therefore, in order to demonstrate some of the anticipated behavior of the full $1+3$ MHD universe, we now study a cascade model of hydromagnetic turbulence.

In ordinary hydrodynamics and hydromagnetics many properties of turbulence, in particular those related to energy transfer and to the spectral properties, including small intermittency corrections, have been studied successfully using a simple cascade model [18]. This is true not only qualitatively, but also quantitatively, which is the reason why the cascade model is now much used in studies of nonlinear physics (see, e.g., [19] and references therein).

The basic idea is that the interactions due to the nonlinear terms in the MHD equations are local in wave number space. In k space the quadratic nonlinear terms, such as $\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B})$, $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$, and $\mathbf{J}\times\mathbf{B}$, become a convolution and have the general form [20]

$$\mathbf{N_k}(\mathbf{v},\mathbf{B})=\int \mathbf{C}_{\alpha\beta}v_{\alpha}(\mathbf{p})B_{\beta}(\mathbf{p}-\mathbf{k})d^3p. \tag{32}$$

(There are similar terms also for the other two nonlinearities.) Where $\mathbf{C}_{\alpha\beta}=\mathbf{C}_{\alpha\beta}(\mathbf{k})$ is a tensor which is linear in $\mathbf{k}$. Interactions in k space involving triangles with similar side lengths have the largest contribution, as discussed in [20]. This has led to the shell model (see, e.g., [19] and references therein), which is formulated in the space of the modulus of the wave numbers. This space is approximated by N shells, where each shell consists of wave numbers with $2^n\leqslant k\leqslant 2^{n+1}$ (in the appropriate units). The Fourier transform of the velocity over a length scale k_n^{-1} ($k_n=2^n$) is given by the complex quantity v_n, and B_n denotes a similar quantity for the B field. Furthermore, the convolution is approximated by a sum over the nearest and the next nearest neighbors:

$$N_n(v,B)=\sum_{i,j=-2}^{2} C_{ij}v_{n+i}B_{n+j}. \tag{33}$$

Here v and B have lost their vectorial character, which reflects the fact that this model is not supposed to be an approximation of the original equations, but should be considered as a toy model that has similar *conservation* properties as the original equations. Thus, e.g., the energy flow should be represented by these equations. It is quite remarkable that such models show several realistic features, including intermittency corrections to the structure function exponents, and are therefore rather popular both in the absence [19] and in the presence [21] of magnetic fields. Therefore we propose to apply such a model also to the early universe.

Velocity and magnetic fields are thus represented by a scalar at the discrete wave numbers $k_n=2^n$ $(n=1,\ldots,N)$, i.e., k_n increases exponentially. Therefore such a model can cover a large range of length scales (typically up to ten orders of magnitude). The important conserved quantity is $E_{\rm tot}R^4$, where $E_{\rm tot}=\int T^{00}d^3x$ is the total energy. Using that the bulk velocity is nonrelativistic, we have $\gamma\rightarrow 1$, so we can expand $\gamma^2\approx 1+\mathbf{v}^2$. Hence

$$E_{\rm tot}\approx\int\left(\rho+\frac{4}{3}\rho\mathbf{v}^2+\frac{1}{2}\mathbf{B}^2\right)d^3x. \tag{34}$$

Since we are here mostly interested in the evolution of the magnetic field we ignore the detailed evolution of ρ and assume $\rho\approx\rho_0R^{-4}$. Thus we require that

$$\int\left(\frac{4}{3}\rho_0\mathbf{v}^2+\frac{1}{2}\mathbf{B}^2R^4\right)d^3x=\text{const.} \tag{35}$$

We use $\mathbf{b}=\mathbf{B}R^2$ and construct equations for v_n and b_n such that

$$\frac{8}{3}\rho_0\sum_{n=1}^{N} v_n^*\frac{dv_n}{d\tilde{t}}+\sum_{n=1}^{N} b_n^*\frac{db_n}{d\tilde{t}}=0. \tag{36}$$

In computing the conservation of the energy, the complex conjugate of this equation should be added. However, it turns out that the ‘‘complex energy’’ (exhibited in the above equation) is conserved by the following construction.

As pointed out, the main idea of the cascade model is to construct a set of equations that share the same basic conservation properties of the nonlinear (quadratic) terms as the original equations. Thus we write equations which mimic equations (9) and (14):

$$\frac{4}{3}\rho_0\frac{dv_n}{d\tilde{t}}=N_n(v,b), \tag{37}$$

$$\frac{db_n}{d\tilde{t}}=M_n(v,b), \tag{38}$$

where

$$\begin{aligned}2N_n(v,b)=&ik_n(A+C)(v_{n+1}^*v_{n+2}^*-b_{n+1}^*b_{n+2}^*)\\&+ik_n(B-\tfrac{1}{2}C)(v_{n-1}^*v_{n+1}^*-b_{n-1}^*b_{n+1}^*)\\&-ik_n(\tfrac{1}{2}B+\tfrac{1}{4}A)(v_{n-2}^*v_{n-1}^*-b_{n-2}^*b_{n-1}^*),\end{aligned} \tag{39}$$

$$\begin{aligned}M_n(v,b)=&ik_n(A-C)(v_{n+1}^*b_{n+2}^*-b_{n+1}^*v_{n+2}^*)\\&+ik_n(B+\tfrac{1}{2}C)(v_{n-1}^*b_{n+1}^*-b_{n-1}^*v_{n+1}^*)\\&-ik_n(\tfrac{1}{2}B-\tfrac{1}{4}A)(v_{n-2}^*b_{n-1}^*-b_{n-2}^*v_{n-1}^*),\end{aligned} \tag{40}$$

with A, B, and C being free parameters. It is straightforward to verify that $2\Sigma v_n^*N_n+\Sigma b_n^*M_n=0$, using that $k_n=2^n$. The $\tilde{t}$ differentiations in Eqs. (37) and (38) are included to mimic closely the nonrelativistic form of Eqs. (9) and (14). In the actual computations we have restored magnetic and kinematic diffusion terms, $-\nu k_n^2v_n$ and $-\eta k_n^2b_n$, on the right hand sides of (37) and (38), respectively. We chose $\nu=\eta$ and as time goes on, lowered ν gradually using the formula $\nu\geqslant(\Sigma k_n^2|v_n|^2)^{1/2}/k_{\rm max}^2$, where $k_{\rm max}=2^N$. This formula estimates the minimum amount of diffusion necessary to prevent the buildup of energy at the smallest resolved scale. We use a third order time step, which is calculated via the formula $\delta_t=0.25\min[(\Sigma k_n^2|v_n|^2)^{-1/2}]$.

B. Results

The numerical study of the cascade model requires, of course, that the parameters A, B, and C are fixed. This problem turns out to be quite interesting, since it allows one to associate the cascade model with a dimension. In hydrodynamics the parameters are fixed by taking into account a conservation law which is nontrivial in the dimension considered. In two dimensions, for example, one uses the re-

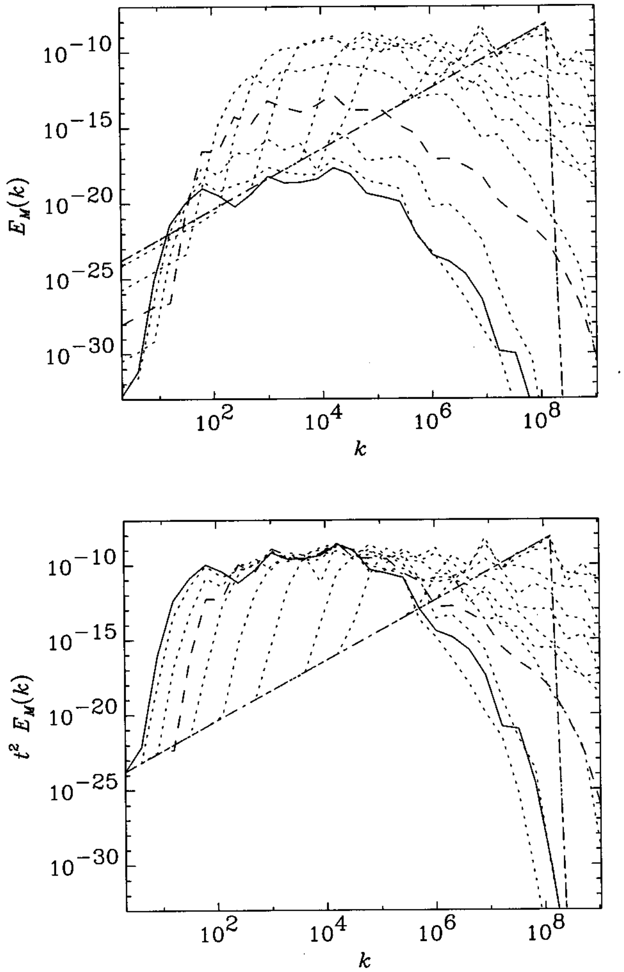

FIG. 2. Spectra of the magnetic energy at different times. The straight dotted-dashed line gives the initial condition ($t_0=1$), the solid line gives the final time ($t=3\times10^4$), and the dotted curves are for intermediate times [in uniform intervals of $\Delta \ln(t-t_0)=0.6$]. $A=1$, $B=-1/4$, and $C=0$.

quirement that the enstrophy is conserved, whereas in three dimensions the helicity should be conserved. In three dimensions Jensen *et al.* [19] used the values $A=1$, $B=-1/4$, and $C=0$, which we have adopted also in several models presented here. We compare the results with another set of parameters for which the quantity

$$H_M=\sum\ (-1)^n k_n^{-1} b_n^* b_n \tag{41}$$

is conserved, in addition to the total energy [21]. This requires that $A=7/5$, $B=-1/10$, and $C=1$. The quantity H_M resembles the magnetic helicity $\mathbf{A}\cdot\mathbf{B}$, which is important, because associated with it is the inverse cascade of magnetic helicity and energy [22]. In Fig. 2 we plot the spectral magnetic energy density $E_M(k_n)=|b_n|^2/(kR^4)$ computed for these values, with the initial field taken to be random [i.e., $E_M(k)=k^2$]. The reason we interpret this expression as the magnetic energy is that we know that $\Sigma b_n^* b_n$ enters in the conserved energy. However, $\Sigma\sim\int dn=\int dk/(k\ln 2)$, so $E_M(k_n)$ is the energy in k space. We used $N=30$, which covers a range of length scales of approximately ten orders of magnitude. As one can clearly see, magnetic energy is transferred from small scales to large scales. This is called the inverse cascade effect. Such an effect is found in many nonlinear systems, for example, in two-dimensional turbulence, relevant, e.g., for the atmosphere.

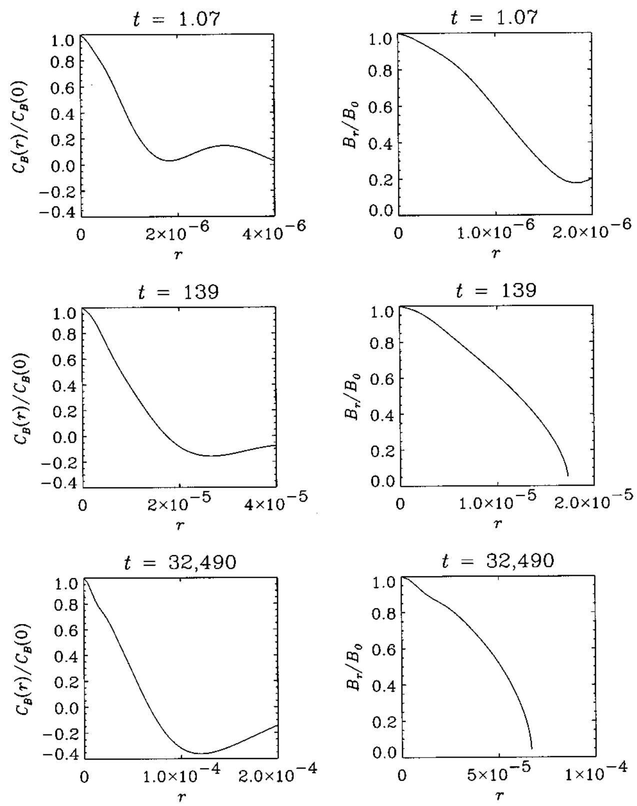

FIG. 3. Left column: the correlation function of B for three different times. Right column: the rms magnetic field as a function of distance for three different times.

The quantity of paramount interest is the magnetic field correlation function

$$C_B(r)\equiv\langle B(r+x)B(x)\rangle, \tag{42}$$

which is related to the power spectrum via a Fourier transform, $C_B(r)=\int E_M(k)\cos(kr)dk$. It is difficult in general to compute this quantity, due to the fluctuations in the spectrum $E_M(k)$. Therefore we have computed $C_B(r)$ from the spectra of the cascade model by interpolating $E_M(k)$ on a uniformly spaced mesh. This, of course, introduces some uncertainty. The result is shown in Fig. 3. Note the clear increase of the widths of the correlation functions.

Note also the anticorrelation at larger length scales. For a magnetic field this is natural, because if one considers the field in some region from a point far away from this region, the magnetic field in the region appears to be approximately a dipole. A negative correlation then arises because the field loop has to close. This would then basically be a consequence of div$\mathbf{B}=0$, which has a tendency to lead to negative correlations. However, the cascade model of course has the difficulty that it does not really operate in ordinary space, but instead it is formulated in the modulus of $\mathbf{k}$ space. Hence we cannot really investigate to what extent the condition div$\mathbf{B}=0$ is satisfied, in contrast to the two-dimensional case discussed in Sec. III B.

Another quantity of interest is the average magnetic field as a function of distance,

$$\mathbf{B}(r)\equiv\frac{1}{r^D}\int d^D x\mathbf{B}(\mathbf{x}), \tag{43}$$

where the integration is over a volume of size r^D in D dimensions. From this definition we have

$$\langle\mathbf{B}(r)^2\rangle=\frac{1}{r^{2D}}\int d^D x\int d^D y\langle\mathbf{B}(\mathbf{x})\mathbf{B}(\mathbf{y})\rangle, \tag{44}$$

where both integrations are over a volume of size r^D. Thus the root mean square magnetic field

$$B_r=\langle\mathbf{B}(r)^2\rangle^{1/2} \tag{45}$$

can be computed directly from the correlation function $C_B(r)$ via

$$B_r=\left(\frac{1}{r^3}\int_0^r r'^2 dr' C_B(r')\right)^{1/2}. \tag{46}$$

For a random field, B_r behaves like $r^{-D/2}$, so the interesting question is whether this initial behavior changes as time passes. In Fig. 3 we show the results. There is a clear broadening of B_r towards larger distances as time passes, as we would expect from the inverse cascade behavior.

The determination of the width of the correlation function above is not very accurate because of the fit involved in computing the Fourier transform of the spectrum. We shall therefore now introduce another length scale that is easier to compute, but whose value is similar to the width of the correlation function. The relevant length scale in turbulence theory is the so-called integral scale, which is the characteristic length associated with the large energetic eddies of turbulence. Roughly speaking one could view it as a measure of the coherence of the magnetic field, too. It is defined by

$$l_0=\int 2\pi k^{-1}E_M(k)dk \bigg/ \int E_M(k)dk, \tag{47}$$

which, in our cascade model, corresponds to $l_0=\Sigma 2\pi k_n^{-1}|b_n|^2/\Sigma|b_n|^2$. If the spectrum is random we get $l_0\simeq 3/22\pi k_{\max}^{-1}$, where $2\pi/k_{\max}$ is the shortest length scale present in the model. This length scale in the initial random spectrum is determined by the mechanism generating the primordial field. In Fig. 4 we show the evolution of $l_0(t)$ in two cases, namely, for the hydromagnetic A,B,C (circular points) and for the MHD A,B,C (diamond-shaped points). Although the two sets of values for A,B,C do not yield identical results, we see that the curves are qualitatively similar. In both cases l_0 increases rapidly by 4–5 orders of magnitude, and there is a plateau structure. The MHD result (diamond-shaped points) has a plateau stretching to $t/t_0=10^8$, but for larger times l_0 keeps increasing. The increase of l_0 by almost five orders of magnitude is important, because it could lead to magnetic fields at the present time at length scales comparable to 1 pc. If we take the electroweak phase transition as the initial state, the QCD phase transition occurs approximately for $t/t_0=10^6$. The maximum time $t/t_0=10^9$ reached in our simulation corresponds to a temperature of 3 MeV, which is close to the nucleosynthesis. It should be noticed that from these results one cannot, of course, say anything about what happens at later times. Therefore it could be that $l_0(t)$ increases further, either by reaching new plateau(s), or otherwise.

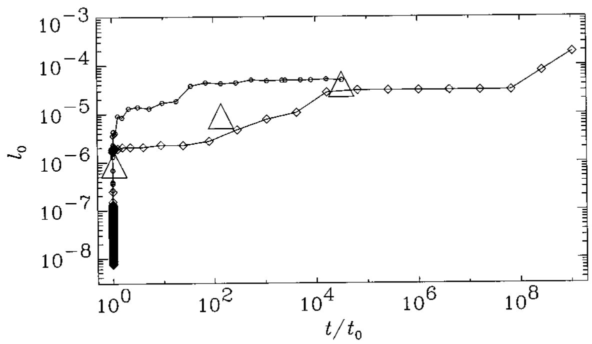

FIG. 4. Evolution of the integral scale. The plot symbols denote uniform intervals of $\Delta\ln(t-t_0)=0.6$. The circular points correspond to the hydrodynamic values $A=1$, $B=-1/4$, and $C=0$, the diamonds correspond to the MHD values $A=7/5$, $B=-1/10$, and $C=1$, and the triangles are the widths computed for the correlation function.

We also measured the integral scale of the magnetic field in the two-dimensional model of Sec. III B and found a clear increase with time. For the three times plotted in Fig. 1 we found for the 64×64 case the values $l_0=0.09$, 0.50, and 0.95, whereas in the 128×128 case the values were $l_0=0.04$, 0.28, and 0.76. The initial difference of a factor of 2 is due to the different resolution. At later times the integral scales for low and high resolution are more similar.

The evolution of l_0 is not straight, but if we make a linear fit through the values given by the diamonds in Fig. 4 we find (ignoring the steep initial increase)

$$l_0(t)\approx r_0(t/t_0)^{1/4}. \tag{48}$$

For further applications of the rough fit formula (48) it may be more sensible to express time in terms of temperature, $T\propto t^{-1/2}$, so

$$l_0(T)\approx r_0(T/T_0)^{-1/2}, \tag{49}$$

where $r_0\approx 10^{-6}$; see Fig. 4. If we wish to extrapolate to the present time we first have to fix the scale r_0 by physical arguments. The various models presented in the literature [1,9] give characteristic scales for the primordial field when it is generated. This scale should be identified with the lowest scale in our calculations which, in the case of the shell model, is about 10^{-8}. The scale r_0 is typically somewhat larger (10^{-6} in the shell model). The reason for this is presumably that a purely random initial condition is not consistent with the MHD equations.

In order to clarify these points we take an example. If we assume that at the time of the electroweak phase transition ($T=100$ GeV) r_0 was 10^{-3} cm (the horizon scale was ≈ 4 cm) then, using our extrapolation (49), we arrive at a scale of 2 pc. If we assume that the initial magnetic field was 10^{18} G, then the present day value would be 10^{-11} G. Such values would lead to sufficiently strong seed magnetic fields to explain the field even in high redshift galaxies by dynamo action [6]. This extrapolation may be too naive, because the nature of turbulence will change as the universe cools down. Furthermore, at later times, when structure formation begins, gravitational energy may lead to additional stirring and enhancement of turbulence in localized regions.

V. DISCUSSION

In the two-dimensional case we found that, starting from a small-scale magnetic field, magnetic structures develop at progressively larger scales. This process of self-organization corresponds to an inverse cascade of magnetic energy and helicity. Using then a cascade (or shell) model to study three-dimensional MHD turbulence we were able to follow this inverse cascade over much longer times. Such a cascade model has been rather successful in the study of hydrodynamic turbulence.

The possibility of an inverse cascade means that the scale of fluctuations of the primordial magnetic field increases much beyond its original scale given by particle physics. Taking the parameter l_0 as a measure of the coherence length of the magnetic field, we see that there is an increase in the coherence of 4–5 orders of magnitude. This means that previous estimates of the field strengths in various mechanisms for generating a primordial field should be revised accordingly. For example, let us consider the estimate by Vachaspati in Ref. [1]. Taking the area average one has the estimate

$$B_r \sim gT^2/4N, \tag{50}$$

where N is the number of steps needed to reach a given scale in terms of the ''fundamental'' scale at which the field is generated. In this case the fundamental scale is the electroweak scale [1]. Proceeding as in Ref. [1] one has $N \sim 10^{24}$ today, if the relevant scale is of order 100 kpc. However, due to the MHD corrections N is, from a conservative point of view, reduced because of the effect of turbulence, and one would instead have $N \leqslant 10^{19}$, which reduces the stochastic decrease of B_r. It should be emphasized that from our calculations one can only say what happens up to a time of order $10^9 t_{\rm EW}$, so presumably N is considerably below 10^{19} today.

The turbulent nature of the magnetic field may have interesting effects on the various phase transitions in the early universe. Also, the inherent shift of energy from small to large scales may be of interest in connection with the density fluctuations due to the magnetic energy.

Of course, the cascade model is a *model* of the real (1+3)-dimensional MHD turbulence. Its successful application in many, widely different nonlinear physical problems suggests, however, that it might also be applicable to the primordial magnetic fields of the early universe. Therefore we believe that its indication of the strong increase in the coherence scale of the primordial field should be taken seriously, and that further, more detailed studies are warranted.

ACKNOWLEDGMENTS

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APPENDIX

We give here the equations for the case where the bulk velocity is small (the gas remains still relativistic, which is important for the scaling properties of $\tilde{\rho}$ and $\tilde{p}$):

$$\frac{\partial \ln\tilde{\rho}}{\partial \tilde{t}} = -\frac{4}{3}(\mathbf{v}\cdot\boldsymbol{\nabla}\ln\tilde{\rho} + \boldsymbol{\nabla}\cdot\mathbf{v}) - \frac{\tilde{\mathbf{J}}\cdot\tilde{\mathbf{E}}}{\tilde{\rho}}, \tag{A1}$$

$$\frac{D\mathbf{v}}{D\tilde{t}} = -\mathbf{v}\left(\frac{D\ln\tilde{\rho}}{D\tilde{t}} + \boldsymbol{\nabla}\cdot\mathbf{v}\right) - \frac{1}{4}\boldsymbol{\nabla}\ln\tilde{\rho} + \frac{\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}}{\frac{4}{3}\tilde{\rho}}, \tag{A2}$$

where $D/D\tilde{t} = \partial/\partial\tilde{t} + \mathbf{v}\cdot\boldsymbol{\nabla}$ is the total derivative, and

$$\frac{\partial\tilde{\mathbf{B}}}{\partial\tilde{t}} = \nabla\times(\mathbf{v}\times\tilde{\mathbf{B}}), \quad \tilde{\mathbf{J}} = \boldsymbol{\nabla}\times\tilde{\mathbf{B}}. \tag{A3}$$

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