PHYSICAL REVIEW D VOLUME 54, NUMBER 2 15 JULY 1996

Scattered light noise in gravitational wave interferometric detectors: Coherent effects

Jean-Yves Vinet
Laboratoire d'Optique Appliquée, Ecole Polytechnique-Ecole Nationale Supérieure de Techniques Avancées, 91120 Palaiseau, France

Violette Brisson
Groupe Virgo, Laboratoire de l'Accélérateur Linéaire, Bat 208, Université Paris-Sud, 91405 Orsay, France

Stefano Braccini
Istituto Nazionale di Fisica Nucleare, Sezione di Pisa, Via Livornese 582/a, San Piero a Grado I-56010 Pisa, Italy
(Received 15 January 1996)

Large interferometric detectors of gravitational waves involve high power light beams stored in optical resonators or Fabry-Perot cavities, installed in km long vacuum pipes. Scattering of light by mirrors, interaction of the scattered light with the walls of the vacuum vessel, and final rescattering on any mirror is a source of noise in such antennas. We present some results obtained within the framework of a coherent approach of scattered light propagation using a coherence function allowing us to study, namely, the effects of small reflecting surfaces along the tube, of diffraction, and of reflection by eventual baffle edges. [S0556-2821(96)05014-X]

PACS number(s): 04.80.Nn, 95.55.Ym

I. INTRODUCTION

Laser interferometers for gravitational wave (GW) detection are presently being developed by at least three collaborations [1–3]. A common feature to all of these future facilities is the storage of light power in long resonant cavities. The optical system must operate in a vacuum and requires at the same time a very efficient seismic isolation. Here, a spurious effect arises: The imperfect mirrors constituting the optical system scatter a weak but finite amount of the stored light over wide angles, allowing it to interact with the vacuum vessel's walls, to be reflected, diffracted, or scattered again, and finally reach any mirror of the system, including the initial one, where a second scattering recombines a part of that scattered light into the stored standing wave. The trouble comes from the fact that, during its interaction with the walls or any linked structure, the light acquires a phase modulation determined by the vibration state of the vacuum tank, sustained by the seismic activity of the ground, and transfers the resulting noise to the standing wave. The result is a bypass of the seismic isolation sytem. This effect was first noticed by the German team operating the Garching prototype of interferometer [4,5].

Owing to the second order scattering process on superpolished mirrors (the best of the present state of the art), the overall effect is expected to be very small, but the shot-noise-equivalent spectral density of phase noise in such optical devices can be as low as $10^{-11}\ \mathrm{Rd}/\sqrt{\mathrm{Hz}}$, and, as customary in this field, even currently negligible effects must be carefully investigated.

Mirrors scatter light because of their roughness. Scattering of electromagnetic waves by rough surfaces has been studied by a number of authors [6] even in the case of large rms roughnesses. Fortunately, we have to deal here only with very weak roughnesses, which allows a significant simplification of the general theory. On the other hand, propagation of stray light in complex systems, and its attenuation, can be treated by well-known techniques based on very general Monte Carlo codes mastered by specialized companies, but not addressing the question of seismic noise injection. The first attempt to make a thorough analysis of the spectral density of noise caused by scattered light in GW interferometers is due to Thorne [7] who introduced the basic concepts some years ago. Since this paper, theoretical, numerical, or even experimental work, often unpublished, has been carried out in the above-mentioned collaborations, for instance by Winkler *et al.* [8,9].

We present here a very special study restricted to effects that can be treated by wave optics and evaluated by a simple analytical calculation, namely, the effects of reflecting surfaces existing in a vacuum pipe, for instance, resulting from imperfections (weldings, scratches), from junctions of secondary pipes to pumping stations, bellows, and even edges of baffles installed for the purpose of scattered light suppression, but being able not only to reflect, but also to diffract scattered light and to couple directly the two mirrors of a cavity by their edges as a residual effect.

We describe in Sec. II the theory to be used; then, for more convenience we give the results of physical interest in Sec. III and the detailed calculations in the Appendix.

II. BASIC THEORY OF COHERENT SCATTERED LIGHT NOISE

We propose here a general formalism allowing us to treat scattered light within the wave optics framework.

A. Emission of scattered light

The scattered light we are faced with is generated by reflection of a Gaussian beam on mirrors with weak roughnesses. The mirrors planned for GW interferometers will have rms roughness small compared to a wavelength of the laser source. Typically, optical surfaces of rms roughness less than 1 Å will be illuminated by the $\lambda \approx 1 \mu$m radiation of a Nd yttrium aluminum garnet (YAG) laser.

We define the rough surface of a mirror by a random departure from its ideal shape, so that, $\vec{x}$ being the coordinates in the mirror's tangent plane, the equation of the mirror's surface can be written as $z = F(\vec{x}) + f(\vec{x})$. $F(\vec{x})$ refers to the nominal geometry of the mirror (possibly spherical), and the function $f(\vec{x})$ will be viewed as a two-dimensional stationnary, centered stochastic process, of standard deviation σ. This means that for fixed $\vec{x}$, $f(\vec{x})$ is a random variable with expectation value $\langle f(\vec{x}) \rangle = 0$ and variance $\langle f(\vec{x})^2 \rangle = \sigma^2$. For two different locations $\vec{x}$ and $\vec{x'}$, $f(\vec{x})$ and $f(\vec{x'})$ are not, in general, independent random variables, and we can define the autocorrelation function $C(\vec{x})$ of the process by

$$C(\vec{x} - \vec{x'}) = \langle f(\vec{x}) f(\vec{x'}) \rangle / \sigma^2, \tag{2.1}$$

so that, obviously, $C(\vec{0}) = 1$. The fact that C depends on the difference $\vec{x} - \vec{x'}$ results from stationnarity. Now, it will be further assumed in what follows, that C depends only on $\|\vec{x} - \vec{x'}\|$, in other words that the roughness is isotropic.

Suppose now that a Gaussian transverse-electromagnetic (TEM_{00}) wave is impinging normally onto the preceding mirror. Let us call $\psi(\vec{x})$ the reemitted wave and $\phi_0(\vec{x})$ the (normalized) TEM_{00} wave which would have been reflected by the smooth version of the same mirror ($\sigma = 0$); we have

$$\psi(\vec{x}) = e^{2ikf(\vec{x})} \phi_0(\vec{x}),$$

where $k \equiv 2\pi/\lambda$. We are now interested in the spatial spectral density of ψ. We can for this purpose compute the squared modulus of its Fourier transform (we note $\vec{p}$ the conjugate variable with respect to $\vec{x}$):

$$|\tilde{\psi}(\vec{p})|^2 = \int e^{i\vec{p}(\vec{x} - \vec{x'})} e^{2ik[f(\vec{x}) - f(\vec{x'})]} \phi_0(\vec{x}) \phi_0(\vec{x'})^* d\vec{x} d\vec{x'}.$$

Owing to the hypothesis that $f \ll \lambda$, we can expand the second exponential and write

$$|\tilde{\psi}(\vec{p})|^2 = \int e^{i\vec{p}(\vec{x} - \vec{x'})} (1 + 2ik[f(\vec{x}) - f(\vec{x'})] - 2k^2[f(\vec{x})^2 + f(\vec{x'})^2 - 2f(\vec{x}) f(\vec{x'})]) \phi_0(\vec{x}) \phi_0(\vec{x'})^* d\vec{x} d\vec{x'},$$

which yields the expectation value

$$\langle |\tilde{\psi}(\vec{p})|^2 \rangle = (1 - 4k^2\sigma^2) |\tilde{\phi}_0(\vec{p})|^2 + 4k^2\sigma^2 \int e^{i\vec{p}(\vec{x} - \vec{x'})} \times C(\vec{x} - \vec{x'}) \phi_0(\vec{x}) \phi_0(\vec{x'})^* d\vec{x} d\vec{x'}.$$

After some elementary algebra, this becomes

$$\langle |\tilde{\psi}(\vec{p})|^2 \rangle = (1 - 4k^2\sigma^2) |\tilde{\phi}_0(\vec{p})|^2 + 4k^2\sigma^2 \frac{1}{4\pi^2} \int \tilde{C}(\vec{q}) |\tilde{\phi}_0(\vec{p} - \vec{q})|^2 d\vec{q}. \tag{2.2}$$

At this point, we introduce the following remarks.

(1) $|\tilde{\phi}_0(\vec{p})|^2$ is a sharply peaked function of $\vec{p}$, taking significant values only in the neighborhood of $\vec{p} = \vec{0}$. This is especially true for the TEM_{00} modes of the giant Virgo or Ligo-type Fabry-Perot cavities, for which the angular divergence of the beam is less than 20 μRd. Denoting by w_0 the waist of the beam (about 2 cm for the Virgo cavities), we have $|\tilde{\phi}_0(\vec{p})|^2 = 2\pi w_0^2 \exp(-p^2 w_0^2/2)$. Identifying spatial frequencies with angles according to $\vec{p} \equiv (k\vartheta\cos\phi, k\vartheta\sin\phi)$ ($\vec{p}$ being interpreted as the projection of the wave vector on the transverse plane) yields $|\tilde{\phi}_0(\vartheta, \phi)|^2 = 2\pi w_0^2 \exp(-2\vartheta^2/\vartheta_g^2)$ where ϑ_g is the angular divergence of the beam, defined by $\vartheta_g \equiv \lambda/\pi w_0$. Thus, clearly, $|\tilde{\phi}_0(\vec{p})|^2$ is negligible for angles noticeably larger than ϑ_g.

(2) We assume that $\tilde{C}(\vec{p})$ does not appreciably vary over an angular interval of the order of ϑ_g.

This allows us to replace Eq. (2.2) by the approximation

$$\langle |\tilde{\psi}(\vec{p})|^2 \rangle = (1 - 4k^2\sigma^2) |\tilde{\phi}_0(\vec{p})|^2 + 4k^2\sigma^2 \tilde{C}(\vec{p}),$$

where we see that the spatial spectral density of reemitted light is the sum of two contributions, one having the angular properties of a specularly reflected beam, with a loss factor of $4k^2\sigma^2$, and a second one directly tied to the statistical properties of the surface. Moreover, the total relative reemitted power (the TEM_{00} mode was assumed normalized) is conserved, but shared between what we identify to specular reflection,

$$P_{\mathrm{spec}} = \frac{P_{\mathrm{main}}}{4\pi^2} \int (1 - 4k^2\sigma^2) |\tilde{\phi}_0(\vec{p})|^2 d\vec{p} = (1 - 4k^2\sigma^2) P_{\mathrm{main}},$$

where P_{main} is the light power flux of the stored wave, and what we identify to scattering,

$$P_{\mathrm{scatt}} = \frac{P_{\mathrm{main}}}{4\pi^2} \int 4k^2\sigma^2 \tilde{C}(\vec{p}) d\vec{p} = 4k^2\sigma^2 P_{\mathrm{main}},$$

the quantity $\epsilon \equiv P_{\mathrm{scatt}}/P_{\mathrm{main}} \equiv 4k^2\sigma^2$ will be called integrated scattering loss rate or scattering loss for brevity. We can, therefore, write, as well,

$$\frac{1}{P_{\mathrm{main}}} \frac{dP_{\mathrm{scatt}}}{d\vec{p}} = \frac{\epsilon}{4\pi^2} \tilde{C}(\vec{p}),$$

by identifying, as above, spatial frequencies and angles by $\vec{p} \equiv (k\sin\vartheta\cos\phi, k\sin\vartheta\sin\phi)$ or

$$\frac{d}{d\vec{p}} \equiv \frac{\lambda}{4\pi^2} \frac{d}{\sin\vartheta\, d\vartheta\, d\phi} \equiv \frac{\lambda}{4\pi^2} \frac{d}{d\Omega}.$$

We obtain

$$\frac{1}{\epsilon P_{\rm main}}\frac{dP_{\rm scatt}}{d\Omega}=\frac{1}{P_{\rm scatt}}\frac{dP_{\rm scatt}}{d\Omega}=\tilde{C}(k\sin\vartheta)/\lambda^2, \quad (2.3)$$

where $C(\vec{x})$ depending only on $x=\|\vec{x}\|$ (isotropy of the roughness), also consequently $\tilde{C}(\vec{p})=\tilde{C}(p)$. Now, $dP_{\rm scatt}/P_{\rm scatt}d\Omega$ is a normalized distribution, isotropic with respect to angle ϕ, and we can write

$$\frac{dP_{\rm scatt}}{P_{\rm scatt}d\Omega}=\frac{p(\vartheta)}{2\pi}, \quad (2.4)$$

with $\int p(\vartheta)\sin\vartheta d\vartheta=1$. And finally, by a comparison between the two last equations,

$$\tilde{C}(k\sin\vartheta)=\frac{\lambda^2}{2\pi}p(\vartheta). \quad (2.5)$$

Information on the normalized angular density of scattered power (ADSP) $p(\vartheta)$ can be obtained by different ways depending on the angular range. For very small angles, corresponding to long correlation distance defects, a direct measurement of the surface by using a profilometer can be carried out. For larger angles, a direct measurement of the ADSP is possible. Here, we shall assume that in the angular region of interest for our study ($\vartheta\gtrsim 10^{-4}$ Rd), we can take a reference model of the form $p(\vartheta)=\kappa/\vartheta^2$ [7,10].

B. Coherence function

The central concept for a wave optics treatment of light scattered from a Gaussian beam is the coherence function. We see from the elementary theory developed above that the light scattered from a Gaussian beam reflected off a mirror of roughness $f(\vec{x})$ can be viewed as emitted from the source $s(\vec{x})=2kf(\vec{x})\phi_0(\vec{x})$ located in the plane $z=0$ of the mirror's surface. The paraxial theory of diffraction provides a means to compute the wave $s_d(\vec{x})$ propagated at a finite distance d from the source. Let us call $G_d(\vec{x})$ the paraxial diffraction kernel

$$G_d(\vec{x})=-\frac{i}{\lambda d}e^{ik\vec{x}^2/2d};$$

we have, thus,

$$s_d(\vec{y})=\int G_d(\vec{y}-\vec{x})s(\vec{x})d\vec{x}.$$

We shall call the coherence function associated with the scattering process, the expectation value:

$$\mathbf{C}(d;\vec{y},\vec{y'})=\langle s_d(\vec{y})s_d(\vec{y'})^*\rangle.$$

After some elementary algebra, we find

$$\mathbf{C}(d;\vec{y},\vec{y'})=\frac{\epsilon}{4\pi^2}\int e^{-i\vec{p}(\vec{y}-\vec{y'})}\tilde{C}(\vec{p})\phi_d(\vec{y}+\vec{p}d/k)\times\phi_d(\vec{y'}+\vec{p}d/k)^*d\vec{p},$$

where $\phi_d(\vec{x})$ is the TEM_{00} diffracted at the distance d. The argument already used in Sec. II A holds again: The integral is in fact restricted to a narrow frequency domain where the neighborhood of $\vec{p}_0=k\vec{y}/d$ intersects the neighborhood of $\vec{p'_0}=k\vec{y'}/d$. Within this elementary region, $\tilde{C}(\vec{p})$ does not vary appreciably, and moreover, as we would expect, the integral vanishes if $k\|\vec{y}-\vec{y'}\|/d$ is larger than $k\vartheta_g$, for in this case, the overlap between the two neighborhoods numerically reduces to void. This allows us to replace $\tilde{C}(\vec{p})$ by $\tilde{C}(\vec{p}_0)\simeq\tilde{C}(\vec{p'_0})$ in the integral, giving

$$\mathbf{C}(d;\vec{y},\vec{y'})=\frac{\epsilon}{4\pi^2}\tilde{C}(k\vec{y}/d)\int e^{-i\vec{p}(\vec{y}-\vec{y'})}\phi_d(\vec{y}+\vec{p}d/k)\times\phi_d(\vec{y'}+\vec{p}d/k)^*d\vec{p},$$

when $\mathbf{C}(d;\vec{y},\vec{y'})$ takes on significant values; $\vec{y},\vec{y'}$ are so close together that we can write $\tilde{C}(ky/d)\simeq\tilde{C}(ky'/d)\simeq\tilde{C}(k\vartheta)=\lambda^2p(\vartheta)/2\pi$, where ϑ is the angle locating the small region around $\vec{y},\vec{y'}$. Now the integral can be explicitly carried out, and we find, after a cumbersome but straightforward calculation using Eq. (2.5):

$$\mathbf{C}(d;\vec{y},\vec{y'})=\frac{\epsilon}{2\pi d^2}p(\vartheta)e^{-(\vec{y}-\vec{y'})^2/2d^2\vartheta_g^2}e^{ik(y^2-y'^2)/2d}. \quad (2.6)$$

This function is equivalent to an autocorrelation function for the scattered field, expressing the size of the speckle. If $\vec{y}=\vec{y'}$, we have the expectation value of the relative scattered intensity:

$$I_s(d,\vartheta)=\frac{\epsilon}{2\pi d^2}p(\vartheta).$$

C. Coupling factor

Consider now a process in which an emitting mirror M_1 sends scattered light to a reflecting or scattering object linked to the ground, which in turn sends a part of that light to a mirror M_2 (eventually, $M_1=M_2$). We first consider the source of scattered light located on M_1: $s_1(\vec{x})=2kf_1(\vec{x})\phi_0(\vec{x})$ where $f_1(\vec{x})$ refers to the roughness of the mirror. After propagation at a distance d_1 where the coupling object is located, the diffracted wave is

$$s_2(\vec{y})=\int G_{d_1}(\vec{y}-\vec{x})s_1(\vec{x})d\vec{x}.$$

The action of the coupling object (either a reflection or a diffraction) will be represented by the complex function $m(t,\vec{y})$ of which some examples will be given in later sections. The time dependence of m expresses its motion caused by the seismic vibrations. The wave diffracted off the element to mirror M_2, at the distance d_2, can be expressed as

$$s_3(\vec{z})=\int G_{d_2}(\vec{z}-\vec{y})m(t,\vec{y})s_2(\vec{y})d\vec{y}.$$

This wave generates a source of scattered light on mirror M_2: namely,

$$s_4(\vec{z}) = 2kf_2(\vec{z})M_2(\vec{z})s_3(\vec{z}),$$

where $f_2(\vec{x})$ refers to the roughness of M_2. f_1 and f_2 are assumed to have the same statistical parameters (including autocorrelation function). Now, we compute the coupling of that scattered light with the main TEM_{00} beam. The Hermitian scalar product being denoted by $(\ldots,\ldots)$, we have

$$\gamma = (\phi_0, s_4) = \int \phi_0(\vec{z})^* M_2(\vec{z}) s_4(\vec{z}) d\vec{z}.$$

Another way of expressing γ is

$$\gamma = \int m(t,\vec{y})\Psi_1(\vec{y})\Psi_2(\vec{y})d\vec{y},$$

with

$$\Psi_1(\vec{y}) = \int G_{d_1}(\vec{y}-\vec{x})2kf_1(\vec{x})\phi_0(\vec{x})d\vec{x},$$

$$\Psi_2(\vec{y}) = \int G_{d_2}(\vec{y}-\vec{x})2kf_2(\vec{x})M_2(\vec{x})\phi_0(\vec{x})^* d\vec{x},$$

but it can be seen that $M_2(\vec{x})\phi_0(\vec{x})^*$ is nothing but $\phi_0(\vec{x})$, because both reflection on a matched mirror and phase conjugation reverse the wave front, so that, as well,

$$\Psi_2(\vec{y}) = \int G_{d_2}(\vec{y}-\vec{x})2kf_2(\vec{x})\phi_0(\vec{x})d\vec{x};$$

so Ψ_1 and Ψ_2 have similar expressions, and γ can be viewed as a scalar product of two waves coming from the two mirrors, taken on the studied surface element, an idea which has been already presented in [7]. If the two mirrors M_1 and M_2 are different, the processes f_1 and f_2 are obviously independent. If M_1 and M_2 are the same mirror, we shall still consider that the roughness at departure is statistically independent of that on return. This is justified by the physical argument that the returning wave is not at all distributed as the initial one on the mirror, and, does not see the same realization of the stochastic process, and that the effects we want to evaluate would remain essentially unchanged if the mirror were suddenly replaced by a statistically equivalent one in between emission and reception of the scattered light. Now, γ is a centered random variable of variance

$$\langle\gamma\gamma^*\rangle = \int m(t,\vec{y})m(t',\vec{y'})^*\langle\Psi_1(\vec{y})\Psi_1(\vec{y'})^*\rangle \times\langle\Psi_2(\vec{y})\Psi_2(\vec{y'})^*\rangle d\vec{y}d\vec{y'}.$$

Note that f_1 and f_2 being independent processes, Ψ_1 and Ψ_2 too, and the expectation value acts separately on the two waves. But the statistical parameters of both processes being the same, we get, using the coherences functions of these scattering processes,

$$\langle\gamma\gamma^*\rangle = \int m(t,\vec{y})m(t',\vec{y'})^*\mathbf{C}(d_1;\vec{y},\vec{y'})\mathbf{C}(d_2;\vec{y},\vec{y'})d\vec{y}d\vec{y'}, \quad (2.7)$$

which is equivalent to a temporal autocorrelation function. After a Fourier transform with respect to time, one obtains the spectral density.

D. Equivalent gravitational noise calculation

The phase noise induced in the interferometer can be related as follows to γ. $\gamma(t)$ represents the amplitude of the contribution to the main wave from recombined scattered light. This means that the total amplitude reflected by mirror M_2 into the TEM_{00} mode is

$$A = A_0[1+\gamma(t)],$$

where A_0 is the unperturbed amplitude. One easily sees that the small change in complex optical amplitude results at first order in γ in a phase modulation of the main wave given by

$$\Delta\Phi(t) = \mathrm{Im}[\gamma(t)].$$

In order to compute the equivalent gravitational signal, we first find the elementary change induced by a translation δL of a mirror on the phase of one interferometer arm: $\Delta\Phi_{\text{elem}} = 4\pi\delta L/\lambda$; now, this translation may be viewed as caused by a GW amplitude h such that $\delta L(t) = \frac{1}{2}h(t)L$, where L is the length ot the interferometer arms. The total phase change in the interferometer when it is due to a GW is twice the elementary one, so that $\Delta\Phi(t) = 4\pi L h(t)/\lambda$ [7]. By comparing with the phase modulation caused by scattered light, we can say that the gravitational signal which would cause the same phase modulation is

$$h(t) = \frac{\lambda}{4\pi L}\mathrm{Im}[\gamma(t)].$$

The h equivalent of the spectral density of scattered light noise is, thus,

$$h(f) = \frac{\lambda}{4\pi L}\gamma(f), \quad (2.8)$$

where $\gamma(f)$ refers to the spectral density of the temporal process $\mathrm{Im}[\gamma(t)]$. In some cases (reflections), we obtain a coupling coefficient of the form $\gamma(t) = \gamma_0 e^{i\Phi_0 + 4i\pi\delta x(t)/\lambda}$ where γ_0 is a real positive constant, Φ_0 a dc phase, and $\delta x(t)$ the motion of the coupling element. In these cases, we have $\gamma(f) = \gamma_0 n(f)$, where $n(f)$ is the spectral density of the process $\sin[\Phi_0 + 4\pi\delta x(t)/\lambda]$. If the maximum amplitude of displacement is small compared to the wavelength of light, taking a uniform random variable for the phase offset, we get

$$n(f) = \frac{2\sqrt{2}\pi\delta x(f)}{\lambda}, \quad (2.9)$$

where $\delta x(f)$ is the spectral density of displacement. If now the amplitude of displacement is large compared to λ, due, for instance, to a resonance at a given frequency f_0, denoting by δx_0 the rms displacement in the width of the resonance, the density $n(f)$ becomes flat until a cutoff frequency $f_{\max}=f_0\times 4\pi\delta x_0/\lambda$, and we can write, approximately,

$$n(f)=\begin{cases}(2f_{\max})^{-1/2} & \text{for } f<f_{\max},\\ 0 & \text{for } f>f_{\max}.\end{cases}$$

Since $\int_0^\infty n(f)^2 df=\frac{1}{2}$, it is clear that in the regime of strong excitation, the spectral density lowers and widens. In other cases (diffraction) we obtain coupling coefficients of the form $\gamma(t)=\gamma_0(t)e^{i\Phi_0}$ where Φ_0 is an unknown constant phase, and $\gamma_0(t)$ a real function. In this case we shall take the spectral density

$$\gamma(f)=\frac{1}{\sqrt{2}}\gamma_0(f). \tag{2.10}$$

III. DISCUSSION OF SOME NOISY EFFECTS IN GW INTERFEROMETERS

A. Small spurious mirrors

We address here the case of small reflecting surfaces located near the internal surface of the vacuum pipe, where scattered light can be directly reflected to the emitting mirror (see Fig. 1). The pipe's internal radius being denoted by R_t, the surface of the element can be represented by $\vec{y}=(R_t+X,Y)$. It is convenient to take a rectangular element such that $-a/2<X<a/2$, $-b/2<Y<b/2$. It is further assumed to have a normal making an angle (α,β) with the optical axis, and a mean curvature radius of r_c, so that its action on any wave front is expressed by means of the complex function

$$m(\vec{y})=\sqrt{R}e^{-2ik\alpha[X\cos(\beta)+Y\sin(\beta)]}\times e^{ik(X^2+Y^2)/2r_c}e^{i[\Phi_0+4\pi\delta x(t)/\lambda]},$$

where R is the intensity reflection coefficient. The static phase Φ_0 represents the mean position of the element along the z axis, and $\delta x(t)$ its longitudinal motion sustained by the seismic noise. We get the spectral density of noise (see Appendix A1),

$$h(f)=h_{\max}(f)F,$$

where F is a form factor $(0<F<1)$ depending on the various parameters (see Figs. 1–4), and

$$h_{\max}(f)=\frac{\lambda}{4\pi L}\frac{\epsilon}{2\pi d^2}p(\vartheta)\sqrt{R}Sn(f),$$

where d is the distance of the reflecting element from the emitting mirror, and $n(f)$ the spectral density discussed in Sec. II D. If we use the reference model of ADSP, so that $p(\vartheta)=\kappa(d/R_t)^2$, and assume that the seismic displacement amplitude is small compared to the wavelength, we get

$$h_{\max}(f)=\frac{\epsilon\kappa}{2\sqrt{2}\pi}\frac{S}{R_t^2}\frac{\delta x(f)}{L}\sqrt{R}, \tag{3.1}$$

with the parameters

λ,	wavelength,	1.06×10^{-6} m,
$\epsilon\kappa$,	scattering,	10^{-6},
S,	reflecting surf.,	10^{-6} m^2,
R_t,	tube radius	0.6 m,
L,	arm length,	3×10^3 m,
R,	reflectivity,	1,
$\delta x(f)$,	seis. noise,	10^{-8} Hz$^{-1/2}\times\left(\frac{10\text{ Hz}}{f}\right)^2$,

we find, for one mirror and one reflecting element

$$h_{\max}=10^{-24}\text{ Hz}^{-1/2}\left[\frac{10\text{ Hz}}{f}\right]^2\left[\frac{S}{10^{-6}\text{ m}^2}\right].$$

The spectral densities of several such elements randomly distributed along the tube should be uncoherently added, i.e., the global noise scales as $\sqrt{n}$, n being the number of spurious reflectors. In a bare vacuum tube, numerous defects or special sites like bellows and junctions, can provide reflecting zones of variable effective areas. The result obtained for a single 1 mm^2 site seems to show that it would be more conservative to hide the tube from the mirrors by a series of baffles.

B. Reflection off baffle edges

For this reason and others, it is generally planned to install light traps or baffles inside the vacuum pipe in order to strongly attenuate scattered light propagation. Some proposed type of baffles (Virgo design) are of conical shape (see Fig. 5). A series of about 80 identical units could typically be installed in each interferometer arm, involving two subseries of 40, symmetrically distributed with respect to the middle point of each arm (see Fig. 6), so as to cover the solid angle spanned by the vacuum pipe seen from any point of a cavity mirror. The problem we address here arises from the edge of the conical surface, which faces the light beam and can reflect the scattered light, as did the small reflectors in the preceding section. The special feature here is that in case of perfect alignment of the baffles there are rings of finite width, coherently reflecting. The edge of any baffle will be modelized by a torus of main curvature radius R_b (the radius of the baffle aperture), and second curvature radius r_c defining the sharpness of the edge. Special machining can give very small r_c, but likely not smaller than 10^{-4} m. Taking z as the abscissa along the optical axis, the equation of our parabolic torus is

$$z=-\frac{(\sqrt{x^2+y^2}-R_b)^2}{2r_c}.$$

The effect of an imperfect machining will be simulated by an extra function $\zeta(\phi)$ representing the departure of the edge from axial symmetry. We shall assume the form $\zeta(\phi)=A\cos(n\phi)$, $n\in\mathbf{N}$. Applying exactly the same formalism as in Sec. III A, and using the reference model of ADSP, we obtain the spectral density of noise as (see Appendix A2):

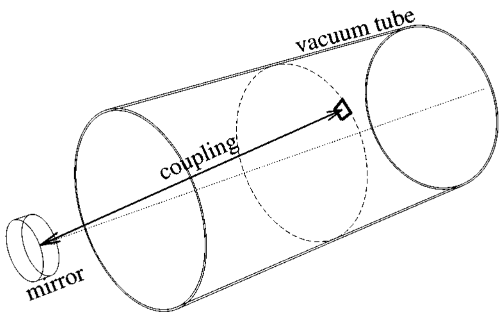

FIG. 1. Long vacuum pipe with a spurious reflecting element.

$$h(f)=\frac{\epsilon\kappa}{8\pi^{7/4}}\sqrt{R_{\rm ref}}n(f)\left[\frac{\lambda^4 r_c d}{R_b^3 w_0 L^2}i_0(2n^2A^2d^2/R_b^2w_0^2)\right]^{1/2},$$

where $n(f)$ refers to the spectral density of phase modulation discussed in Sec. II D. For values of A around 1 mm, even for $n=1$, the function $i_0(x)$ can be replaced by its asymptotic form $i_0(x)\sim 1/\sqrt{2\pi x}$, and the preceding formula becomes independent of d:

$$h(f)=\frac{\epsilon\kappa}{8\sqrt{2}\pi^2}\sqrt{R_{\rm ref}}\;n(f)\frac{\lambda^2}{R_bL}\sqrt{\frac{r_c}{nA}}. \qquad (3.2)$$

If now, the seismic amplitude is small compared to the wavelength, and if $\mathcal{N}$ is the number of baffles facing the mirror, we obtain

$$h(f)=\frac{\epsilon\kappa}{4\pi}\sqrt{\mathcal{N}\mathcal{R}_{\rm ref}}\;\frac{\delta x(f)}{L}\frac{\lambda}{R_b}\sqrt{\frac{r_c}{nA}}\quad (m>0.2).$$

Using the parameters

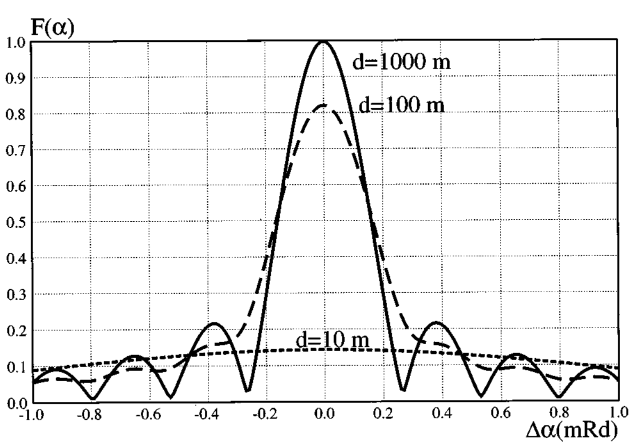

FIG. 2. Form factor vs alignment of the reflector ($a=b=2$ mm) ($\beta=0,\ \alpha=R_t/d+\Delta\alpha$).

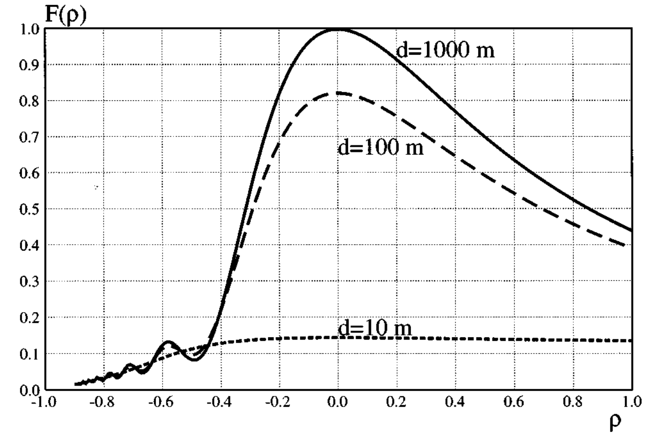

FIG. 3. Form factor vs optical matching of the reflector surface ($a=b=2$ mm) $\rho\equiv r_c/d-1$.

λ,	wavelength ,	1.06×10^{-6} m,
$\epsilon\kappa$,	scattering,	10^{-6},
R_b ,	baffle rad.,	0.5 m,
A,	tilt,	10^{-3} m,
$R_{\rm ref}$,	refl. coef.,	4×10^{-2} (glass),
L,	arm length,	3×10^{3} m,
r_c ,	curv. rad.,	10^{-4} m,
$\delta x(f)$,	seis. noise,	10^{-8} mHz$^{-1/2}\times\left(\frac{10\ \rm Hz}{f}\right)^2$,
$\mathcal{N}$,	nb. of baffles,	40,
n,	nb. of nodes,	1,

and considering the effect of the four mirrors constituting the two cavities being in the same situation, yields

$$h(f)=4.5\times10^{-25}\ \ {\rm Hz}^{-1/2}\times\left[\frac{10\ \ \rm Hz}{f}\right]^2.$$

The choice of $n=1$ corresponds to the most likely defect of a baffle, i.e., a global tilt of its structure, so that a point of the edge is shifted by a length A along the z axis, and the oppo-

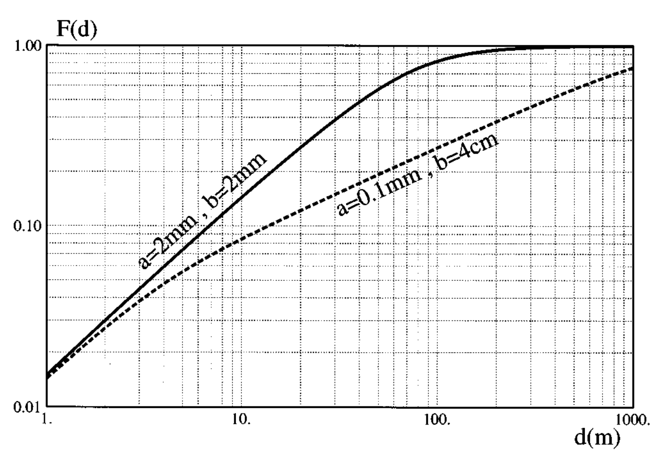

FIG. 4. Form factor vs distance for two aspect ratios.

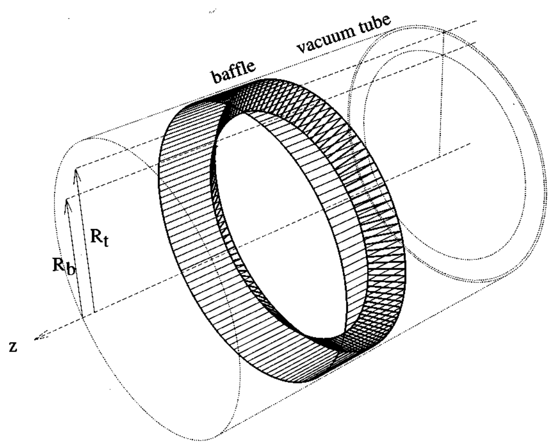

FIG. 5. Conical baffle structure.

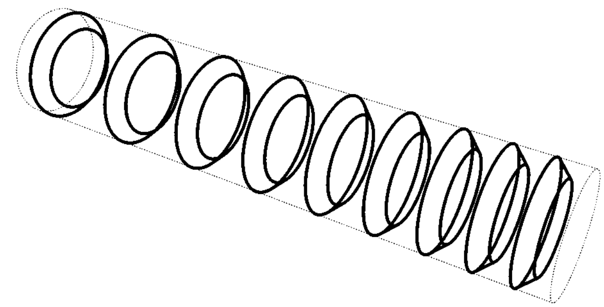

FIG. 6. Series of baffles.

site one by a length $-A$. It is clear that these sorts of defects, breaking the coherence of the reflected field, are a benefit, so far as coherent effects are concerned, but it remains to determine what tilts are compatible with the purpose of the baffle. As a global effect, the previous result is very small, and far from the sensitivity of present and even advanced interferometers. In the strong excitation regime, the spectral density is slightly decreased, but the frequency dependence is modified as indicated in Sec. II D.

C. Diffraction by edges

We now address the question of direct propagation of scattered light from an emitting mirror to the opposite one in a cavity of length L, through the baffling system. Each baffle can be seen as a circular aperture in a perfectly absorbing screen, it receives scattered light, and its edge becomes a source of diffracted light. Let us consider a particular baffle at a distance d_1 from the emitter mirror (and $d_2=L-d_1$ from the receiver mirror). The fact that the edge is vibrating in the transverse plane causes a modulation of that source. The spectral density of noise is derived in Appendix A3. We assume the baffle to have an offset ($\Delta x=E\cos\mu,\Delta y=E\sin\mu$) in the tranverse plane. This offset, is, however, moderate in the sense that the scattered light intensity can be regarded as roughly constant along the edge of the baffle (say a few cm). Let us call $\xi(f,\phi)$ the spectral density of displacement of the baffle's edge vs azimuthal angle, and set

$$X_k^2(f)\equiv\frac{1}{2\pi}\int_0^{2\pi}\xi^2(f,\phi)\cos(2k\phi-2k\mu)d\phi.$$

Let us define the following notation, using the reduced distance $\rho=d_1/L$:

$$m=\frac{E^2}{[\rho^2+(1-\rho)^2]w_0^2},$$

$$F(\rho)\equiv\frac{2\sqrt{2}\rho(1-\rho)}{\sqrt{\rho^2+(1-\rho)^2}}$$

(so that $F_{\max}=1$ for $\rho=\frac{1}{2}$),

$$X(\rho,f)=\left[i_0(m)X_0^2(\rho,f)+2\sum_{k=1}^{\infty}i_k(m)X_k^2(\rho,f)\right]^{1/2},$$

where $i_k(m)\equiv\exp(-m)I_k(m)$, I_k being the modified Bessel function. We have the result (see Appendix A3)

$$h(f)=2^{-1/4}(2\pi)^{-7/4}\epsilon\kappa\left(\frac{\lambda}{R_b}\right)^{3/2}\frac{X(\rho,f)}{\sqrt{Lw_0}}F(\rho)^{1/2}. \quad (3.3)$$

A special but useful example can be solved when the motion of the baffle is rigid, its edge being simply displaced so that

$$\xi^2(f,\phi)=\xi_0^2(f)\cos^2(\phi-\beta),$$

where $\xi_0(f)=\delta x(f)$, i.e., the spectral density of displacement is equal to that of the seismic noise; β gives the polarization of the motion. In this case we obtain

$$X^2(\rho,f)=\frac{1}{2}\delta x^2(f)\left[i_0(m)+\frac{1}{2}\cos(2\mu-2\beta)i_1(m)\right],$$

which is a maximum when the vibration polarization is aligned with the offset of the baffle. Assume the baffles (numbered by $k=1,\ldots,80$) distributed inside the vacuum pipe according to the law

$$\rho_k=\rho_1(1+\tau)^{k-1},\quad k=1,\ldots,40,$$

$$\rho_k=1-\rho_{81-k},\quad k=41,\ldots,80,$$

imposed by the requirement that a baffle be located at the end of the shadow of its predecessor, with a small overlap. This yields, with $\rho_1=2\times10^{-3}$ (the first baffle is assumed at 6 m) and $\tau=0.15$,

$$\left(\sum_{k=1}^{80}F(\rho_k)\right)^{1/2}\simeq4.3.$$

With $m=0$ (on axis baffles) and two arms, we find the following spectral density whatever the amplitude of motion

$$h_{\text{global}}(f)=2^{-3/4}(2\pi)^{-7/4}\epsilon\kappa\left(\frac{\lambda}{R_b}\right)^{3/2}\frac{\delta x(f)}{\sqrt{Lw_0}}\left(\sum_{k=1}^{80}F(\rho_k)\right)^{1/2}. \quad (3.4)$$

With the parameters

λ,	wavelength,	1.06×10^{-6} m,
$\epsilon\kappa$,	scattering,	10^{-6},
E,	offset,	0 m,
R_b,	baffle rad.,	0.5 m,
L,	arm length,	3×10^3 m,
w_0,	beam waist,	2×10^{-2} m,
$\delta x(f)$,	seis. noise,	10^{-8} mHz$^{-1/2}\times\left(\frac{10\ \text{Hz}}{f}\right)^2$.

We obtain

$$h_{\text{global}}(f)=4.1\times10^{-25}\ \text{Hz}^{-1/2}\left(\frac{10\ \text{Hz}}{f}\right)^2.$$

This result seem rather negligible, as the preceding one did, but we must keep in mind that contrarily to the preceding results, the spectral density of noise will remain proportional to that of the tube's motion even in the strong excitation regime. It is therefore conservative to pay attention to damping of the mechanical resonances of the tube, especially for the transverse modes bending sections of pipe in between two anchor points. Let us emphasize that the given formula is valid for small offsets. If the beam is strongly decentered (the offset is not small compared to the edge's radius), an other regime arises, which we have not addressed here.

IV. CONCLUSION

We have given a general method, based on a coherence function, for the study of Gaussian beams scattering by supermirrors, within the framework of wave optics. This allowed us to solve some reflection and diffraction problems arising in gravitational wave interferometric detectors, and to compute the spectral density of noise caused by the seismic vibrations of the ground transmitted this way to the stored main standing wave. It has been shown that, if the vacuum pipe were to remain empty of any baffling system, special attention should be paid to the surface state of the walls: scratches, weldings, junctions, and metal waves of bellows can generate small surface elements having unfortunate orientations. The elementary level of noise is such that an abundance of such defects could generate a global noise not so far from the present sensitivity ($\sim10^{-21}$ Hz$^{-1/2}$ at 10 Hz) of GW antennas. If baffles are installed, it has been shown that coherent effects of these baffles on scattered light (reflection on the edge, diffraction off the edge) are negligible at least for the present generation of GW antennas, including eventual advanced versions at least for almost centered beams. All problems related to scattered light in interferometric antennas can not be solved this way. Many other questions are better studied by means of a statistical uncoherent approach, and will be discussed in a foregoing paper.

APPENDIX: COUPLING FACTORS COMPUTATIONS

1. Reflecting elements

If we use Eq. (2.7) with the mirror operator $m(\vec{y})$ described in Sec. III A, we get a coupling factor γ such that

$$\langle\gamma\gamma^*\rangle=\int m(\vec{y})m(\vec{y'})^*\mathbf{C}(d;\vec{y},\vec{y'})^2d\vec{y}d\vec{y'}$$

or, explicitly, using Eq. (2.6),

$$\langle\gamma\gamma^*\rangle=\left(\frac{\epsilon p(\vartheta)}{2\pi d^2}\right)^2\int m(\vec{y})m(\vec{y'})^*e^{i(\vec{y}-\vec{y'})^2/d^2\vartheta_g^2}\times e^{ik(y^2-y'^2)}d\vec{y}d\vec{y'}.$$

By substituting the expression of $m(\vec{y})$, we find

$$\langle\gamma\gamma^*\rangle=\frac{\epsilon^2}{4\pi^2d^4}p(\vartheta)^2R\Gamma'\Gamma'',$$

with

$$\Gamma'=\int_{-a/2}^{a/2}dX\int_{-a/2}^{a/2}dX'e^{-(X-X')^2/d^2\vartheta_g^2}e^{2ik(R_t/d-\alpha\cos\beta)(X-X')}\times e^{ik(1/d-1/r_c)(X^2-X'^2)},$$

$$\Gamma''=\int_{-b/2}^{b/2}dY\int_{-b/2}^{b/2}dY'e^{-(Y-Y')^2/d^2\vartheta_g^2}e^{-2ik\alpha\sin\beta(Y-Y')}\times e^{ik(1/d-1/r_c)(Y^2-Y'^2)}.$$

After the change of variables $U\equiv(X-X')/2$ and $V\equiv(X+X')/2$, we get

$$\Gamma'=2\int_{-a/2}^{a/2}dU\int_{-a/2+|U|}^{a/2-|U|}dVe^{-4U^2/d^2\vartheta_g^2}e^{4ik(R_t/d-\alpha\cos\beta)U}\times e^{4ik(1/d-1/r_c)UV}$$

and, similarly,

$$\Gamma''=2\int_{-b/2}^{b/2}dU\int_{-b/2+|U|}^{b/2-|U|}dVe^{-4U^2/d^2\vartheta_g^2}e^{-4ikU\alpha\sin\beta}\times e^{4ik(1/d-1/r_c)UV}.$$

Performing the V integration yields, for instance,

$$\Gamma'=4\int_{-a/2}^{a/2}e^{-4U^2/d^2\vartheta_g^2}e^{4ik(R_t/d-\alpha\cos\beta)U}\times\frac{\sin[4k(1/d-1/r_c)U(a/2-|U|)]}{4k(1/d-1/r_c)U}dU,$$

which can be written as

$$\Gamma'=\frac{2a^2}{q'}\int_0^1e^{-x^2/\sigma'^2}\frac{\sin[q'x(1-x)]\cos(p'x)}{x}dx,$$

with $\sigma'=\vartheta_gd/a$, $p'=2ka(R_t/d-\alpha\cos\beta)$, and $q'=ka^2(r_c-d)/dr_c$. An analogous expression can be found for Γ'':

$$\Gamma''=\frac{2b^2}{q''}\int_0^1e^{-x^2/\sigma''^2}\frac{\sin[q''x(1-x)]\cos(p''x)}{x}dx,$$

with $\sigma''=\vartheta_g d/b$, $p''=-2kb\alpha\sin\beta$, and $q''=kb^2(r_c-d)/dr_c$. The integrals Γ' and Γ'' can be easily evaluated by the simplest numerical integration algorithm. We can define a form factor by

$$F=\frac{\sqrt{\Gamma'\Gamma''}}{ab}.$$

Its maximum value of 1 is reached when $p'=q'=0$. Then, using Eqs. (2.8) and (2.9), we find

$$h(f)=\frac{\lambda}{4\pi L}\frac{\epsilon}{2\pi d^2}p(\vartheta)\sqrt{R}Sn(f)F.$$

2. Reflecting edges

By applying the same method as in Appendix A1, we obtain

$$h(f)=\frac{\lambda}{4\pi L}\frac{\epsilon}{2\pi d^2}p(\vartheta)\sqrt{R_{\rm ref}\Gamma}n(f),$$

where as above, L is the interferometer's arm length, d the distance between mirror and reflecting edge, $R_{\rm ref}$ the reflection coefficient of the baffle's material, R_b the radius of the circular edge and ϑ the angle of the baffle aperture seen from the emitter mirror, $\vartheta=R_b/d$, and

$$\Gamma=\int e^{-(\vec{y}-\vec{y'})^2/d^2\vartheta_g^2}e^{ik(y^2-y'^2)/d}m(\vec{y})m(\vec{y'})^*d\vec{y}d\vec{y'},$$

where we now perform the variable change $\vec{y}\to(R,\phi)$ defined by

$$\vec{y}=((R_b+R)\cos\phi,(R_b+R)\sin\phi),$$

$$\vec{y'}=((R_b+R')\cos\phi',(R_b+R')\sin\phi').$$

The element $d\vec{y}$ will be approximated by $d\vec{y}=R_b dR d\phi$. $m(\vec{y})$ represents the action of the reflecting edge on an optical amplitude

$$m(\vec{y})=e^{-2ik\zeta(\phi)}e^{ikR^2/r_c}.$$

We have $y^2-y'^2=2R_b(R-R')+R^2-R'^2$ and $(\vec{y}-\vec{y}')^2=(R-R')^2+4(R_b+R)(R_b+R')\sin^2[(\phi-\phi')/2]$. Here, some approximations are convenient: R is formally taken in the range $]-\infty,\infty[$, but the region effectively contributing the integral is practically restricted to a finite interval $[-\sqrt{\lambda r_c},\sqrt{\lambda r_c}]$. Now, $d\vartheta_g$ being less than 6×10^{-2} m, $2R_b/d\vartheta_g$ is larger than 16, so that the difference $u\equiv(\phi-\phi')/2$ is in practice very small. We get, keeping only the significant terms,

$$\Gamma=R_b^2\int e^{-(R-R')^2/d^2\vartheta_g^2}e^{-(2R_b u/d\vartheta_g)^2}e^{ik(R^2-R'^2)(1/d-1/r_c)}$$
$$\times e^{2ikR_b(R-R')/d}e^{2ik[\zeta(\phi)-\zeta(\phi')]}dRdR'd\phi d\phi',$$

which can be split into two factors $\Gamma=\Gamma'\times\Gamma''$ with

$$\Gamma'=R_b^2\int e^{-(R-R')^2/d^2\theta_g^2}e^{ik(1/d-1/r_c)(R^2-R'^2)}$$
$$\times e^{2ikR_b(R-R')/d}dRdR'$$

and

$$\Gamma''=\int e^{-(2R_b u/d\vartheta_g)^2}e^{2ik[\zeta(\phi)-\zeta(\phi')]}d\phi d\phi'.$$

We find

$$\Gamma'=\frac{1}{2}R_b^2\lambda r_c\frac{1}{1-r_c/d}\simeq\frac{1}{2}R_b^2\lambda r_c\quad(r_c\ll d).$$

We now assume $\zeta(\phi)$ of the form $\zeta(\phi)=A\cos(n\phi)$ $(n\in\mathbf{N})$, so that

$$\Gamma''=\int e^{-(2R_b u/d\vartheta_g)^2}e^{4ikA\sin(nu)\sin(nv)}d\phi d\phi',$$

with $v\equiv(\phi+\phi')/2$; it can be numerically checked that the replacement of $\sin u$ by u has a negligible effect on the value of Γ'', even for relatively high values of n, so that we obtain a simple result

$$\Gamma''=2\int_0^{2\pi}dv\int_{-\infty}^{\infty}e^{-(2R_b u/d\vartheta_g)^2}e^{4inkA\sin(nv)u}du$$
$$=\sqrt{\pi}\frac{d\vartheta_g}{R_b}\int_0^{2\pi}dv\,e^{-(n\vartheta_g kAd/R_b)^2\sin^2(nv)},$$

so that

$$\Gamma''=\sqrt{\pi}\frac{d\vartheta_g}{R_b}2\pi i_0\left[2\left(\frac{ndA}{R_b w_0}\right)^2\right],$$

where the function $i_0(z)$ is related to the modified Bessel function by $i_0(z)\equiv\exp(-z)I_0(z)$. Finally, it turns out that

$$h(f)=\frac{\epsilon\kappa}{8\pi^{7/4}}\sqrt{R_{\rm ref}}n(f)\left[\frac{\lambda^4 r_c d}{R_b^3 w_0 L^2}i_0(2n^2A^2d^2/R_b^2w_0^2)\right]^{1/2}.$$

3. Diffracting edges

The function $m(t,\vec{y})$ introduced in Sec. II C can be written as

$m(t,\vec{y})=1$ if $\vec{y}$ is in the aperture at time t,

$m(t,\vec{y})=0$ otherwise.

The coupling coefficient introduced in Sec. II C takes the form

$$\gamma=\int m(t,\vec{y})\Psi_1(\vec{y})\Psi_2(\vec{y})^*d\vec{y},$$

where

$$\Psi_1(\vec{y})=\int_{\text{emitter}} G_{d_1}(\vec{y}-\vec{x})2kf_1(\vec{x})\phi_0(0,\vec{x})d\vec{x},$$

$$\Psi_2(\vec{y})=\int_{\text{receiver}} G_{d_2}(\vec{y}-\vec{z})^*2kf_2(\vec{z})\phi_0(L,\vec{z})d\vec{z},$$

d_1 being the distance from the emitter to the baffle, d_2 from the baffle to the receiver mirror, and L the length of the cavity, so that $d_1+d_2=L$. The moving aperture can be represented by a mean static domain D_0, plus a thin region ∂D swept by the oscillating border, so that the dynamical part of the coupling coefficient has a variance

$$\langle\gamma\gamma^*\rangle=\int_{\partial D}\mathbf{C}(d_1;\vec{y},\vec{y'})\mathbf{C}(d_2;\vec{y},\vec{y'})d\vec{y}d\vec{y'}.$$

Inserting the definition (2.6) of a coherence function into the this last equation yields

$$\langle\gamma\gamma^*\rangle=\frac{\epsilon p(\vartheta_1)}{2\pi d_1^2}\frac{\epsilon p(\vartheta_2)}{2\pi d_2^2}\Gamma,$$

with $\vartheta_1\equiv R_b/d_1$, $\vartheta_2\equiv R_b/d_2$, and

$$\Gamma=\int_{\partial D}\exp\left[-\frac{(\vec{y}-\vec{y'})^2}{2\vartheta_g^2}\left(\frac{1}{d_1^2}+\frac{1}{d_2^2}\right)+\frac{ik}{2}(y^2-y'^2)\times\left(\frac{1}{d_1}\ \frac{1}{d_2}\right)\right]d\vec{y}d\vec{y'}.$$

We can write

$$\vec{y}=\begin{cases}E\cos\mu+R_b\cos\phi\\ E\sin\mu+R_b\sin\phi\end{cases}\qquad d\vec{y}=R_b\xi(t,\phi)d\phi,$$

where R_b is the radius of the aperture, (E,μ) accounts for a possible location of the baffle off the optical axis, and $\xi(t,\phi)$ is the radial motion of the border. We have

$$(\vec{y}-\vec{y'})^2=4R_b^2\sin^2\left(\frac{\phi-\phi'}{2}\right),$$

$$y^2-y'^2=-4ER_b\sin\left(\frac{\phi+\phi'}{2}-\mu\right)\sin\left(\frac{\phi-\phi'}{2}\right).$$

After the change of variables $u\equiv(\phi-\phi')/2$ and $v\equiv(\phi+\phi')/2$, we have

$$\Gamma=2R_b^2\int\xi(t,u,v)\xi(t',u,v)e^{-M\sin^2(u)}\times e^{-iP\sin(v-\mu)\sin u}dv\,du,$$

with

$$M=\frac{2R_b^2}{\vartheta_g^2}\left(\frac{1}{d_1^2}+\frac{1}{d_2^2}\right),\quad P=2kER_b\left(\frac{1}{d_1}+\frac{1}{d_2}\right).$$

The minimum value of M (when $d_1=d_2=L/2$) is very large ($\simeq 10^3$), so that only very small values of u contribute the integral ($|u|<1/\sqrt{M}$). For not too high values of E, the maximum value of P (when d_1 or $d_2\simeq 10$ m) allows the substitution $\sin u\to u$. Moreover, we shall assume that the displacement ξ does not change appreciably over the angle $|u|<1/\sqrt{M}$ because the seismic noise will only excite low frequency deformation modes for which the number of nodes is small. Thus

$$\Gamma=2R_b^2\int_0^{2\pi}\xi(t,v)\xi(t',v)\sqrt{\frac{\pi}{M}}e^{(P^2/4M)\sin^2(v-\mu)}dv$$
$$=2R_b^2\sqrt{\frac{\pi\vartheta_g^2d_1^2d_2^2}{2R_b^2(d_1^2+d_2^2)}}\int_0^{2\pi}\exp\left(-\frac{2E^2(d_1+d_2)^2}{w_0^2(d_1^2+d_2^2)}\times\sin^2(v-\mu)\right)\xi(t,v)\xi(t',v)dv.$$

By taking the temporal expectation value then performing the Fourier transform with respect to time, we get the spectral density

$$\Gamma(f)=\sqrt{2\pi}\frac{\lambda R_b}{\pi w_0}\frac{d_1d_2}{\sqrt{d_1^2+d_2^2}}\times e^{-m}\int_0^{2\pi}e^{m\cos(2v-2\mu)}\xi^2(f,v)dv,$$

where we have set $m\equiv E^2(d_1+d_2)^2/w_0^2(d_1^2+d_2^2)$. An expansion of the exponential in a series of modified Bessel functions gives

$$\Gamma(f)=\sqrt{2\pi}\frac{\lambda R_b}{\pi w_0}\frac{d_1d_2}{\sqrt{d_1^2+d_2^2}}\times e^{-m}\left(I_0(m)X_0^2(f)+2\sum_{k=1}^{\infty}I_k(m)X_k^2(f)\right),$$

with the definition

$$X_k^2(f)=\frac{1}{2\pi}\int_0^{2\pi}\xi^2(f,v)\cos(2kv-2k\mu)dv.$$

The h-equivalent noise is

$$h(f)=2\frac{\lambda}{4\pi L}\sqrt{\frac{1}{2}\langle\gamma\gamma^*\rangle(f)}.$$

There is an extra factor of 2 with respect to the preceding sections due to the fact that the coupling between the two mirrors is coherent in the two directions. It gives

$$h(f)=\frac{\lambda}{2\sqrt{2}\pi L}\frac{\epsilon}{2\pi d_1d_2}\sqrt{p(\vartheta_1)p(\vartheta_2)\Gamma(f)}.$$

By setting $\rho\equiv d_1/L$, $d_2=(1-\rho)L$,

$$m = \frac{E^2}{[\rho^2 + (1-\rho)^2] w_0^2},$$

$p(\vartheta_1) = \kappa(d_1/R_b)^2$, $p(\vartheta_2) = \kappa(d_2/R_b)^2$, and

$$X(\rho,f) = \left(e^{-m} I_0(m) X_0^2(f) + 2 \sum_{k=1}^{\infty} e^{-m} I_k(m) X_k^2(f) \right)^{1/2},$$

we get the result

$$h(f) = 2^{-3/4} (2\pi)^{-7/4} \epsilon \kappa \left(\frac{\lambda}{R_b} \right)^{3/2} \frac{X(\rho,f)}{\sqrt{L w_0}} |P) F(\rho)^{1/2},$$

with

$$F(\rho) \equiv \frac{2\sqrt{2}\rho(1-\rho)}{\sqrt{\rho^2 + (1-\rho)^2}}.$$

[1] A. Abramovici, W.E. Althouse, R.W.P. Drever, Y. Gürsel, S. Kawamura, F.J. Raab, D. Shoemaker, L. Sievers, R.E. Spero, K.S. Thorne, R.E. Vogt, R. Weiss, S.E. Whitcomb, and M.E. Zucker, Science **256**, 325 (1992).

[2] K. Danzmann, H. Lück, A. Rüdiger, R. Schilling, M. Schrempel, W. Winkler, J. Hough, G.P. Newton, N.A. Robertson, K.A. Strain, J.R.J. Bennett, V. Kose, M. Kühne, B.F. Schutz, D. Nicholson, J. Shuttleworth, H. Welling, P. Aufmuth, R. Rinkleff, A. Tünnermann, and B. Willke, Internal Report MPQ **190** (1994).

[3] C. Bradaschia, R. Del Fabbro, A. Di Virgilio, A. Giazotto, H. Kautzsky, V. Montelaticci, D. Passuello, A. Brillet, O. Cregut, P. Hello, C.N. Man, P.T. Manh, A. Marraud, D. Shoemaker, J.Y. Vinet, F. Barone, L. Di Fiore, L. Milano, G. Russo, J.M. Aguirregabiria, L. Bel, J.P. Duruisseau, G. Le Denmat, P. Tourrenc, M. Capozzi, M. Longo, M. Lops, I. Pinto, G. Rotoli, T. Damour, S. Bonazzola, J.A. Marck, Y. Gourgoulhon, L.E. Holloway, F. Fuligni, V. Iafolla, and G. Natale, Nucl. Instrum. Methods A **289**, 518 (1990).

[4] H. Billing, K. Maischberger, A. Rüdiger, R. Schilling, L. Schnupp, and W. Winkler, J. Phys. E **12**, 1043 (1979).

[5] The Munich Gravitational Wave Detector Using Laser Interferometry, Part II. Scattered Light, W. Winkler, Proceedings of the NATO ASI Quantum Optics and Experimental General Relativity, Bad Windsheim 1981; in *Quantum Optics, Experimental Gravity and Measurement Theory* (Plenum, New York, 1983), pp. 533–543.

[6] P. Beckmann and A. Spizzichino, *The Scattering of Electromagnetic Waves From Rough Surfaces* (Pergamon, New York, 1963).

[7] K.S. Thorne, ''Light scattering and proposed baffle configuration,'' Caltech report, 1989 (unpublished).

[8] W. Winkler, R. Schilling, K. Danzmann, J. Mizuno, A. Rüdiger, and K. Strain, Appl. Opt. **33**, 7547 (1994).

[9] W. Winkler, in *The Detection of Gravitational Waves*, edited by David Blair (Cambridge University Press, Cambridge, England, 1991), pp. 293–299.

[10] J.-Y. Vinet, V. Brisson, and S. Braccini, ''Scattered light noise,'' VIRGO Report No. NTS26, 1995 (unpublished).