Scalar susceptibility in QCD and the multiflavor Schwinger model

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We evaluate the leading infrared behavior of the scalar susceptibility in QCD and in the multiflavor Schwinger model for a small nonzero quark mass m and/or small nonzero temperature as well as the scalar susceptibility for the finite-volume QCD partition function. In QCD, it is determined by one-loop chiral perturbation theory, with the result that the leading infrared singularity behaves as $\sim \ln m$ at zero temperature and as $\sim T/\sqrt{m}$ at finite temperature. In the Schwinger model with several flavors we use exact results for the scalar correlation function. We find that the Schwinger model has a phase transition at $T=0$ with critical exponents that satisfy the standard scaling relations. The singular behavior of this model depends on the number of flavors with a scalar susceptibility that behaves as $\sim m^{-2/(N_f+1)}$. At finite volumes V we show that the scalar susceptibility is proportional to $1/m^2V$. Recent lattice calculations of this quantity by Karsch and Laermann are discussed. [S0556-2821(96)04111-2]

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I. INTRODUCTION

The scalar susceptibility in QCD is defined as

$$\chi=\int d^4x\left\langle \sum_{i=1}^{N_f} \overline{q}q_i(x)\sum_{i=1}^{N_f} \overline{q}_iq_i(0)\right\rangle - V\left\langle \sum_{i=1}^{N_f} \overline{q}_iq_i\right\rangle^2 =\frac{1}{V}\partial_m^2 \ln Z|_{m=0}, \quad (1.1)$$

where V is the four-dimensional Euclidean volume and the averaging is performed either over the vacuum state or over the thermal ensemble. (In the Euclidean approach the latter corresponds to an asymmetric box with imaginary time extension of $\beta=1/T\ll L$, i.e., $V=L^3\beta$.) The definition (1.1) is for a diagonal mass matrix with equal quark masses. It is especially interesting to study this quantity in the neighborhood of the thermal phase transition point. It is expected that for QCD with two massless flavors a second-order phase transition occurs leading to restoration of chiral symmetry [1] (see [2,3] for recent reviews) with a diverging susceptibility. This can be understood simply in terms of Landau mean field theory. For a system with order parameter η coupled to external field h with a second-order phase transition at $T=T_c$, the fluctuations of the order parameter are described by the effective potential

$$V^{\rm eff}(\eta)=A(T-T_c)\eta^2+B\eta^4+C\eta h. \quad (1.2)$$

In QCD η is the chiral condensate and h is the quark mass. At $T=T_c$, the minimum of the potential occurs at $\eta\sim h^{1/3}$ which gives the law $\chi=\partial\eta/\partial h\sim h^{-2/3}$ for the susceptibility. On the other hand, if T is close to T_c but $T\neq T_c$ and the external field is weak enough,

$$h\ll|T-T_c|^{3/2}, \quad (1.3)$$

the quartic term in (1.2) is irrelevant, and the scaling law is $\chi\sim|T-T_c|^{-1}$ both above and below T_c (the proportionality constant in these two regions differs by a factor of 2).

Recently, on the basis of lattice simulations of the three-dimensional Gross-Neveu model it has been suggested that a second-order phase transition involving soft modes consisting of fermions has critical exponents given by mean field theory [4]. In particular, if this is also true for QCD with two massless flavors, we get

$$\langle\overline{q}q\rangle\sim m^{1/3}, \quad (1.4)$$

$$\chi\sim m^{-2/3}, \quad (1.5)$$

at $T=T_c$, and

$$\chi\sim\frac{1}{|T-T_c|} \quad (1.6)$$

at

$$\Lambda^{1/3}m^{2/3}\ll|T-T_c|\ll T_c, \quad (1.7)$$

where Λ is a typical hadronic mass scale. These scaling laws have been reproduced by a simple stochastic matrix model [5,6]. The scalar susceptibility was recently measured for lattice QCD with two light flavors [7]. They found a diverging susceptibility at $T=T_c$ with critical exponent $\delta^{-1}=0.24\pm0.03$, which does not agree with the prediction $\delta^{-1}=1/3$ of the Landau mean field theory. Obviously, further numerical measurements of the critical indices in QCD are highly desirable.

In this paper we will study the scalar susceptibility in three different cases. First, in Sec. II the scalar susceptibility is evaluated for the multiflavor massive Schwinger model which shares many common qualitative features with QCD and, in particular, it shows a phase transition with ''restoration'' of chiral symmetry at $T=0$. All other critical expo-

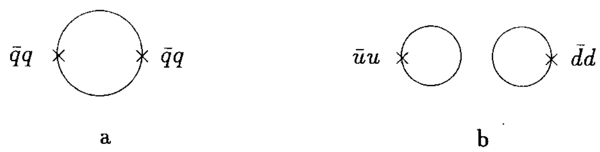

FIG. 1. Connected (a) and disconnected (b) graphs contributing to the scalar susceptibility of quarks propagating in a background gluon field.

nents that can be defined in the Schwinger model (SM) will be evaluated as well, and it will be shown that they satisfy the scaling relations modified for a phase transition at zero temperature. Second, we evaluate the scalar susceptibility in QCD at low temperatures using chiral perturbation theory. Third, in order to estimate finite-size effects in lattice calculations, we calculate the scalar susceptibility in volumes with spatial length below the Compton wavelength of the pion.

Before proceeding further, we should note that the scalar susceptibility as defined in (1.1) involves a quadratic ultraviolet divergence due to a trivial perturbative graph depicted in Fig. 1(a). The situation is the same as with the chiral condensate which involves a trivial divergent perturbative contribution $\sim m\Lambda_{\rm UV}^2$. The predictions (1.4)–(1.6) hold for the infrared-sensitive part of $\langle \overline{q}q\rangle$ and χ.

The susceptibility (1.1) can be written as the sum of a connected and a disconnected contribution corresponding to the graphs of Figs. 1(a) and 1(b), respectively:

$$\chi = N_f\chi^{\rm con} + N_f^2\chi^{\rm dis}. \tag{1.8}$$

The connected contribution to the susceptibility was calculated in [8]. The disconnected contribution is defined by

$$\chi^{\rm dis} = \int d^4x \langle \overline{u}u(x)\overline{d}d(0)\rangle - V\langle \overline{u}u\rangle\langle \overline{d}d\rangle = \frac{1}{V}\partial_{m_u}\partial_{m_d}\ln Z, \tag{1.9}$$

where m_u and m_d are two different quark masses that are set to zero after differentiation. We will obtain the latter contribution from the difference of χ and $\chi^{\rm con}$.

Both scalar susceptibilities can be expressed in terms of the eigenvalues λ_k of the Dirac operator,

$$\chi^{\rm dis} = \frac{1}{V}\left[\left\langle \left(\sum_k \frac{1}{i\lambda_k + m}\right)^2\right\rangle - \left\langle \sum_k \frac{1}{i\lambda_k + m}\right\rangle^2\right], \tag{1.10}$$

$$\chi^{\rm con} = -\frac{1}{V}\left\langle \sum_k \frac{1}{(i\lambda_k + m)^2}\right\rangle. \tag{1.11}$$

A related susceptibility is the pseudoscalar susceptibility which in terms of the Dirac eigenvalues is given by

$$\chi^{\pi} = \frac{1}{V}\left\langle \sum_k \frac{1}{\lambda_k^2 + m^2}\right\rangle. \tag{1.12}$$

II. THE SCHWINGER MODEL

In this section we discuss the Schwinger model (SM) with N_f light flavors, and determine the critical exponents from known results for the correlation functions. The Lagrangian of the model is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2 + i\sum_{f=1}^{N_f} \overline{q_f}\gamma_\mu(\partial_\mu - igA_\mu)q_f - m\sum_{f=1}^{N_f} \overline{q_f}q_f, \tag{2.1}$$

where g is the coupling constant and all quark masses are chosen equal.

In the standard Schwinger model ($N_f = 1$), there is no nonanomalous global symmetry to be broken spontaneously and no reason for the phase transition to occur, e.g., the scalar susceptibility is just a finite constant [9]. The theory with $N_f > 1$ and zero fermion masses has the global symmetry $\mathrm{SU}_L(N_f)\times\mathrm{SU}_R(N_f)$ (much as in QCD) and the potential possibility of its spontaneous breaking with generation of fermion condensate exists. However, because of Coleman's theorem [10] a QCD-like phase transition cannot occur in a two-dimensional (2D) theory at a nonzero temperature. Nevertheless, the dynamics of the SM at small T resembles a theory with second-order phase transition in the region of temperatures slightly above critical. One can say that the phase transition *does* occur at $T = 0$.

For $m \ll g$ the particle spectrum of the SM involves a massive photon [11] with mass $\mu_+ \sim g\sqrt{N_f/\pi} + O(m)$ and ''quasi Goldstone'' particles[1] with the mass [12,13]

$$\mu_- \sim g^{1/(N_f+1)} m^{N_f/(N_f+1)}. \tag{2.2}$$

This gives us the critical exponent $\mu = N_f/(N_f+1)$.

At nonzero m, the $\mathrm{SU}_L(N_f)\times\mathrm{SU}_R(N_f)$ symmetry of the massless SM Lagrangian is broken explicitly, and the formation of the chiral condensate becomes possible. The chiral condensate involves an UV-divergent piece $\sim m\ \ln\Lambda_{\rm UV}$ and an infrared contribution (sensitive to the small eigenvalues of the Euclidean Dirac operator). The latter has been determined in [13,14]:

$$\langle \overline{q}q\rangle \sim m^{(N_f-1)/(N_f+1)} g^{2/(N_f+1)}, \tag{2.3}$$

providing us with the critical exponent $\delta = (N_f+1)/(N_f-1)$. The susceptibility is

$$\chi = \frac{\partial\langle \overline{q}q\rangle_m}{\partial m} \sim \left(\frac{g}{m}\right)^{2/(N_f+1)}. \tag{2.4}$$

In the region $\mu_+^{-1} \ll |x| \ll \mu_-^{-1}$ the vacuum scalar correlator is given by [11,15,13]

$$\langle \overline{q}q(x)\overline{q}q(0)\rangle_0 \sim \frac{g^{2/N_f}}{|x|^{2-2/N_f}}. \tag{2.5}$$

The associated critical exponent is $\zeta = 2 - 2/N_f$. At $|x| \gg \mu_-^{-1}$ the correlator levels off at the value of the square of the chiral condensate (2.3).

[1] It is better to use the term ''quasi Goldstone'' than the term ''pseudo Goldstone'' commonly used for pions because quasi Goldstone states in the SM become sterile in the chiral limit $m \to 0$. That conforms with Coleman's theorem, which forbids the existence of massless interacting particles in two dimensions.

TABLE I. The critical exponents for mean field theory (MFT) and the Schwinger model (SM). Conventions are as in Landau and Lifshitz [32].

Exponent	MFT	SM
α	0	-1
β	1/2	
γ	1	$2/N_f$
δ	3	$(N_f+1)/(N_f-1)$
ε	0	
μ	1/3	$N_f/(N_f+1)$
ν	1/2	1
ζ	0	$2-2/N_f$

In the region $\mu_- \ll T \ll \mu_+$ (weak field limit) the condensate is given by [14]

$$\langle \overline{q}q \rangle \sim m \left(\frac{g}{T} \right)^{2/N_f}, \tag{2.6}$$

which leads to the susceptibility

$$\chi \sim \left(\frac{g}{T} \right)^{2/N_f}, \tag{2.7}$$

and the critical exponent $\gamma = 2/N_f$.

The correlation length is the inverse fermion Matsubara frequency $\sim 1/T$ which gives the critical exponent $\nu = 1$. For $\mu_- \ll T$ the energy density is that of a gas of massless particles. Therefore $\epsilon \sim (\pi/3)T^2$ and the specific heat for zero field is given by

$$C = \frac{d\epsilon}{dT} \sim T, \tag{2.8}$$

leading to $\alpha = -1$.

The above results for the critical exponents have been summarized in Table I. Note that for $N_f = 2$ *some*, but not all, of the exponents (namely, δ and γ) coincide with the predictions of the mean field theory (MFT) (second column). These values of critical exponents can be summarized by the effective nonlocal Lagrangian

$$\mathcal{L} = A\,\eta(\Delta + BT^2)^{1/N_f}\eta + C\,\eta^{2N_f/(N_f-1)} + D\,\eta h. \tag{2.9}$$

Of course, (2.9) is just a shorthand for the values of critical indices obtained and cannot be used for *serious* loop calculations.

In the standard theory of second-order phase transitions with a *nonzero* critical temperature the above eight critical exponents satisfy five universal relations and the hyperscaling relation. A zero critical temperature brings about the following modifications.

The critical exponent β refers to the broken phase and therefore cannot be defined.

In the strong field limit $h \gg t^{\nu/\mu}$ the partition function $\propto \exp\{-h^{\mu/\nu}/t\}$. Then the critical exponent ε becomes singular for $t \to 0$ and cannot be defined. (For the SM the free energy remains exponentially small until $t \sim \mu_- \propto m^{\mu}$.)

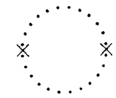

FIG. 2. Pseudo Goldstone loop determining the scalar susceptibility.

Because $c_p = -T\partial^2\Phi/\partial T^2$ and $T_c = 0$, the Rushbrooke scaling relation reads $\alpha + 2\beta + \gamma = 1$ instead of 2. After elimination of β and ϵ we are left with only three relations:

$$\alpha + \frac{\gamma(\delta+1)}{\delta-1} = 1,$$

$$\nu(2-\zeta) = \gamma,$$

$$\mu(1+\gamma-\alpha) = 2\nu. \tag{2.10}$$

The hyperscaling relation follows from the condition that the total free energy is of the order of TV/ξ^d, where V is the spatial volume of dimension d and ξ is the correlation length. For $T_c = 0$, we obtain, instead of $\nu d = 2 - \alpha$,

$$\nu d = -\alpha. \tag{2.11}$$

The meaning of the hyperscaling relation is that loop corrections to the Green's functions estimated from the effective Lagrangian (2.9) are of the same order, as far as powers are concerned, as the tree expressions. We leave it to the reader to verify that the critical exponents of the SM satisfy the relations (2.10) and (2.11).

III. QCD AT LOW TEMPERATURES

The primary interest of the quantity (1.1) is that its critical behavior can provide information on the physics of the phase transition in QCD. However, our second remark is that the leading infrared behavior of χ can be determined *exactly* in the low-temperature region. The proper technique to extract it is chiral perturbation theory [16]. The leading infrared behavior is determined by the graph in Fig. 2 involving a loop of quasimassless pseudo Goldstone bosons. The effective low-energy Lagrangian of QCD has the form

$$\mathcal{L}^{\rm eff} = \tfrac{1}{4}F_\pi^2 \mathrm{Tr}(\partial_\mu U^\dagger)(\partial_\mu U) + \Sigma \ \mathrm{Re\ Tr}\{\mathcal{M}U\} + \cdots, \tag{3.1}$$

where $U = \exp\{2i\phi^a t^a/F_\pi\}$ and ϕ^a are the pseudo Goldstone fields. The chiral condensate is denoted by $\Sigma = |\langle \overline{q}q \rangle_0|$ (no summing over colors assumed), and $\mathcal{M}$ is the quark mass matrix. The partition function with $\mathcal{M} = \mathrm{diag}(m, m, \ldots, m)$ has been calculated by Gasser and Leutwyler [17]. Their expression for the free energy density of lukewarm pion gas is

$$f = \epsilon_0(M_\pi) - \frac{N_f^2-1}{2} g_0(T, M_\pi) + \frac{N_f^2-1}{4N_f} \frac{M_\pi^2}{F_\pi^2} g_1^2(T, M_\pi) \tag{3.2}$$

where

$$g_0(T,M_\pi)=-\frac{T}{\pi^2}\int p^2dp\ \ln[1-\exp(-E/T)], \quad (3.3)$$

$E=\sqrt{p^2+M_\pi^2}$, and

$$g_1(T,M_\pi)=\frac{1}{2\pi^2}\int \frac{p^2dp}{E[\exp(E/T)-1]}=-\frac{\partial g_0(T,M_\pi)}{\partial M_\pi^2}. \quad (3.4)$$

Here, M_π stands for the common mass of the N_f^2-1 pseudo Goldstone modes given by the Gell-Mann–Oakes–Renner relation $F_\pi^2 M_\pi^2=2m\Sigma$.

Substituting in Eq. (3.2) the expansion [18,19]

$$g_0=\frac{\pi^2}{45}T^4-\frac{T^2M_\pi^2}{12}+\frac{TM_\pi^3}{6\pi}, \quad (3.5)$$

we obtain

$$f=-\frac{\pi^2}{90}(N_f^2-1)T^4-\frac{N_f}{2}F_\pi^2M_\pi^2\left(1-\frac{N_f^2-1}{12N_f}\frac{T^2}{F_\pi^2}-\frac{N_f^2-1}{288N_f^2}\frac{T^4}{F_\pi^4}\right)-\frac{N_f^2-1}{12\pi}TM_\pi^3\left(1+\frac{1}{8N_f}\frac{T^2}{F_\pi^2}\right)-\frac{N_f^2-1}{64\pi^2}M_\pi^4\ln\left(\frac{\Lambda^2}{M_\pi^2}\right). \quad (3.6)$$

Consider first the case of zero temperature. For the infrared singular contribution to the scalar susceptibility we obtain [33]

$$\chi^{\rm IR}=\frac{N_f^2-1}{8\pi^2}\left(\frac{\Sigma}{F_\pi^2}\right)^2\ln\frac{\Lambda^2}{M_\pi^2}. \quad (3.7)$$

The infrared singular connected contribution to the susceptibility was calculated in [8]. [One should just substitute 1 for $\mathrm{Tr}\{t^at^b\}=\delta^{ab}/2$ in Eq. (2.13) of Ref. [8].] The result is

$$\chi^{\rm IR\ con}=\frac{N_f^2-4}{16\pi^2N_f}\left(\frac{\Sigma}{F_\pi^2}\right)^2\ln\frac{\Lambda^2}{M_\pi^2}. \quad (3.8)$$

Using (1.8) we immediately obtain the disconnected contribution

$$\chi^{\rm IR\ dis}=\frac{N_f^2+2}{16\pi^2N_f^2}\left(\frac{\Sigma}{F_\pi^2}\right)^2\ln\frac{\Lambda^2}{M_\pi^2}. \quad (3.9)$$

Instead of using the partition function of Gasser and Leutwyler we can calculate the susceptibility also using the same technique as in [8]. Choosing $\mathcal{M}=\mathrm{diag}(m,m,\ldots,m)$, one immediately obtains the vertex

$$\left\langle 0\left|\sum_f \overline{q_f}q_f\right|\phi^a\phi^b\right\rangle=\frac{2\Sigma}{F_\pi^2}\delta^{ab}, \quad (3.10)$$

which, by evaluation of the diagram in Fig. 2 reproduces the result (3.7).

The infrared singular contribution to the susceptibility at finite temperature can be obtained directly from (3.6) as well:

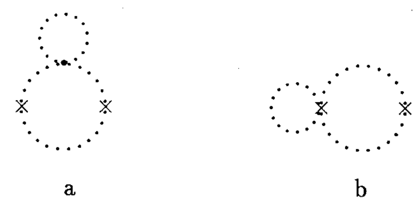

FIG. 3. Two-loop contributions to the scalar susceptibility: (a) mass renormalization; (b) vertex renormalization.

$$\chi_T^{\rm IR}=\frac{(N_f^2-1)}{4\pi}\frac{T}{\sqrt{2m}}\left(\frac{\Sigma}{F_\pi^2}\right)^{3/2}, \quad (3.11)$$

where the Gell-Mann–Oakes–Renner relation has been used. The connected contribution to the susceptibility was not calculated in [8]. However, the zero-temperature result of Ref. [8] can be extended immediately to finite T by making the substitution

$$\int\frac{d^4p}{(2\pi)^4}\rightarrow T\sum_n\int\frac{d^3p}{(2\pi)^3}, \quad (3.12)$$

where the sum is over all Matsubara frequencies $(p_0=2\pi n/\beta)$. The infrared singular part comes only from the term $n=0$ and the result is

$$\chi_T^{\rm IR\ con}=\frac{N_f^2-4}{8\pi N_f}\frac{T}{\sqrt{2m}}\left(\frac{\Sigma}{F_\pi^2}\right)^{3/2}. \quad (3.13)$$

This implies that at finite temperature the spectrum of the Dirac operator is nonanalytic for small eigenvalues. The disconnected contribution is

$$\chi_T^{\rm IR\ dis}=\frac{N_f^2+2}{8\pi N_f^2}\frac{T}{\sqrt{2m}}\left(\frac{\Sigma}{F_\pi^2}\right)^{3/2}. \quad (3.14)$$

The relations (3.11), (3.13), and (3.14) are quite analogous to the relations for the magnetic susceptibility for ferromagnets known for a long time. They are also determined by a loop of pseudo Goldstone particles (the magnons) depicted in Fig. 2 and have the same behavior[2] [21]:

$$\chi^{\rm magnon}=\left.\frac{\partial M}{\partial H}\right|_{H=0}\sim\frac{T}{\sqrt{H}}. \quad (3.15)$$

The relations (3.11), (3.13), and (3.14) hold in the low-temperature region,

$$M_\pi \ll T \ll T_c. \quad (3.16)$$

This makes a direct comparison with recent lattice calculations [7] impossible. At not so low temperatures, in addition to the graph in Fig. 2, higher-order graphs in chiral perturbation theory also contribute. The relevant two-loop graphs are depicted in Fig. 3, but the results for the susceptibility

[2]Recently, a full nonlinear effective Lagrangian has been constructed [20] in a way which makes the analogy between the theory of ferromagnets and *CPT* most transparent.

can be obtained directly from the partition function as well (3.6):

$$\chi_T^{\rm IR}=\frac{N_f^2-1}{4\pi}\frac{T}{\sqrt{2m}}\left(\frac{\Sigma}{F_\pi^2}\right)^{3/2}\left(1+\frac{1}{8N_f}\frac{T^2}{F_\pi^2}\right). \quad (3.17)$$

The two-loop temperature dependence can be absorbed in the one-loop temperature modification of the condensate and the pion decay constant as obtained in [17]:

$$\Sigma(T)=\Sigma(0)\left(1-\frac{N_f^2-1}{12N_f}\frac{T^2}{F_\pi^2}\right), \quad (3.18)$$

$$F_\pi(T)=F_\pi(0)\left(1-\frac{N_f}{24}\frac{T^2}{F_\pi^2}\right). \quad (3.19)$$

It is not clear what happens to next order in T. Temperature corrections to the condensate have been found to three-loop level in [19]. Two-loop effects in $M_\pi^2(T)$ have been extracted in [22,23], but $F_\pi^2(T)$ at the two-loop level is currently unknown. It is not clear whether one can just substitute the temperature-dependent Σ, M_π, and F_π^2 in (3.11) to all orders in the temperature expansion. The same question can be asked concerning the Gell-Mann–Oakes–Renner relation. Taking into account corrections of order T^2/F_π^2, it also holds at nonzero temperatures. Whether it holds at higher orders is an open question [24].

IV. THE SCALAR SUSCEPTIBILITY IN FINITE VOLUMES

The results (3.7), (3.11), (3.13), and (3.14) are valid in the thermodynamic limit, which means that the spatial length of the box where the theory is defined is much larger than the pion Compton wavelength. However, realistic boxes used in lattice calculations can never be made so large if the pion mass is small enough (which is in turn necessary for the quantitative analytic predictions to be possible). In this section we consider the opposite limit,

$$\Lambda^{-1}\ll L\ll M_\pi^{-1}\sim\frac{1}{\sqrt{m\Lambda}}, \quad (4.1)$$

where the susceptibility can be found from the exact results for the finite-volume partition function of [25]. In this range the susceptibility is expected to be determined by finite-volume effects and to be of order

$$\frac{\Sigma}{m}\left[a_0+a_1\frac{1}{mV\Sigma}+O\left(\frac{1}{(mV\Sigma)^2}\right)\right], \quad (4.2)$$

which is much larger than the thermodynamic limit $\sim\Lambda^2$ for the susceptibility. If $a_0\neq 0$, the result becomes independent of the volume, suggesting that it holds for $V\to\infty$ outside the range (4.1) provided that we are sufficiently close to the chiral limit so that $\Sigma/m\gg\Lambda^2$. This is indeed what happens for the pion susceptibility but not for the scalar susceptibility (see below). In general, the results of this section have the same status as in Ref. [25]. They say little about the properties of the theory in the physical infinite-volume limit but can be used to test the validity of numerical calculations in QCD which are performed for finite volumes.

The theoretically simplest susceptibility is the pseudoscalar (or pion) susceptibility. Using the Banks-Casher relation [26], it can be related immediately to the chiral condensate [see Eq. (1.12)]

$$\chi^\pi=\frac{\Sigma}{m}, \quad (4.3)$$

in the limit that $\Sigma mV\gg 1$. The validity of this result extends into the thermodynamic limit outside the range (4.1).

As follows immediately from (1.11) and (1.10) $\chi^{\rm con}$ can be calculated from the average spectral density,

$$\rho(\lambda)=\langle\omega(\lambda,A)\rangle=\left\langle\frac{1}{V}\sum_n\delta(\lambda-\lambda_n)\right\rangle. \quad (4.4)$$

On the other hand, $\chi^{\rm dis}$ can be expressed in the connected two-point level correlation function:

$$\rho_c(\lambda,\lambda')=V[\langle\omega(\lambda,A)\omega(\lambda',A)\rangle-\rho(\lambda)\rho(\lambda')]. \quad (4.5)$$

If, apart from the pairing $\pm\lambda_k$, the eigenvalues of the Dirac operator are uncorrelated [27], i.e., if $\langle f(\lambda_n)g(\lambda_m)\rangle=\langle f(\lambda_n)\rangle\langle g(\lambda_m)\rangle$ for $n\neq m$ and $\lambda_n,\lambda_m>0$, we have, in the limit of large V and for positive λ,λ',

$$\rho_c(\lambda,\lambda')=\rho(\lambda)\delta(\lambda-\lambda'). \quad (4.6)$$

Using the $U_A(1)$ symmetry of the Dirac spectrum the disconnected susceptibility (1.10) can be written as

$$\chi^{\rm dis}=\int_0^\infty d\lambda\int_0^\infty d\lambda'\frac{4m^2\rho_c(\lambda,\lambda')}{(\lambda^2+m^2)(\lambda'^2+m^2)}, \quad (4.7)$$

which after insertion of the correlation function (4.6) leads to

$$\chi^{\rm IR\ dis}=\frac{\pi\rho(0)}{m}=\frac{\Sigma}{m}, \quad (4.8)$$

where the second equality is the Banks-Casher relation [26].

Because the diagonalization of the Dirac operator induces correlations between the eigenvalues, we expect quite a different prediction from the finite-volume partition function.

For $N_f=2$ the finite-volume partition function is known [25]. For $\theta=0$,

$$Z=\frac{2}{V\Sigma(m_u+m_d)}I_1[V\Sigma(m_u+m_d)] \quad (4.9)$$

where $V=L^3/T$ in the four-dimensional (4D) Euclidean volume. The susceptibilities $\chi^{\rm con}$ and $\chi^{\rm dis}$ can be disentangled by differentiating with respect to *different* quark masses. Because the partition function depends on the quark masses via the sum m_u+m_d, we find

$$\chi^{\rm con}=0. \quad (4.10)$$

In the limit $\kappa\equiv V\Sigma m\gg 1$ the disconnected contribution simplifies to

$$\chi^{\rm dis}=\frac{3}{8m^2V}. \tag{4.11}$$

In present day lattices the zero-mode states are completely mixed with the much larger number of nonzero-mode states. Therefore the partition function is effectively calculated in the sector $\nu=0$. Existing numerical calculations in the model of the instanton–anti-instanton liquid [28] are also done for $\nu=0$. In a sector with fixed topological charge the finite-volume partition function has been calculated analytically for an arbitrary number of flavors with equal mass. To leading order in κ^{-1}, the partition functions in the sector of zero topological charge is given by

$$Z^{\rm eff}_{\nu=0}\sim\frac{\exp(N_f\kappa)}{\kappa^{N_f^2/2}}. \tag{4.12}$$

For $\kappa\gg1$ we find

$$N_f\chi^{\rm con}+N_f^2\chi^{\rm dis}=\frac{N_f^2}{2}\frac{1}{m^2V}. \tag{4.13}$$

In general we have not been able to calculate the connected and disconnected contributions to the susceptibility separately. However, for $N_f=2$ the partition function is known for different quark masses [25]. For $\nu=0$ we find

$$Z_{\nu=0}=\frac{1}{2\pi}\int_0^{2\pi}d\theta\ \frac{2I_0[V\Sigma(m_u^2+m_d^2+2m_um_d\cos\theta)^{1/2}]}{V\Sigma(m_u^2+m_d^2+2m_um_d\cos\theta)^{1/2}}, \tag{4.14}$$

which allows us to calculate the connected and disconnected pieces of the susceptibility separately. The result for $\kappa\gg1$ is $\chi^{\rm con}=1/2m^2V$ and $\chi^{\rm dis}=1/4m^2V$. For $N_f=0$ the two contributions to the susceptibility can be obtained from the spectral density and the two-level spectral correlation function which can be derived from chiral random matrix theory. The result for $\kappa\gg1$ is [29] $\chi^{\rm con}=0$ and $\chi^{\rm dis}=1/4m^2V$. Our conjecture for arbitrary N_f and $\nu=0$ consistent with all the above results is

$$\chi^{\rm con}=\frac{N_f}{4}\frac{1}{m^2V}F^{\rm con}(\kappa), \tag{4.15}$$

$$\chi^{\rm dis}=\frac{1}{4m^2V}F^{\rm dis}(\kappa), \tag{4.16}$$

in agreement with the simplest possible flavor dependence consistent with (4.13). Both $F^{\rm con}(\kappa)$ and $F^{\rm dis}(\kappa)$ approach 1 for $\kappa\gg1$, but will in general depend on N_f for finite κ. The disconnected contribution is suppressed by a factor $1/mV\Sigma$ with respect to the result for uncorrelated eigenvalues. The coefficient a_0 in (4.2) turns out to be zero, which is in agreement with the fact that there are no massless scalar particles.

The quark mass dependence of the scalar susceptibility has been calculated by lattice QCD simulations only for relatively large quark masses [7]. In this work with $m^2V\approx1$ (in units of the lattice spacing) a quark mass dependence of $\sim1/m$ is found not only at the critical point but also at lower temperatures where the susceptibility levels off at a significantly lower value. This result is in between the finite-volume prediction (4.13) and the result from chiral perturbation theory (3.11), with mass dependence of $1/m^2$ and $1/\sqrt{m}$, respectively. The lattice results for $\chi^{\rm dis}$ agree with the mass dependence (4.8), suggesting that the eigenvalues of the Dirac operator are only weakly correlated. Two different types of correlations can be considered, namely, correlations between eigenvalues corresponding to different gauge field configurations and correlations between eigenvalues obtained from the same lattice gauge configuration. It has been shown [30] that lattice eigenvalues show strong spectral correlations (the latter type), but this does not exclude the possibility that correlations between eigenvalues corresponding to different members of the ensemble are absent. In random matrix theory it has been shown [31] that spectral averages and ensemble averages are the same. The exciting possibility that this type of generalized ergodicity does not hold for the lattice Dirac eigenvalues deserves further attention.

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