Focusing and the holographic hypothesis

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(Received 1 March 1996)

The ''screen mapping'' introduced by Susskind to implement 't Hooft's holographic hypothesis is studied. For a single screen time, there are an infinite number of images of a black-hole event horizon, almost all of which have a smaller area on the screen than the horizon area. This is consistent with the focusing equation because of the existence of focal points. However, the *boundary* of the past (or future) of the screen obeys the area theorem, and so always gives an expanding map to the screen, as required by the holographic hypothesis. These considerations are illustrated with several axisymmetric static black-hole spacetimes.
[S0556-2821(96)50312-7]

PACS number(s): 04.70.Bw, 04.70.Dy

I. INTRODUCTION

The generalized second law of thermodynamics [1] is the statement that the entropy outside event horizons plus the Bekenstein-Hawking entropy $A/4$ (in Planck units) of all event horizons cannot decrease. The law seems to be correct, at least in quasistationary processes [2]. If it is true, it must be that $A/4$ is the most entropy that could possibly be contained in a region bounded by an area A [1,3,4]. There has been much debate over the past 20 years about whether or not this bound really holds, and part of the problem in proving it is that it is not precisely clear what the statement means. Nevertheless, there are many reasonable arguments in support of it.

If the $A/4$ bound is indeed valid, then the number of states is vastly overestimated by any ordinary flat space density of states in three dimensions. The reason is that almost all of those states would, because of gravity, create a black hole whose entropy is fixed at $A/4$. One is thus led to the *holographic hypothesis* [3–6], according to which the true degrees of freedom are enumerated on a surface enclosing the volume of interest, at an information density less than or equal to one bit per Planck area.

Susskind proposed to implement 't Hooft's holographic hypothesis by mapping all the points of space, by light rays that impinge perpendicularly, onto a flat two-dimensional *screen* $\mathcal{S}$ in a distant asymptotically flat region. We call this mapping the *screen map*. This particular idea was partly motivated by properties of string theory in the light cone gauge, but the mapping between surface and volume degrees of freedom is not really specified in Susskind's proposal. The proposal seems more intended to get some kind of picture on the table so one can begin thinking about it. The first test to which Susskind subjected the screen map was to ask whether the horizons of any black holes that are present are necessarily mapped onto sets of *larger area* on the screen. This is required by the holographic hypothesis since the black hole has the maximal bit density of one per unit area. We call a screen map with this property an *expanding map*. Susskind argued that his screen map is indeed expanding. He also noted that it is a one to many map from horizon to screen, and suggested that at the quantum level one should perhaps superpose all of the images.

In this Rapid Communication we take a closer look at the definition and properties of the screen map. We find that, for a single screen time, there are an infinite number of multiple images, corresponding to *different* horizon times. Since they arise from different time slices of the horizon, it does not seem appropriate to superpose them. Moreover, almost all of these single image maps are *contracting* rather than expanding, which is allowed by the focusing equation on account of the presence of focal points. However, we prove in general and illustrate in several examples that there is always at least one expanding image of the horizon. This proof is essentially Hawking's area theorem, applied to the boundary of the past (or future) of the screen.

II. THE SCREEN MAP

We are interested in light rays that hit the screen orthogonally at one screen time, that is, at one particular spacelike slice $\mathcal{S}$ of the screen's history. The first thing to clarify is whether the light rays leave the screen towards the future or towards the past. Lacking the holographic theory, we do not have a way to decide this, so we shall consider both possibilities. If the rays from the screen are *past directed*, then they never actually cross the future black-hole horizon. They can however cross the ''stretched horizon'' [7,8], which in any case might be argued to be more relevant than the true event horizon. If the rays from the screen are *future directed*, then they can indeed cross the future black-hole horizon, so we get a precise map from a subset of $\mathcal{S}$ to the horizon by following the rays.

We define the *future screen map* to be the past directed congruence of null rays orthogonal to $\mathcal{S}$. A point on a given ray is mapped to the point where the ray hits the screen. Similarly we define the *past screen map* as reversing the roles of past and future. In a static spacetime, such as the maximally extended Schwarzschild metric, these are equivalent. In particular, although the future screen map does not

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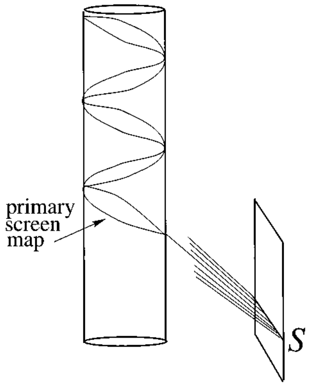

FIG. 1. Past screen map in black-hole spacetime, showing intersection with horizon. The multiple coverings of the horizon come from rays that orbit the hole before crossing the horizon. Only the first covering lies in the primary screen map.

intersect the future horizon, it does intersect the past horizon. In such cases the past horizon can serve as a surrogate for the stretched horizon. Of particular interest will be the part of this congruence that lies on the *boundary* $\dot{I}^{-}[\mathcal{S}]$ of the past of $\mathcal{S}$. We refer to this part as the *primary* future screen map, and similarly for the past screen map.

The definition of the past screen map is illustrated in Fig. 1 for the case in which a single black hole is present. The intersection of the screen map with the horizon is shown. The multiple coverings of the horizon come from rays that orbit the hole before crossing the horizon. Only the first covering lies in the primary screen map.

Rather than taking the screen to be planar, it might be natural to think of it as a large sphere in an asymptotically flat spacetime, which can be taken all the way out to future or past null infinity. The distinction between planar and spherical screens makes no difference for our general arguments (as long as the rays orthogonal to the screen are not diverging), but the specific examples considered below will refer to a planar screen at infinity.

A. Focal points and expanding maps

In [5] it was argued that the screen map is necessarily expanding as a consequence of the *focusing equation*

$$\rho' = \rho^2/2 + \sigma_{ab}\sigma^{ab} + R_{ab}k^a k^b, \tag{1}$$

which governs the rate of change with respect to affine parameter of the convergence ρ of a nontwisting congruence of null geodesics. σ_{ab} is the shear tensor, R_{ab} is the Ricci tensor, and k^a is the tangent vector to the congruence. According to the Einstein equation one has $R_{ab}k^a k^b = 8\pi G T_{ab}k^a k^b$, so the rate of change ρ' is positive as long as the null energy condition $T_{ab}k^a k^b \geqslant 0$ holds. Thus, assuming the null energy condition, one knows that if the light rays have vanishing convergence at the screen, ρ must be non-negative everywhere from screen to horizon (with affine parameter increasing from screen to horizon), so the screen map is expanding.

There is a serious flaw in this reasoning, however, since ρ may become infinite somewhere, after which point it can be *negative*. The focusing equation still says it must increase after that, but that is of no help in establishing the expanding character of the map. A point where ρ is infinite is a *focal point* (also often called a ''caustic'' or a ''conjugate point'' to the screen). To be sure of the expanding character of the map one must show that there are no focal points between the screen and the horizon.

B. Primary screen map and the area theorem

The *boundary* of the past of the screen $\dot{I}^{-}[\mathcal{S}]$ is what we have called the primary screen map. This boundary is similar to a black-hole event horizon, and shares with such a horizon the property that the area of its cross sections cannot decrease toward the future [9]. This property follows from the focusing equation, the null energy condition, and a key property of all past boundaries: each point lies on a null geodesic that runs all the way up the boundary to $\mathcal{S}$ with no focal points along the way. The proof assumes that no ''naked singularities,'' i.e., singularities visible from the screen, are encountered along the way up to $\mathcal{S}$.

Thus one has an ''area theorem'' for the primary screen map, which guarantees that this map is expanding. In particular, if the null rays cross a stretched horizon, then the stretched horizon will necessarily be mapped to a larger area on the screen.

All of the above applies equally well to the boundary of a past screen map, with the roles of future and past interchanged. The boundary has an area that must *decrease* toward the future, but still *increases* toward the screen, so again one obtains an expanding map to the screen (assuming now that no singularities are encountered on the way from the screen).

III. STATIC AXISYMMETRIC EXAMPLES

In this section we look at a number of specific examples that illustrate the general principles already discussed. In some of the examples we can actually calculate the areas of the screen images of the black-hole horizon and show that a finite number (often only one) are greater but the remaining infinite number are less than the horizon area. We explain this fact by identifying the focal points. We also identify the primary screen map in all the examples and show explicitly that it has no focal points. Since these are static spacetimes the future and past screen maps are equivalent. For definiteness, we adopt here the past screen map, which intersects the future black-hole horizon.

A. Single black hole

For a single spherically symmetric black hole we can easily analyze the screen map explicitly by use of the integrated geodesic equation. Three orbits from the screen to a Schwarzschild hole are shown in Fig. 2. The orbits can be labeled by the angle at which they hit the horizon, measured from the perpendicular from the screen to the hole. The orbits with angles between 0 and π cover the horizon once, when rotated around the axis. Those with angles between π and 2π give a second covering, and so on. There are an infinite number of coverings of the horizon. There is an upper bound for the impact parameter for capture by the hole, so the areas of the coverings on the screen must converge to

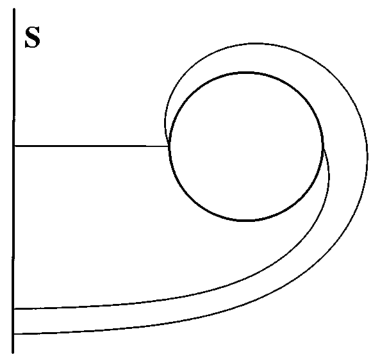

FIG. 2. Three orbits from screen to a Schwarzschild black hole, drawn in (r,ϕ) plane.

zero. These coverings are illustrated in Fig. 3, which is drawn approximately to scale for the case of a Schwarzschild black-hole. The primary cover has area $\simeq 1.24 A_H$, greater than the horizon area, but all the rest have lesser area. In fact the sum of all the rest has lesser area, $0.45 A_H$.

Since the higher-order covers are not expanding, there must be at least one focal point somewhere on each of the corresponding null congruences. This focal point is shown in Fig. 4 for the second cover in the Schwarzschild case. All the null rays will intersect at a point directly behind the black-hole (as viewed from the screen). At this point the convergence goes to infinity and just past this point it is negative (although in general the convergence may become positive again and may even diverge again if the congruence goes through another focal point). It is interesting to note that the separation of the null rays in the transverse direction is larger on the screen than on the horizon, so it is not the angular focusing but the radial focusing that makes the map fail to be expanding, even though it is the angular focusing that produces the focal point.

We also analyzed the screen map for an extremally charged Reissner-Nordström hole, where it turns out that

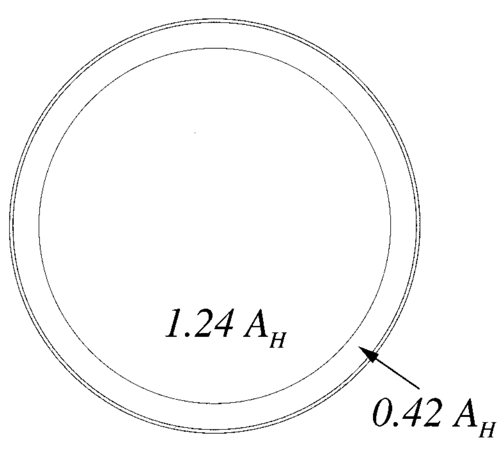

FIG. 3. Screen image of Schwarzschild horizon. The inner disc is the primary cover, and all the rest of the covers are annuli. The higher-order covers accumulate at an impact parameter of $3^{3/2}M$, the capture radius. Only the first and second covers are shown explicitly. The rest are too narrow to show to scale. Even the second cover has smaller area (0.42 A_H) than the horizon.

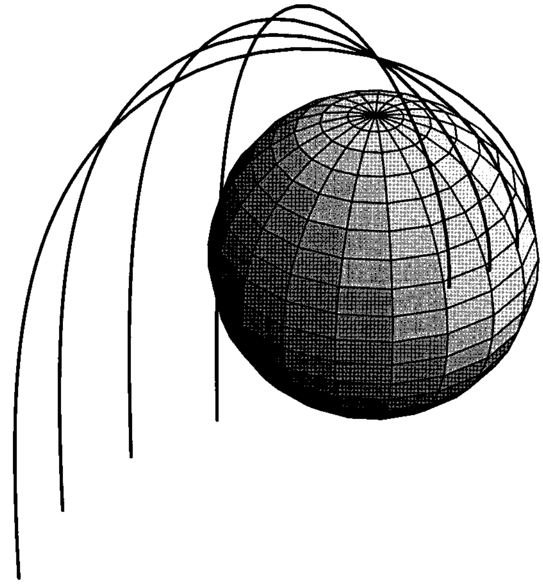

FIG. 4. Focal point for the second cover of the screen map to a Schwarzschild horizon.

both the primary and the second cover are expanding, but the rest are not.

B. Two distant black holes

It was pointed out in [5] that one black hole cannot be hidden behind another, and that the screen map will still be expanding for both horizons in this case. We investigated this situation explicitly for the case where the two black holes are very far from one another and the screen is perpendicular to the axis joining them. In this case we could approximate the capture orbits for the second hole analytically by simply composing with the scattering by the first hole. The screen map pattern in this case is rather complicated. A general orbit can alternate between the two holes, wrapping any number of times around on each visit, before finally crossing the horizon of one of the holes. The primary map for the second hole is given by the orbits that never cross the axis (see Fig. 5), and it traces out an annulus on the screen. We estimated the area of this annulus for large separation d, and found that it grows like $d^{1/2}$: $A \sim (M_1 d/M_2^2)^{1/2} A_2$.

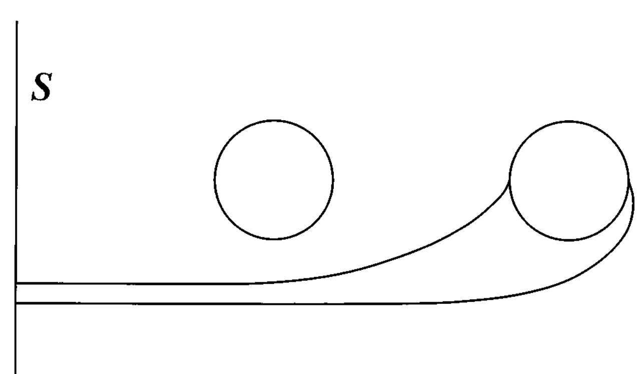

FIG. 5. Extreme rays of the primary cover of the second black hole. None of the rays in this cover cross the axis.

Here M_1, M_2 are the masses of the first and second holes, respectively, A_2 is the horizon area of the second, and we have assumed that $d \gg M_2^2/M_1$.

C. General static axisymmetric spacetime

For two nearby black holes one cannot just compose the asymptotic scatterings, and also one must decide what metric to take for the two black holes. In order to have a static situation, it is natural to consider a pair of extremally charged black holes with like sign charges, one of the Majumdar-Papapetrou solutions [10]. Generalizing this, one can consider a linear array of any number of such black holes. We shall now show by explicit consideration of the geodesics that the primary map has no focal points, so the map is expanding.

It is most natural to discuss this problem for an arbitrary static axisymmetric geometry with the screen perpendicular to the axis. Because of the axisymmetry, focal points will occur on any rays that cross the axis, so we restrict attention to those rays that do not cross the axis. These can only have focal points if they are focused to crossing points in the radial direction. To establish the absence of such crossing points we adopt a technique from the analysis by Yurtsever [11] of chaos in the orbits around a pair of extremal black-holes.

Consider a line element of the form $g = f dt^2 - g d\phi^2 - h_{ij} dx^i dx^j$, where $i = 1,2$, and f, g, and h_{ij} are functions only of x^i. The rays from the screen have constant ϕ, and, since they are null geodesics, they are the same as for the conformally rescaled three-dimensional line element $\tilde{g} = dt^2 - \tilde{h}_{ij} dx^i dx^j$, where $\tilde{h}_{ij} = f^{-1} h_{ij}$. The geodesics of $\tilde{g}$ project to geodesics of the Riemannian metric $\tilde{h}_{ij}$, and the arc length along the projection agrees with the affine parameter along the null geodesic. Computation reveals that the curvature of the two-dimensional metric $\tilde{h}_{ij}$ is negative everywhere outside the horizon of a general multi-extremal black-hole solution, as well as for a Reissner-Nordström black hole with any charge to mass ratio. Thus in all these cases the projected curves are receding from each other. The spacetime null geodesics are therefore also receding with respect to $\tilde{g}$. While this does not imply that they are receding with respect to the physical metric g, it does at least imply that they will not reach a focal point.

IV. REMARKS

The main lesson of this geometric exercise is the conclusion that, to guarantee the expanding nature of the holographic map from horizon to screen, one should restrict to the *primary* screen map, i.e., the boundary of the past (or future) of the screen. This boundary generalizes the concept of a black-hole event horizon and satisfies the area theorem.

Even the primary screen map is only guaranteed to be expanding if the null energy condition is satisfied. In the presence of quantum fields this condition can be locally violated by the expectation value of the stress-energy tensor, so one should not rely on the null energy condition. Perhaps an *averaged* null energy condition would be sufficient to establish the expanding nature of the primary screen map. (Several recent works [12] have studied the extent to which averaged energy conditions hold in quantum field theory.) Alternatively, the expanding property may be lost, which may reveal something about the holographic principle.

Our proof of the expanding nature of the primary screen map relies heavily on the assumption that the rays orthogonal to the screen are not diverging. This is guaranteed in asymptotically flat space by taking a flat or spherical screen at infinity. In a closed expanding universe it seems one must choose a future screen map rather than a past one in order to have any chance of finding an expanding map from all points of space to a single screen. Strangely, if the universe recollapses, one must then switch to a past screen map. Perhaps some insight into the holographic hypothesis can be gleaned from an investigation of its compatibility with closed universes.

ACKNOWLEDGMENTS

This work was supported in part by the University of Utrecht, by grants from the University of Maryland, and by NSF Grant No. PHY-94-13253.

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