Multi-black-hole geometries in (2+1)-dimensional gravity

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Generalizations of the black hole geometry of Bañados, Teitelboim, and Zanelli (BTZ) are presented. The theory is three-dimensional vacuum Einstein theory with a negative cosmological constant. The n-black-hole solution has n asymptotically anti–de Sitter "exterior" regions that join in one "interior" region. The geometry of each exterior region is identical to that of a BTZ geometry; in particular, each contains a black hole horizon that surrounds (as judged from that exterior) all the other horizons. The interior region acts as a closed universe containing n black holes. The initial state and its time development are discussed in some detail for the simple case when the angular momentum parameters of all the black holes vanish. A procedure to construct n black holes with angular momentum (for $n \geqslant 4$) is also given.

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I. INTRODUCTION

Since the discovery by Bañados, Teitelboim, and Zanelli (BTZ) [1,2] of black holes in (2+1)-dimensional Einstein theory there has been considerable interest in finding solutions that describe several black holes. Such solutions exist in (3+1)-dimensional general relativity under special circumstances. The best-known solution of this type is probably the static configuration of n charged black holes whose gravitational and electromagnetic forces balance [3]. Time-dependent multi-black-hole (MBH) solutions are also known, for instance the MBH cosmologies of Kastor and Traschen [4]. The essential requirement for simple, closed-form MBH solutions appears to be absence of specific interactions between the black holes, due to some special balance condition, for example on the charge/mass ratio. In (2+1)-dimensional Einstein theory, where spacetime curvature is uniquely determined by the cosmological constant Λ, there can be no specific interaction between bodies. One would therefore expect such MBH solutions to exist in 2+1 dimensions.

For positive or vanishing Λ the global structure of spacetime is too rigidly determined by the curvature to admit even single black holes. For negative Λ the BTZ solution [1] has all the expected properties of a black hole in a negative curvature, asymptotically anti–de Sitter (adS) environment; for a comprehensive review see Ref. [5].

In this paper we shall construct MBH generalizations of the BTZ solution. We will find it convenient to approach the solutions first from the point of view of the initial value problem. Existence of solutions of the initial value constraints establishes existence of the corresponding spacetime, at least for a finite interval in time. We will then discuss this time development of the MBH's.

In 3+1 dimensions the constraints are easier to solve than the full dynamics, and MBH initial values can be given for a variety of masses, charges, and topologies (see, for example, Refs. [6] and [7]). There is also a variety of horizon structures, as determined by the apparent horizons [7,8]; for example, two black holes may initially exhibit only the apparent horizon of each hole, or they may have three horizons, with the extra one surrounding both holes. In 2+1 dimensions we obtain only the latter scenario with regular initial values and asymptotically anti–de Sitter exterior regions. This is to be expected because the exterior regions are static and therefore cannot contain more than one regular black hole [9].

In Sec. II we recall the single BTZ black hole and show how its initial geometry can be represented in the case of vanishing angular momentum J. Section III gives a construction of initial data for MBH's without angular momentum ($J=0$). The time development of these MBH's is discussed in Sec. IV. Section V shows how to construct MBH's with nonvanishing angular momentum.

II. BTZ INITIAL VALUES

In three dimensions the Ricci tensor determines algebraically the full Riemann tensor. A three-dimensional Einstein space therefore has constant curvature proportional to Λ. The BTZ solution is of this type and takes the form, in Schwarzschild coordinates [1,5],

$$ds^2 = -\left(\frac{r^2}{l^2} - M\right)dt^2 + \frac{dr^2}{f^2} + r^2 d\phi^2 - J d\phi dt. \qquad (1)$$

Here $l^2 = -1/\Lambda$, ϕ is an angular coordinate with period 2π, and

$$f^2 = \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right).$$

The zeros of f^2 are denoted by r_+ and r_-, so that

$$M = \frac{r_+^2 + r_-^2}{l^2} \quad \text{and} \quad J = \frac{2r_+ r_-}{l}.$$

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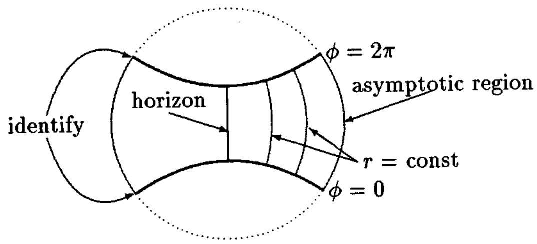

FIG. 1. The BTZ black hole (region between the thick circular arcs), cut along $\phi = 0$, as part of the Poincaré disk (dotted).

When $J = 0$ the BTZ geometry is time symmetric about $t =$const, so this initial two-dimensional spacelike surface must also have constant negative curvature. Its universal covering is therefore two-dimensional hyperbolic space, H. Most of the subsequent discussion will concern the geometry of H and the identifications implied by the periodicity of ϕ, rather than coordinate expressions such as (1).

We shall find it convenient to use two representations of H. One is its isometric embedding in three-dimensional Minkowski space, for example as the past spacelike hyperboloid of constant spacetime distance from the origin. The isometries of H are the boosts and rotations of the three-dimensional homogeneous Lorentz group. As applied to H, the boosts are called *transvections*. Each transvection leaves one geodesic invariant. In the embedding this geodesic is the intersection of the hyperboloid with the plane through the origin orthogonal to the boost's axis. Conversely, any (directed) geodesic segment determines a transvection.

The other representation is Poincaré's disk model. This can be regarded as a "stereographic" projection of the Minkowski hyperboloid on its tangent plane, the projection center being the point diametrically opposite to the tangent plane's point of tangency. The map is conformal (like the stereographic projection of the sphere), the boundary of the disk represents ideal points at infinity, circles look like circles, and geodesics are represented by circles that meet the boundary orthogonally. Two geodesics that meet at infinity are said to be parallel, and two geodesics that do not intersect are called ultraparallel. Two ultraparallel geodesics determine a unique geodesic segment that is normal to both and represents the shortest distance between them. The segment's transvection moves one of the geodesics into the other.

The initial state of black holes with $J = 0$ can now be described in terms of the geometry of H: any two ultraparallels represent a $t = 0$ surface of a BTZ black hole, cut open along a geodesic ϕ=const. Namely, the common normal to the two ultraparallels determines a one-parameter family of transvections corresponding to ϕ displacement in the BTZ metric. The BTZ initial geometry itself corresponds to the region between the ultraparallels. (These should be identified in order to reassemble the uncut BTZ geometry.) Because the $r =$const curves of BTZ space are metrically circles, they are also (parts of) circles on the Poincaré disk, and they are orthogonal to the ultraparallels. The unique circle (or straight line) of this set that is a geodesic corresponds to the black-hole horizon. So the horizon defines the transvection by which $\phi = 0$ and $\phi = 2\pi$ are identified. These features are illustrated in Fig. 1. (It is worth noting that such diagrams on the Poincaré disk are not cartoons, but represent geometric relationships as precise as those shown by diagrams in Euclidean geometry.)

By means of a suitable isometry almost any $J = 0$ black hole can be put in the symmetrical position of Fig. 1. It is clear that the only remaining difference between $J = 0$ black holes is then the "width" of the region between the ultraparallels, which can be measured by the length of the horizon, $2\pi r_+ = 2\pi\sqrt{M/|\Lambda|}$, or by the black-hole mass M. The exceptional case occurs when the two geodesics are parallel. Since they meet at infinity, their minimum distance, and the parameter M, is zero. Thus the horizon of this "massless black hole" may be said to be at infinity, corresponding to an infinitely long (and infinitely thin) throat.

III. MBH INITIAL VALUES

Instead of cutting the BTZ space along one radial geodesic ($\phi = 0$), we can cut it along $\phi = 0$ and $\phi = \pi$. The resulting "strips" on the Poincaré disk are congruent. We can imagine them laid one on top of the other, and sewn together at the radial boundaries, thus restoring the BTZ geometry. (The identification is smooth because the extrinsic and intrinsic geometries of these boundaries agree.) We denote by *doubling* this procedure whereby two congruent copies of a region are identified along totally geodesic boundaries.

This suggests a simple way to construct MBH geometries: Take any set of mutually ultraparallel geodesics that bound a region in H, such as the thick arcs in Fig. 2(a), and double it [Fig. 2(b)]. The figure has threefold symmetry, so we obtain an initial geometry with three black holes having equal masses. In general the masses can have arbitrary values, including zero. [If all three masses are zero, Fig. 2(a) is an "ideal triangle" with vertices on the boundary of the Poincaré disk.] If it is understood that any figure is to be doubled, we can take a picture like Fig. 2(a) as a representation of the MBH geometry. We shall call such a picture, showing the n ultraparallels, a *diagram* for the n-BH geometry. The unique orthogonal geodesics to adjacent ultraparallel geodesics of a diagram determine the horizon between those geodesics.

The region outside of each horizon is isometric to that of a BTZ black hole—one needs only to eliminate all but the two geodesics bounding that region in order to obtain that "single" BTZ black-hole initial state. Thus the exterior regions have all the properties of single black holes, in particular the mass of each region is well defined. The interior region, between the horizons, is a new type of initial value for 2+1 Einstein theory. It can be interpreted as an n-black-hole universe that is closed except for the throats of the black holes (with $n = 3$ in the example above). The diagram of the interior region is a polygon with $2n$ sides that meet at right angles. We call this the *polygon* of the diagram [Fig. 3(a) gives an example]. Alternate sides of the polygon represent the lengths of the black-hole horizons, and the distances between adjacent horizons.

Because the side lengths of a closed 90° polygon cannot be assigned arbitrarily, there are three relations between the masses and distances of a MBH geometry so constructed. If we give an orientation to the polygon we can regard the sides as a sequence of transvections, or a sequence of boosts

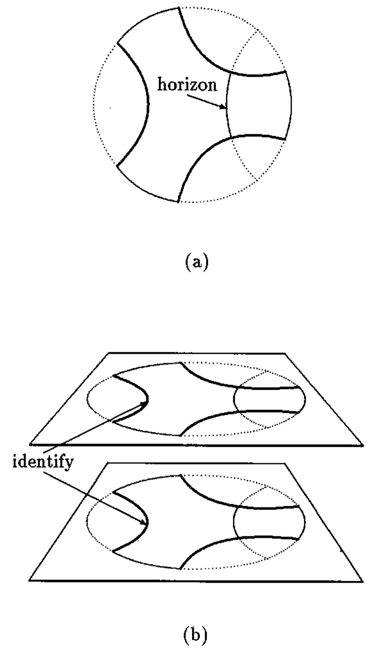

FIG. 2. Construction of a three-black-hole initial geometry by doubling a region bounded by three geodesics in the Poincaré disk. (a) Half of the initial geometry is represented by the region bounded by the thick circular arcs. The horizon of one black hole region (minimal geodesic between the two geodesics on the right) is shown. The other two horizons can be obtained by 120° rotations. (b) Two disks are placed one above the other, and the thick boundaries are identified vertically, as shown explicitly for the boundary on the left. The result is an initial state with three asymptotically adS regions and three horizons.

$\{L_i\}$ in the Minkowski space embedding. The relation then demands that successive application of all these boosts yield unity, $\Pi_{i=1}^{2n} L_i = \mathbb{1}$.

If the lengths (or mass parameters) of two horizons are equal we can identify two such horizons, rather than attaching the asymptotically adS region on the other side. Thus one can also construct truly closed "wormhole" universes. (This is just one set of the many identifications possible in H to form closed spaces.) However, the initial data as constructed by doubling are not the most general $J=0$ MBH geometries. For example, we can cut the four-hole diagram of Fig. 3(a) along the dotted circle and reattach after turning through some angle [Fig. 3(b)]. It is conjectured that the general $J=0$ MBH geometry can be obtained from a diagram type by rotations along such closed geodesics.

The black-hole or wormhole universes are related by a curious duality. The dual universe is obtained by exchanging the roles of the horizons and of the identification lines in the polygon. (A polygon with pairwise equal sides is its own dual.)

IV. TIME DEVELOPMENT

Because the initial data of each exterior region are the same as that of some single BTZ black hole, the exterior time development will also be the same. The Killing vector of that development can be extended even beyond the horizon, but it does not extend to a global symmetry. To discuss

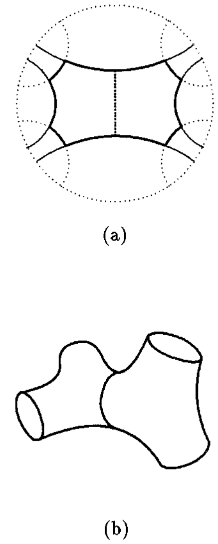

FIG. 3. A diagram that allows rotation. (a) A diagram for four black holes. Its polygon consists of the thick arcs. The thick dotted line becomes a circle after the identifications. The resulting geometry is to be cut along the thick dotted line, rotated by some angle, and reassembled. (b) Three-dimensional picture to give an idea of the result (cut off at the horizons).

the global time development it is therefore more appropriate to choose a constant lapse, $N=1$, $N_i=0$ ("time orthogonal" development). With this choice the diagram's identification lines develop into totally geodesic timelike surfaces. These can therefore still be smoothly identified in the three-dimensional time development.

Locally the time orthogonal development is the same as that of adS space: successive spacelike surfaces have increasingly negative intrinsic curvature, and acquire increasing, spatially constant, extrinsic curvature (with normals converging in the advancing time direction). Globally, then, the intrinsic geometry of the time surfaces is characterized by the same diagram as the initial surface, but corresponding to larger negative curvature. The total "volume" of the interior region (the area of the polygon) decreases, as does the distance between black-hole horizons. Physically the interior MBH universe collapses, and the relative motion of the black holes is similar to that of test particles in a background adS space. [The same behavior was found in (3+1)-dimensional black-hole universes [4,10].]

After a finite time the area tends to zero and the curvatures of the time slices become infinite. In the adS universe this is not a spacetime singularity, but in the BTZ and MBH geometry there is a singularity of the Misner [11] type, where the spacetime fails to be Hausdorff [2]. Physically we can say that, in these time-orthogonal coordinates, each black-hole horizon as well as the entire interior collapse simultaneously. Just before the collapse the spacelike geometry looks like n thin horns that flare out at infinity and are connected on their thin ends. In this sense the singularity consists of n lines meeting at a point. (This point is the end point of the unstable geodesic between the black holes that avoids falling into any of them.)

Because there is no global Killing vector that moves the initial surface in the MBH spacetimes we constructed, this initial surface is unique. In this respect the MBH spacetimes differ from the BTZ black holes, where the time-symmetric initial surfaces can be moved by the timelike Killing vector.

Instead of generating MBH spacetimes from initial values and their time development, we can also obtain them from an identification of the three-dimensional adS spacetime. We still choose an initial time-symmetric spacelike surface, and a diagram in it. We construct the totally geodesic timelike surfaces (*identification surfaces*) normal to the initial surface and intersecting it along a diagram geodesic. We then double the spacetime between these surfaces by sewing along the surfaces [a three-dimensional version of the procedure of Fig. 2(b)]. The identification surfaces will intersect somewhere in the future and past of the initial surface. This causes the non-Hausdorff singularity mentioned above.

V. MBH WITH ANGULAR MOMENTUM

MBH's with angular momentum can be constructed by a similar identification of three-dimensional regions of adS space. We imitate the construction of the BTZ "single" black hole [1,5]. We cut this geometry along $\phi=0$ and embed the resulting three-dimensional "slab" in three-dimensional adS space. The line $r=r_+$ is an extremum of distance between the identification surfaces $\phi=0$ and $\phi=2\pi$. We call it the *horizon* even in the three-dimensional context. We replace the identification surfaces by new, totally geodesic identification surfaces that are generated by all geodesics orthogonal to the horizon. This transforms the general metric (1) with $J\neq 0$ into one with $J=0$, mass μ, and coordinates ρ,φ,τ (and a different identification between $\varphi=0$ and $\varphi=2\pi$):

$$\mu=\frac{r_+^2}{l^2},\quad \frac{\rho^2}{r_+^2}=\frac{r^2-r_-^2}{r_+^2-r_-^2},$$

$$\varphi=\phi-\frac{r_-}{r_+l}\,t,\quad \tau=t-\frac{r_-l}{r_+}\,\phi. \tag{2}$$

Consider the transvection whose invariant geodesic is the horizon. It maps one new (φ=const) identification surface into the other, as for the case $J=0$. The BTZ case $J\neq 0$ is different because the identification is not simply given by this transvection. Rather, there is an additional "twist," a boost about the horizon. Let the boost take place in the surface $\varphi=2\pi$. According to Eq. (2) it can be described in that surface as a time translation by $2\pi r_- l/r_+$. Thus the BTZ black hole with angular momentum can be obtained from a $J=0$ BTZ black hole with the same horizon length by shifting the $J=0$ BTZ time by this amount on the second identification surface.

To obtain an analogous MBH solution we must find several identification surfaces and horizons that can be consistently doubled in the presence of twists, and in three-dimensional adS spacetime. These surfaces and horizons can again be specified by a series of isometries of adS spacetime. In addition to transvections that move one surface into the next, and transvections along the geodesics adjoining adjacent horizons, there should now also be boosts about the horizons. Consistency demands that the product of all these transformations be unity, a set of six conditions among the masses, distances, and angular momenta.

We can again associate a diagram with these transformations, consisting of mutually orthogonal geodesics that represent alternately the connections between horizons and the horizons themselves. The geodesics are the invariant geodesics of the transvections; the boosts are measured by the amount by which the directions of the connecting geodesics at the two ends of a horizon fail to be parallel, as judged by parallel transport along the horizon. Thus such a diagram cannot lie in a plane, totally geodesic spacelike surface and is therefore harder to visualize. The identification surfaces are then generated by all the geodesics orthogonal to the horizons at the horizons' ends. (The same surface is generated from either of the horizons ending on it.) Finally the spacetime region bounded by these surfaces is doubled by identifying along the surfaces with a second, similar region. In order that the twist not cancel when going all the way around a horizon in the doubled spacetime, the second region should be identically constructed but with twists in the opposite directions.

If there are three identification surfaces, the horizon lines always lie in the plane through the surfaces' three centers, so the diagram is planar and corresponds to $J=0$. For nonzero angular momenta we therefore need at least four black holes in our construction. To show that such diagrams and identification surfaces exist we give one example. Consider the diagram of Fig. 3(a) embedded in three-dimensional adS spacetime. Let four identification surfaces be generated by all geodesics orthogonal to the horizons at their end points. Boost the part to the right of the thick dotted line by a Lorentz transformation that has this dotted line as its axis. The left identification surface and the two horizons on the left are not affected by this boost of the right part, and the top and bottom identification surfaces are invariant under this boost; however, the right identification surface and the two right horizons move. Likewise, the geodesics connecting the horizons on the left with those on the right move. As a result there is now a twist between the long and the short connecting geodesics, the diagram is no longer planar, and there is angular momentum associated with each horizon.

VI. CONCLUSION

We have seen that it is possible to construct out of pieces of adS spacetime a spacetime that has many asymptotically adS regions containing many horizons. Each of these regions is isometric to the corresponding region of a BTZ black hole. It is therefore appropriate to regard such spacetimes as MBH spacetimes. For the case of zero angular momentum our construction can be characterized by a polygon whose sides represent the distances and masses involved. The closure condition yields three relations between these parameters, but other, "nonpolygonal" arrangements can also be constructed. In the case $J\neq 0$ we get six rather less transparent conditions between masses, distances, and angular momenta, but more general MBH configurations, not obtained by simple doubling, presumably exist.

Note added. After this paper was finished I heard from Dr. Alan Steif of UC Davis that he has also found many of the results of the present paper's Secs. II and III (see Ref. [12]).

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