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Generalized Wick transform for gravity

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Using a key observation due to Thiemann, a generalized Wick transform is introduced to map the constraint functionals of Riemannian general relativity to those of the Lorentzian theory, including matter sources. This opens up a new avenue within ''connection dynamics'' where one can work, throughout, only with real variables. The resulting quantum theory would then be free of complicated reality conditions. The ramifications of this development to the canonical quantization program are discussed.

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This work is motivated by two related but independent considerations. The primary motivation comes from canonical quantum gravity. The approach based on self-dual connections has the advantage that all equations are low order polynomials. However, to ensure that one recovers real, Lorentzian general relativity one has to impose rather complicated ''reality conditions'' on the basic canonical variables. (See, e.g., [1,2].) Quantization would be significantly easier if one could work entirely with real variables and yet have manageable constraints. Within connection dynamics, this is indeed possible in the Riemannian (i.e., positive definite) signature because, in this case, self-dual connections are real [3]. Therefore, one way to achieve the desired goal would be to try to define a generalized Wick transform which would map the Riemannian constraint functionals to the Lorentzian ones. To be useful, the transform has, of course, to be sufficiently simple. The purpose of this communication is to show that a recent result of Thiemann's [4] implies that a transform with desired features exists and, furthermore, fits rather well with recent developments in quantum connection dynamics [5–10]. I should emphasize that the Riemannian theory plays only a mathematical role in our description. The philosophy is the same as the one that underlies exactly soluble models: The physical, Lorentzian, theory is complicated but can be tackled by mapping it to a mathematically simpler theory. It just happens that the simpler theory can be identified with Riemannian general relativity.

Our secondary motivation comes from the fact that the availability of such a transform would also be useful in other approaches to quantum gravity, notably the ones based on path integrals. In Minkowskian field theories, one often works with a Euclidean framework based on the Osterwalder-Schrader axioms, constructs the theory, and then recovers the Wightman functions from the Schwinger ones by a Wick rotation of the time coordinate. This simple route is not available in theories of gravity. The question therefore arises if there is a more general transform which will map the Riemannian action to the Lorentzian. We will answer this question affirmatively using the phase space form of the action. However, whether this result can be used to develop a full fledged path integral approach is still unclear.

Let us begin by specifying the phase space. Fix an orientable, smooth three-manifold Σ. The ADM phase space consists of pairs (E^a_i, K^i_a) of real fields on Σ where E^a_i are the (nondegenerate) triads of density weight one and K^i_a, the conjugate momenta. Thus, the three-metric q_{ab} is defined via $E^a_i E^{bi} = q q^{ab}$ where q is the determinant of q_{ab} and the extrinsic curvature K_{ab} is defined by $K_{ab} = (1/\sqrt{q}) K^i_a E^c_i q_{bc}$. The Gauss and the vector constraints

$$\mathscr{G}_i := \epsilon_{ijk} K^j_a E^{ak} = 0$$

and

$$\mathscr{V}_b := 2E^a_i D_{[a} K^i_{b]} = 0, \tag{1}$$

where D is defined by $D_a E^b_i = 0$, are independent of the signature. The scalar (or Hamiltonian) constraint, on the other hand, does depend on the signature:

$$\mathscr{S}_L = -qR + 2E^{[a}_i E^{b]}_j K^j_a K^i_b = 0$$

and

$$\mathscr{S}_R = -qR - 2E^{[a}_i E^{b]}_j K^j_a K^i_b = 0, \tag{2}$$

where R is the scalar curvature of the metric q_{ab} defined by E^a_i and where the subscripts L and R stand for "Lorentzian" and "Riemannian."

To pass to connection dynamics one can make a canonical transformation [3,1]

$$(E^a_i, K^i_a) \mapsto (A^i_a := \Gamma^i_a + K^i_a, E^a_i), \tag{3}$$

where Γ^i_a is the SU(2) spin connection determined by the triad E^a_i. Note that the new configuration variable A^i_a is real; it is again an SU(2) connection on Σ. It is easy to verify that the Gauss and the vector constraints (1) can be expressed as

$$\mathscr{G}_i = \mathscr{D}_a E^a_i = 0 \quad \text{and} \quad \mathscr{V}_b \approx E^a_i F^i_{ab} = 0, \tag{4}$$

where $\mathscr{D}$ is the gauge covariant operator defined by A and where $\approx$ stands for "equals, modulo Gauss constraint." Note incidentally that these are the simplest equations one can write down without reference to background fields: Among the nontrivial gauge covariant expressions, the left side of the first equation is the only one which is at most linear in E and A and that of the second is the only one which is at most linear in E and quadratic in A. The next simplest equation one can write is

$$\mathscr{S}'_R := \epsilon^{ijk} E^a_i E^b_j F_{abk} = 0. \tag{5}$$

When translated in terms of (E,K), $\mathscr{S}'_R$ reduces to $\mathscr{S}_R$ modulo the Gauss constraint. Note that while the left sides of (2) are nonpolynomial in E and K due to the presence of R, the left side of (5) is at worst quadratic in E and quadratic in A. These are the primary simplifications of connection dynamics. Using the close similarity of $\mathscr{S}_R$ and $\mathscr{S}_L$, one can readily translate the Lorentzian scalar constraint to connection dynamics [11]: $\mathscr{S}_L \approx -2qR - \epsilon^{ijk} E^a_i E^b_j F_{abk} = 0$. However, due to the presence of R the equation is again complicated and difficult to deal with in the quantum theory.

Now, if one uses a complex connection $A^{\mathbb{C}i}_a = \Gamma^i_a - iK^i_a$ in place of the real A^i_a, the Lorentzian scalar constraint does simplify [1]: it takes the same form as (5), i.e., becomes

$$\mathscr{S}_L \approx \mathscr{S}'_L := \epsilon^{ijk} E^a_i E^b_j F^{\mathbb{C}}_{abk} = 0, \tag{6}$$

where $F^{\mathbb{C}}$ is the curvature of $A^{\mathbb{C}}$. However, now the connection is complex and, to recover real general relativity, one has to impose reality conditions which also seem complicated at first sight. However, recently Thiemann [4] has provided an approach to incorporate such reality conditions in the quantization of a wide class of theories, including general relativity. This is achieved through the introduction of a "complexifier" which, in the classical theory, maps real connections A to complex ones $A^{\mathbb{C}}$. The resulting quantum complexifier can be regarded as a nontrivial generalization of the coherent state transform of [12] and (modulo certain technical issues that are being investigated) maps the Hilbert space of square-integrable functions of A to an appropriate Hilbert space of holomorphic functions of $A^{\mathbb{C}}$. For the class of theories in which the Hamiltonian (or the Hamiltonian constraint) is simpler in the holomorphic representation, Thiemann's complexifier should make dynamics as well as reality conditions manageable in the quantum theory.

In this Rapid Communication, I will restrict myself to general relativity but work entirely with real connections. The resulting framework seems to be technically simpler and conceptually more transparent for the case under consideration. I will also present an extension of the generalized Wick transform to the case when matter sources are present and discuss several ramifications of these results.

The idea is to construct a Poisson bracket preserving automorphism on the algebra of functions on the phase space which maps the Riemannian constraints to the Lorentzian ones (modulo constant rescalings). Recall first that, given a real function T on the classical phase space, the one-parameter family of diffeomorphisms generated by its Hamiltonian vector field X_T induces the map $W(t)$ on the algebra of functions on the phase space:

$$\begin{aligned} f \mapsto W(T) \circ f &= f + t\{f,T\} + \frac{t^2}{2!}\{\{f,T\},T\} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\{f,T\}_n, \end{aligned} \tag{7}$$

where $\{,\}$ denotes the Poisson brackets. For each value of the real parameter t, $W(t)$ preserves the $\star$-algebra structure as well as the Poisson brackets on the space of functions on the phase space. Now, if we let T be complex valued, the vector field X_T becomes complex and no longer generates motions on the phase space. However (assuming the series converges) the map $W(t)$ of (7) continues to be a Poisson bracket-preserving automorphism on the algebra of complex-valued functions on the phase space (although it no longer preserves the $\star$ relation). Following [4] let us set

$$T := \frac{i\pi}{2} \int_\Sigma d^3x \; K^i_a E^a_i . \tag{8}$$

Then, regarding E^a_i and K^i_a as (coordinate) functions on the phase space, and setting $W = W|_{t=1}$, we have

$$W \circ E^a_i = iE^a_i \quad \text{and} \quad W \circ K^i_a = -iK^i_a . \tag{9}$$

The automorphism property now implies $(W \circ f)(E,K) = f(iE, -iK)$, so that the constraint functions transform via

$$W \circ \mathscr{G}_i = \mathscr{G}_i, \quad W \circ \mathscr{V}_b = \mathscr{V}_b, \quad \text{and} \quad W \circ \mathscr{S}_R = -\mathscr{S}_L . \tag{10}$$

Thus, the automorphism W defined by the Thiemann generating function T maps the Riemannian constraints to the Lorentzian ones. It will therefore be referred to as a "generalized Wick transform."

The quantization strategy for the Lorentzian theory is then as follows. Begin with the real phase space $\mathcal{P}$ of pairs (A_a^i, E_i^a). The classical configuration space is then $\mathcal{A}/\mathcal{G}$, the space of connections modulo gauge transformations and the quantum configuration space is a suitable completion $\overline{\mathcal{A}/\mathcal{G}}$ thereof [5]. By now, integral [5–8] and differential [9] calculus on $\overline{\mathcal{A}/\mathcal{G}}$ is well developed and we can use it to define the Hilbert space of states and quantum operators. The heuristic requirement that the configuration and momentum operators $\hat{A}$ and $\hat{E}$ (when expressed as usual by multiplication by A and $-i\hbar\,\delta/\delta A$, respectively) be self-adjoint can be made precise and essentially suffices to select a unique measure μ_0 on $\overline{\mathcal{A}/\mathcal{G}}$. The resulting Hilbert space $\mathcal{H}_0 := L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ serves as the space of kinematic states of quantum gravity, the quantum analogue of the full phase space $\mathcal{P}$. (Using integration theory one can also define a loop transform from $\mathcal{H}_0$ to a space of suitable functions of loops and thus provide a rigorous basis for the Rovelli-Smolin loop representation [13].) Using differential calculus on $\overline{\mathcal{A}/\mathcal{G}}$ one can introduce geometric operators on $\mathcal{H}_0$, e.g., corresponding to areas of two-surfaces and volumes of three-dimensional regions [14,10]. (See also [15].) These can be shown to be self-adjoint with purely discrete spectra, showing that quantum geometry is very different from what the continuum picture suggests.

To tackle dynamics one has to solve the quantum constraints. Since we work on $\overline{\mathcal{A}/\mathcal{G}}$, the Gauss constraint is already taken care of. (Alternatively, it could also be imposed in the manner of Dirac; the final result is the same.) The vector or diffeomorphism constraint can be solved [10] using a "group averaging" technique [16]; there are no anomalies. The space of solutions is naturally endowed with a Hilbert space structure [10], which we will denote by $\mathcal{H}_d$.

The last and the key step is to solve the Hamiltonian constraint. The presence of the generalized Wick transform suggests the following strategy. One can begin with the Riemannian constraint $\mathcal{S}_R'$ of Eq. (5). (Since the Gauss constraint has already been imposed, $\mathcal{S}_R$ and $\mathcal{S}_R'$ are on the same footing.) Because it has a simple expression in terms of A and E one can hope to regularize the corresponding quantum operator. For technical reasons (associated with regularization), one is led to work not with $\mathcal{S}_R'$ itself but rather with its square root. Let $\hat{\mathcal{S}}$ be the corresponding quantum operator. The idea now is to exploit the generalized Wick transform. Since W is a Poisson-brackets preserving automorphism on the algebra of phase space functions, its quantum analogue $\hat{W}$ would be an automorphism on the algebra of quantum operators. In view of Eq. (10) we can simply define the Lorentzian operator $\hat{\mathcal{S}}_L$ by

$$\hat{\mathcal{S}}_L = \hat{W} \circ \hat{\mathcal{S}} \circ \hat{W}^{-1} \equiv \exp\left(-\frac{1}{i\hbar}\hat{T}\right) \circ \hat{\mathcal{S}} \circ \exp\left(\frac{1}{i\hbar}\hat{T}\right), \tag{11}$$

where $\hat{T}$ is an operator version of T. Equation (7) implies that $\hat{\mathcal{S}}_L$ so defined will automatically have the correct classical limit. Physical quantum states $|\Psi\rangle_L$ can now be obtained by Wick transforming the kernel of $\hat{\mathcal{S}}$:

$$\hat{\mathcal{S}}|\Psi\rangle = 0 \Leftrightarrow \hat{\mathcal{S}}_L(\hat{W}|\Psi\rangle) \equiv \hat{\mathcal{S}}_L|\Psi\rangle_L = 0. \tag{12}$$

Thus, the availability of the Wick transform could provide considerable technical simplification: the problem of finding solutions to all quantum constraints is reduced essentially to that of regulating relatively simple operators $\hat{\mathcal{S}}$ and $\hat{T}$.

Indeed, significant progress has already been made on both these problems. First, using a key idea due to Rovelli and Smolin [17], the operator $\hat{\mathcal{S}}$ has been made well defined on diffeomorphism invariant states, i.e., on a dense subspace of $\mathcal{H}_d$ [18]. (Since $\mathcal{S}_R'$ itself is only diffeomorphism covariant, the image of $\hat{\mathcal{S}}$ is also only diffeomorphism covariant. However, we are interested only in the kernel of this operator.) This regularization is not yet fully satisfactory. Nonetheless, it holds considerable promise; it is the first systematic attempt at a nonperturbative regularization of the "Wheeler-DeWitt equation." As for $\hat{T}$, note first that the classical T can be expressed as $T = (i\pi/2)\{V, H_E\}$ where V is the total volume of Σ and $H_E := \int d^3x (1/\sqrt{q}) \mathcal{S}_E'$ is the "Riemannian Hamiltonian." Hence, it is natural to set $\hat{T} = (1/i\hbar)[\hat{V}, \hat{H}_E]$. Now, $\hat{V}$ has already been regularized rigorously and the regularization of $\hat{\mathcal{S}}$ provides an avenue to regularize $\hat{H}_E$. If this last regularization can be completed one would be able to extract solutions to all quantum constraints via Eq. (12).

The final step in the program is to introduce the appropriate inner product on the space of physical states. If one uses the analogue of the generalized Wick transform for simple model systems one finds that, to obtain interesting physical states, one has to allow solutions $|\Psi\rangle$ (to the analogue of the Riemannian constraint) which are far from being "tame"; for example, they may diverge at "infinity" (i.e., at the boundary of the configuration space). Therefore, the problem of finding the correct inner product is, in general, quite nontrivial. However, if these concrete steps can be completed in the case under consideration, one would have a consistent nonperturbative quantization of general relativity. The focus will shift to developing approximation methods to extract physical predictions of the theory.

The "real" strategy adopted here is of course closely related to the "complex" strategy of Thiemann's [4]. At the classical level, the two are completely equivalent; only the emphasis is different. Thus, in the complex approach, one notes that the generalized Wick transform W has the action $W \circ A_a^i = A_a^{\mathbb{C}i}$ on connections and concludes that W sends Riemannian scalar constraint $\mathcal{S}_R'$ of Eq. (5) to the Lorentzian $\mathcal{S}_L'$ of Eq. (6). Since $A^{\mathbb{C}}$ and $\mathcal{S}_L'$ are complex valued, in the quantum theory, one is then naturally led to the holomorphic representation. In the real approach, by contrast, one works exclusively with real phase space variables and real constraint functions. (In particular, the classical Wick transform could be useful also in geometrodynamics.) In the quantum theory, the use of holomorphic representation is no longer essential. However, there is nothing that prevents one from constructing this representation using techniques from [4]. Indeed, it is desirable to construct it because of its closeness to coherent states; it could, for example, play an important role in semiclassical considerations. In both approaches, the issue of introducing the physically appropriate inner product remains open, although a general direction for completing this task has been suggested in [4].

For general relativity, Thiemann introduced the generating function T only in the source-free case. Can the strategy of using a generalized Wick transform be extended consistently to incorporate the presence of matter? The answer is in the affirmative. Perhaps the most concise way to see this is to use the phase space action functional. Since this is in part a restatement of the main results, for the convenience of readers who may be more familiar with the Arnowitt-Deser-Misner (ADM) framework, I will use this opportunity to state these result using geometrodynamical variables.

The space-time action for general relativity with a cosmological constant, coupled with a scalar and a Maxwell field, can be expressed in terms phase space variables. Modulo surface terms, one obtains

$$\begin{aligned} S_R^L=&\frac{1}{2}\int d^4x\ N[\mp(q_{ab}q_{cd}-q_{ac}q_{bd}-q_{ad}q_{bc})P^{ab}P^{cd} \\ &+2q(R-2\Lambda)+(\mp P^aP^b+B^aB^b)q_{ab} \\ &+(\mp\pi^2+qq^{ab}D_aD_b\phi+\mu^2q\phi)], \end{aligned} \tag{13}$$

where the superscript L and the subscript R stand for "Lorentzian" and "Riemannian," respectively; q_{ab} is the three-metric and P^{ab} its canonical momentum; A_a is the Maxwell three-potential and P^a its canonical conjugate momentum; ϕ is the Klein-Gordon field and $\tilde{\pi}$ its canonically conjugate momentum; N is the lapse; R, the scalar curvature of q_{ab}, and Λ the cosmological constant. To see the explicit form of the constraints one can reexpress the action in the canonical phase space form as

$$\begin{aligned} S_R^L=&\int dt\int d^3x\ P^{ab}\dot{q}_{ab}+P^a\dot{A}_a+\pi\dot{\phi}+N\mathbf{S}_R^L \\ &+2N^a(\mathbf{V}_R^L)_a+({}^4A\cdot t)D_aP^a, \end{aligned} \tag{14}$$

where N, N^a, and ${}^4A\cdot t$ are the Lagrange multipliers representing the lapse, the shift, and the Maxwell scalar potential; and $\mathbf{S}$ and $\mathbf{V}_a$ are the scalar and vector constraints. These are given by

$$\begin{aligned} 2\mathbf{S}_R^L=&\pm(q_{ab}q_{cd}-q_{ac}q_{bd}-q_{ad}q_{bc})P^{ab}P^{cd} \\ &+2q(R-2\Lambda)+(\pm P^aP^b+B^aB^b)q_{ab} \\ &+(\pm\pi^2+qq^{ab}D_a\phi D_b\phi+\mu^2q\phi^2) \end{aligned} \tag{15}$$

and

$$2(\mathbf{V}_R^L)_a=2q_{ab}D_cP^bc-\pi D_a\phi-P^bF_{ab}, \tag{16}$$

where $B^a=\frac{1}{2}\ \eta^{abc}\partial_{[b}A_{c]}$ is the magnetic field of the vector potential A_a. Note that the scalar constraint is a density of weight two and the vector potential, of weight one. Therefore, the lapse N in Eqs. (13) and (14) is a scalar density of weight -1 while the shift N^a is just a vector field.

Now, let us consider the generalized Wick transform $\mathbf{W}$ on the Einstein-Maxwell-Klein-Gordon phase space generated by the function $\mathbf{T}$:

$$\mathbf{T}:=\frac{i\pi}{2}\int d^3x\ q_{ab}P^{ab}+\frac{i\pi}{4}\int d^3x\ A_aP^a. \tag{17}$$

It is then straightforward to compute the action of $\mathbf{W}$ on the canonical pairs, regarded as (coordinate) functions on the phase space. However, since the action functionals of Eqs. (13) and (14) depend also on the Lagrange multipliers and coupling constants, we need to specify how $\mathbf{W}$ acts on them. Can we choose transformation properties of these nondynamical variables so that Riemannian action functional is mapped to the Lorentzian one? Not only does such a choice exist but is in fact unique:

$$\begin{aligned} &\mathbf{W}\circ(N,N^a,{}^4A\cdot t)=(-N,N^a,e^{i\pi/4}\ {}^4A\cdot t), \\ &\mathbf{W}\circ(\Lambda,\mu^2)=(-i\Lambda,-i\mu^2). \end{aligned} \tag{18}$$

(Note that $\mathbf{W}$ has the same action on the Lagrange multiplier ${}^4A\cdot t$ as it has on the dynamical variable A.) With this specification, it is straightforward to verify that $\mathbf{W}\circ S_R=S_L$. Thus, $\mathbf{W}$ serves as the generalized Wick transform. General considerations outlined in the source-free case suggest that the corresponding quantum operator $\hat{\mathbf{W}}$ should send the kernel of the Riemannian constraint operators to that of the Lorentzian constraints. However, to make these heuristic considerations precise, it is essential to regulate the Riemannian constraint operators and the generator $\mathbf{T}$. These problems are yet to be investigated. Finally, note that the classical generator $\mathbf{T}$ has a suggestive form:

$$\mathbf{T}=\sum_{s=0}^{s=2}\left(\frac{i\pi}{4}\right)s\int d^3x\ Q\circ P, \tag{19}$$

where s is the spin of the field, Q its configuration variable, and P its momentum variable. This form continues to hold for spin $\frac{1}{2}$ fields as well. It may well be a reflection of a deeper structure underlying the generalized Wick transform.

To conclude, let me summarize a few features of the classical generalized Wick transform $\mathbf{W}$. The fact that $\mathbf{W}$ sends the Riemannian action functional to the Lorentzian one may tempt one to look for a simple space-time interpretation of the transform. However, I believe that such an interpretation does not exist. Note in particular that the lapse-shift pairs transform in a way that is different from what a space-time interpretation would suggest (i.e., from the common usage in quantum cosmology). The natural home for the transform appears, rather, to be the phase space. However, care is needed even in this picture: As was already emphasized, because the generating function $\mathbf{T}$ is imaginary, $\mathbf{W}$ does not arise from a canonical transform on the real phase space. We could complexify the phase space and consider the Hamiltonian flow generated by $\mathbf{T}$. As far as I can see, however, one cannot interpret $\mathbf{W}$ as mapping a real subspace of this complex phase space which can be called "the phase space of the Riemann theory" to a real subspace which can be identified with the "phase space of the Lorentzian theory." Thus, the interpretation of W as a generalized Wick transform refers only to its role as an automorphism on the algebra of functions on the common real phase space of the two theories.

Next, while the quantum operator $\hat{\mathbf{W}}$ sends solutions of the Riemannian quantum constraints to solutions of the Lorentzian quantum constraints, there is no obvious sense in which the classical **W** maps solutions to constraints of one theory to those of the other again because **W** is not associated with a diffeomorphism of the phase space. However, the classical **W** does have an interesting "dynamical" role. Fix a lapse shift pair (N,N^a) and consider the "Hamiltonian" functional $H_R := \int d^3x [N\mathbf{S}_R + N^a(\mathbf{V}_R)_a]$ of the Riemannian theory. Denote the corresponding Hamiltonian vector field by $X^R_{N,\vec{N}}$. Since the vector fields can be regarded as derivations on the ring of smooth functions, the automorphism **W** on the algebra of smooth functions induces a map on the space of vector fields which we will denote again by **W**. Now, because **W** sends H_R to $H_L := \int d^3x[-N\mathbf{S}_L + N^a(\mathbf{V}_L)_a]$ and because it preserves Poisson brackets, it follows that $\mathbf{W}\circ X_{N,\vec{N}} = X^L_{-N,\vec{N}}$. Thus, the Riemannian dynamical trajectories are sent to the Lorentzian dynamical trajectories. Recall however that only the integral curves of the Hamiltonian vector fields which lie on the constraint surface can be identified with physical solutions. Hence, in general **W** does not send four-dimensional Riemannian solutions to four-dimensional Lorentzian ones. On general grounds, one does not expect any map with this stronger property to exist on the full solution space. Indeed it is surprising that even a map that sends $X^R_{N,\vec{N}}$ to $X^L_{-N,\vec{N}}$ should exist. That this is achieved by an explicit and relatively simple generator **T** is very striking. It is quite possible that this fact will have some powerful applications already in classical gravity.

Finally, the ideas discussed here should be applicable also to integrable models obtained by dimensional reduction of general relativity [19].

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