Exact solutions for null fluid collapse

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Exact nonstatic spherically symmetric solutions of the Einstein equations for a null fluid source with pressure P and density ρ related by $P=k\rho^a$ are given. The $a=1$ metrics are asymptotically flat for $1/2<k\leqslant 1$ and cosmological for $0<k<1/2$. The $k=1$ metric is the known charged Vaidya solution. In general the metrics have multiple apparent horizons. In the long time limit, the asymptotically flat metrics are hairy black hole solutions that "fall between" the Schwarzschild and Reissner-Nordström metrics.

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The problem of gravitational collapse in general relativity is of much interest. One would like to know whether, and under what initial conditions, gravitational collapse results in black hole formation. In particular, one would like to know if there are physical collapse solutions that lead to naked singularities. If found, such solutions would be counterexamples of the cosmic censorship hypothesis, which states that curvature singularities in asymptotically flat spacetimes are always shrouded by event horizons.

Since the general problem appears intractable due to the complexity of the full Einstein equations, metrics with special symmetries are used to construct gravitational collapse models. One such case is the two-dimensional reduction of general relativity obtained by imposing spherical symmetry. Even with this reduction however, there are very few inhomogeneous nonstatic exact solutions known. One example is the Vaidya metric [1,2]. It describes the collapse of pressureless null dust and is asymptotically flat. For the minimally coupled massless scalar field, the only inhomogeneous nonstatic examples are cosmological solutions which have naked singularities [3,4]. For perfect fluids with equation of state relating pressure P and energy density ρ given by $P=k\rho$, self-similar solutions (which are not asymptotically flat) have been studied [5].

Recently there have been a number of numerical studies of the spherically symmetric collapse problem [6–10]. The initial data in these works are a compact ingoing pulse. It was found that a black hole forms if any of the parameters c in the initial data are above certain critical values c_*, otherwise the pulse scatters back out to infinity leaving flat space. In particular, the black hole mass formula $M_{\rm BH}=k(c-c_*)^{0.36}$ was discovered in Ref. [6]. This shows that black holes initially form with zero mass—that is, no mass gap is found.

There have been a number of attempts aimed at an analytical understanding of these results [11–18]. One approach is to look for exact collapse solutions and see what black hole mass formulas may be extracted from them.

In this Rapid Communication we give exact inhomogeneous and nonstatic spherically symmetric solutions of the Einstein equations for a collapsing null fluid. The equation of state of the fluid is $P=k\rho^a$. As we will see, while these collapse solutions do not have direct relevance for the critical behavior mentioned above, they do have a number of interesting features, including hair on black holes.

An inverted approach is used to find the solutions. First the stress-energy tensor is determined from the metric. Then the equation of state and the dominant energy condition are *imposed* on its eigenvalues. This leads to an equation for the metric function, which is easily solved. The precise form of the stress-energy tensor is then displayed. Some interesting properties of the solutions are then discussed.

For the general spherically symmetric metric

$$ds^2=-e^{2\psi(r,v)}F(r,v)dv^2+2e^{\psi(r,v)}dv\,dr+r^2d\Omega^2, \quad (1)$$

where $0\leqslant r\leqslant\infty$ is the proper radial coordinate, $-\infty\leqslant v\leqslant\infty$ is an advanced time coordinate, and $d\Omega^2$ is the metric on the unit two sphere, the Einstein equations $G_{ab}=8\pi T_{ab}$ give

$$m'\equiv\frac{\partial m}{\partial r}=-4\pi r^2 T_v{}^v, \quad (2)$$

$$\dot m\equiv\frac{\partial m}{\partial v}=4\pi r^2 T_v{}^r, \quad (3)$$

$$\psi'=4\pi r T_{rr}. \quad (4)$$

The mass function $m(r,v)$ is defined by $F(r,v)=1-2m(r,v)/r$, and is a measure of the mass contained within radius r. We will consider the special case $\psi(r,v)=0$, which means from (4) that $T_{rr}=0$.

The stress-energy tensor derived from the above metric may be diagonalized to give the energy density and the principal pressures. The eigenvalue problem is $T_a{}^bU_b=\lambda U_a$. The θ-ϕ part of $T_a{}^b$, which is determined from $G_{\theta\theta}$ and $G_{\phi\phi}=\sin^2\theta\, G_{\theta\theta}$, is already diagonal with pressure eigenvalues

$$P\equiv T_\theta{}^\theta=T_\phi{}^\phi=-\frac{m''}{8\pi r}. \quad (5)$$

Since $T_{rr}=0$ (by choice), we have $T_v{}^v=T_r{}^r$ and $T_r{}^v=0$. Therefore the v-r part of the matrix to be diagonalized is

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$$T_a{}^b = \begin{pmatrix} T_v{}^v & T_v{}^r \\ 0 & T_v{}^v \end{pmatrix}. \tag{6}$$

This has one eigenvalue λ which gives the energy density ρ: namely,

$$-\rho \equiv \lambda = T_v{}^v = -\frac{m'}{4\pi r^2}. \tag{7}$$

The corresponding eigenvector is $v_a = (1,0,0,0)$ [in the coordinates (v,r,θ,ϕ)], and is lightlike. Therefore the stress-energy tensor, (which follows from $\psi=0$), is of Type II [19]. Its nonvanishing components are

$$T_{vv} = \rho\left(1-\frac{2m}{r}\right)+\frac{\dot{m}}{4\pi r^2}, \quad T_{vr} = -\rho, \tag{8}$$

$$T_{\theta\theta} = P g_{\theta\theta}, \quad T_{\phi\phi} = P g_{\phi\phi}.$$

These components may be succinctly written using the two linearly independent future pointing lightlike vectors $v_a = (1,0,0,0)$ and $w_a = (F/2,-1,0,0)$ as

$$T_{ab} = \frac{\dot{m}(r,v)}{4\pi r^2} v_a v_b + \rho(r,v)(v_a w_b + v_b w_a) + P(r,v)(g_{ab} + v_a w_b + v_b w_a). \tag{9}$$

[This tensor should be compared with the perfect fluid one $T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b)$, where u_a is timelike.]

The stress-energy tensor (9) has support along both the two future pointing lightlike vectors v_a and w_a, and as we will see below, it is of precisely the form which gives the charged Vaidya solution [2]. For $P=\rho=0$, (9) reduces to the stress-energy tensor which gives the uncharged Vaidya metric. We also note that while $T_{ab}w^a w^b = \dot{m}/4\pi r^2$, $T_{ab}v^a v^b = 0$. Therefore there is energy flux only along one of the null directions.

A static observer with four-velocity $S^a = (1/\sqrt{F},0,0,0)$ and a rotating observer with four-velocity $R^a = (\sqrt{2/F},0,0,1/r \sin\theta)$ see, respectively, the energy densities $T_{ab}S^a S^b = \dot{m}/4\pi F r^2 + \rho$ and $T_{ab}R^a R^b = 2(\dot{m}/4\pi F r^2 + \rho) + P$.

The stress-energy tensor (9) satisfies the dominant energy condition if the following three conditions are met:

$$P \geqslant 0, \ \ \rho \geqslant P, \ \text{ and } \ T_{ab}w^a w^b > 0. \tag{10}$$

The first two of these imply that $m' \geqslant 0$ and $m'' \leqslant 0$. The former just says that the mass function either increases with r or is constant, which is a natural physical requirement on it.

To satisfy the first two of the dominant energy conditions (10), we *impose* the equation of state $P = k\rho^a$, with $k \leqslant 1$ and $a \leqslant 1$. The $a<1$ case will be discussed later. For $a=1$ this gives the equation

$$-\frac{m''}{8\pi r} = k\frac{m'}{4\pi r^2} \tag{11}$$

for the mass function, which is easily integrated to give

$$m(r,v) = \begin{cases} f(v) - g(v)/[(2k-1)r^{2k-1}], & k \neq \frac{1}{2}, \\ f(v) + g(v)\ln r, & k = \frac{1}{2}, \end{cases} \tag{12}$$

where $f(v)$ and $g(v)$ are arbitrary functions (which are restricted only by the energy conditions). Therefore we have explicitly that

$$P = k\frac{g(v)}{4\pi r^{2k+2}} = k\rho. \tag{13}$$

Therefore we must have $g(v) \geqslant 0$ for positive pressure and energy density.

The last requirement in (10) for the dominant energy condition leads, for $k \neq 1/2$, to

$$\dot{m} = \dot{f}(v) - \frac{\dot{g}(v)}{(2k-1)r^{2k-1}} > 0. \tag{14}$$

Physically this means that the matter within a radius r increases with time, which corresponds to an implosion. This condition is most easily satisfied if $\dot{f}>0$, and either $\dot{g}>0$ and $k<1/2$, or $\dot{g}<0$ and $k>1/2$.

In summary, for the dominant energy condition, we must have $g(v) \geqslant 0$ and either $\dot{g}>0$ for $k<1/2$ or $\dot{g}<0$ for $k>1/2$. For the weak or strong energy conditions (which are equivalent for Type II stress-energy tensors), we only need $\rho \geqslant 0$, $P \geqslant 0$, and $T_{ab}w^a w^b > 0$, but not $\rho > P$. Therefore for the latter energy conditions we can have $k>1$ as well.

For $k=1/2$, neither the weak nor dominant energy conditions can be satisfied for all r because $\dot{m} = \dot{f}(v) + \dot{g}(v)\ln r$, which always becomes negative for sufficiently small r. Therefore we will not consider this case further.

In summary, we have shown that the metric

$$ds^2 = -\left(1 - \frac{2f(v)}{r} + \frac{2g(v)}{(2k-1)r^{2k}}\right)dv^2 + 2dv\ dr + r^2 d\Omega^2 \tag{15}$$

is a solution of the Einstein equations for the null fluid stress-energy tensor (9) with $P=k\rho$, where P is given by (13).

There are two special cases of this solution which are already known. One is the Vaidya metric [1], which arises for $g(v)=0$ (vanishing ρ and P). Then the only nonvanishing component of the stress-energy tensor (9) is $T_{vv} = \dot{m}/4\pi r^2 = \dot{f}(v)/4\pi r^2$. The other is the charged Vaidya metric, where the charge depends on v [2]. This arises when $k=1$ in (15).

We therefore see that the stress-energy tensor we have determined by imposing the equation of state $P=k\rho$ is a one parameter $(k>0)$ generalization of the stress-energy tensor which gives the charged Vaidya metric. The corresponding metric must therefore depend on k as in equation (15). (We note that the parameter k in the metric or the stress-energy tensor cannot be eliminated by a coordinate transformation because it is the constant of proportionality between the eigenvalues of the stress-energy tensor—exactly the same reason that this constant is not ‘‘gauge’’ for the ordinary perfect fluid solutions with equation of state $P=k\rho$.)

A metric is considered to be asymptotically flat [20,21] if in the vicinity of a spacelike hypersurface its components behave as

$$g_{ab}\rightarrow \eta_{ab}+\frac{\alpha_{ab}(x^c/r,t)}{r}+O\left(\frac{1}{r^{1+\epsilon}}\right) \tag{16}$$

as $r\rightarrow\infty$. ($\epsilon>0$, η_{ab} is the Minkowski metric, α_{ab} is an arbitrary symmetric tensor, and x^c is a flat coordinate system at spacelike infinity.)

According to this definition, our metrics (15) are asymptotically flat for $k>1/2$ and are cosmological for $k<1/2$. (In particular, for $k=1$, $f(v)=M$ and $2g(v)=Q^2$, the metric is just Reissner-Nordström.)

When the imposed equation of state on the eigenvalues (5) and (7) of the stress-energy tensor (9) is $P=k\rho^a$, the equation for the mass function is more complicated:

$$-\frac{m''}{8\pi r}=k\left(\frac{m'}{4\pi r^2}\right)^a. \tag{17}$$

This has solution

$$m(r,v)=f(v)+\int dr[g(v)-k(4\pi)^{1-a}r^{2(1-a)}]^{1/1-a}. \tag{18}$$

Since $a<1$, this mass function gives only cosmological metrics—the pressures are too small to make them asymptotically flat. However, if the dominant energy condition is not imposed but only the weak (or strong) one is, then $a>1$ is possible, which will give asymptotically flat metrics.

We now consider in turn the asymptotically flat ($k>1/2$) and the cosmological ($k<1/2$) metrics.

$k>1/2$: Since the dominant energy condition is satisfied only for $g(v)\geqslant 0$ and $\dot{g}(v)\leqslant 0$, if $g(v)$ is zero initially, it must remain zero. On the other hand, if it is nonzero initially, it can decrease to zero. Therefore the asymptotically flat metrics can be flat in the $v\rightarrow-\infty$ limit only if $g(v)\equiv 0$. But this is just the Vaidya case. However, the following two parameter ($A\geqslant 0, 0\leqslant B\leqslant 1$) family of metrics (where we have set $k=1$ for concreteness), demonstrates an interesting feature:

$$ds^2=-\left(1-\frac{A(1+\tanh v)}{r}+\frac{(1-B\ \tanh v)}{r^2}\right)dv^2 + 2dv\ dr+r^2 d\Omega^2. \tag{19}$$

In the $v\rightarrow-\infty$ limit this metric has a naked singularity at $r=0$. However, in the $v\rightarrow+\infty$ limit it may have horizons depending on the relative values of A,B. Specifically, these horizons are given by $r=A\pm\sqrt{A^2+B-1}$. Therefore a black hole first forms at nonzero mass when $A^2=1-B$. This mass gap separates a black hole from a naked singularity, and may be contrasted with the critical behavior solutions [6–8] where there is no mass gap between flat space and a black hole metric. Similar results hold for other values of $k>1/2$. What is happening physically is that initially only the "charge" term is present in the metric, and subsequently, infalling fluid reduces the "charge" and adds a "mass" term.

These solutions also include metrics that give the evolution of a naked singularity at $v=-\infty$ into flat space at $v=\infty$. This case occurs if we set $A=0$, $B=1$ in (19).

Another feature of these solutions is that they give black holes with null fluid hair for $1/2<k<1$. An example of such a static solution results from the $v\rightarrow\infty$ limit of the metric:

$$ds^2=-\left(1-\frac{A(1+\tanh v)}{r}+\frac{(1-B\ \tanh v)}{r^{3/2}}\right)dv^2 + 2dv\ dr+r^2 d\Omega^2, \tag{20}$$

with B as the "hair." (We have put $k=3/4$.)

$k<1/2$: The dominant energy condition is now satisfied for $g(v)\geqslant 0$ and $\dot{g}(v)\geqslant 0$. Therefore these metrics can be flat as $v\rightarrow-\infty$, and have black holes as $v\rightarrow\infty$. A specific two parameter ($A,B\geqslant 0$) example for $k=1/3$ (radiation) is

$$ds^2=-\left(1-\frac{C+A(1+\tanh v)}{r}-\frac{B(1+\tanh v)}{r^{2/3}}\right)dv^2 + 2dv\ dr+r^2 d\Omega^2. \tag{21}$$

The apparent horizons in the $v\rightarrow\infty$ limit (with $C=0$) are now given by the cubic equation $(r-2A)^3-(2B)^3 r=0$. This equation always has a solution for the ranges of A and B allowed by the energy conditions. Thus, this solution describes the collapse of radiation ($P=\rho/3$) from flat space at $v=-\infty$ to a black hole at $v=\infty$.

Another possibility occurs when C is negative in (21). Then, as for the $k>1/2$ case above, we have a naked singularity at $v=-\infty$, which becomes a black hole at $v=\infty$ as the collapse proceeds.

In the general case, if $\lim_{v\rightarrow\infty}f(v)=A$ and $\lim_{v\rightarrow\infty}g(v)=B$ the radii of the apparent horizons in the $v\rightarrow\infty$ limit of the metric (15) are given by

$$r=2A-\frac{2B}{(2k-1)r^{2k-1}}, \tag{22}$$

which in general may have multiple solutions.

In conclusion, we have given a new class of null fluid collapse solutions (15) and (18) of the Einstein equations for the stress-energy tensor (9). These include new asymptotically flat black hole solutions ($a=1$, $1/2<k<1$) with multiple apparent horizons and hair. The general metric depends on one parameter (k), and two arbitrary functions of v (modulo energy conditions). The long time limits of the asymptotically flat solutions "fall between" the Schwarzschild and Reissner-Nordström metrics in the sense that $1<2k\leqslant 2$ in (15).

Physically, the $k>1/2$ solutions describe the evolution of a naked singularity into the same, or into a black hole. The parameters in the solution (19) determine which of these possibilities occurs, and the black hole always forms at a finite nonzero mass. The $k<1/2$ solutions describe the evolution of either flat space or a naked singularity into a black hole in a cosmology. All of the new solutions support the cosmic censorship conjecture.

Because of these physical properties, the exact solutions we have given are not of relevance for the collapse solutions that exibit critical behavior [6–8]. The physical reason for this is that, although our stress-energy tensor (9) has nonzero components along both ingoing and outgoing null directions, there is energy flow only along one direction (because, as noted above $T_{ab}v^a v^b = 0$).

It should be possible to find more exact solutions by imposing other equations of state on the eigenvalues of the stress-energy tensor (9).

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