How generic are null spacetime singularities?

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The spacetime singularities inside realistic black holes are sometimes thought to be spacelike and strong, since there is a generic class of solutions (BKL) to Einstein's equations with these properties. We show that null, weak singularities are also generic, in the following sense: there is a class of vacuum solutions containing null, weak singularities, depending on 8 arbitrary (up to some inequalities) analytic initial functions of 3 spatial coordinates. Since 8 arbitrary functions are needed (in the gauge used here) to span the generic solution, this class can be regarded as generic.

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One of the most fascinating outcomes of general relativity is the observation that the most fundamental concept in physics, the fabric of space and time, may become singular in certain circumstances. A series of singularity theorems [1] imply that spacetime singularities are expected to develop inside black holes. The observational evidence at present is that black holes do exist in the Universe. The formation of spacetime singularities in the real world is thus almost inevitable. However, the singularity theorems tell us almost nothing about the nature and location of these singularities. Despite a variety of investigations, there is today still no consensus on the structure of singularities inside realistic black holes.

At issue are the following features of singularities: their location, causal character (spacelike, timelike, or null), and, most importantly, their strength. We use here Tipler's terminology [2] for weak and strong singularities. In typical situations, if the spacetime can be extended through the singular hypersurface so that the metric tensor is C^0 and nondegenerate, then the singularity is weak [2]. The strength of the singularity has far-reaching physical consequences. A physical object which moves toward a strong curvature singularity will be completely torn apart by the diverging tidal force, which causes unbounded tidal distortion. On the other hand, if the singularity is weak, the total tidal distortion may be finite (and even arbitrarily small), so that physical observers may possibly not be destroyed by the singularity [2,3].

The main difficulty in determining the structure of black hole singularities is that the celebrated exact black hole solutions (the Kerr-Newman family) do not give a realistic description of the geometry inside the horizon, although they do describe well the region outside. This is because the well-known no-hair property of black holes, that arbitrary initial perturbations are harmlessly radiated away and do not qualitatively change the spacetime structure, only applies to the *exterior* geometry. The geometry inside the black hole (near the singularity and/or the Cauchy horizon) is unstable to initial small perturbations [4,5], and consequently we must go beyond the classic exact solutions to understand realistic black hole interiors. To determine the structure of generic singularities, it is necessary to take initial data corresponding to the classic black hole solutions, make generic small perturbations to the initial data, and evolve forward in time to determine the nature of the resulting singularity. For this purpose a linear evolution of the perturbations may be insufficient; the real question is what happens in full nonlinear general relativity.

The simplest black-hole solution, the Schwarzschild solution, contains a central singularity which is spacelike and strong. For many years, this Schwarzschild singularity was regarded as the archetype for a spacetime singularity. Although this particular type of singularity is known today to be unstable to deviations from spherical symmetry (and hence unrealistic) [6], another type of a strong spacelike singularity, the so-called Balinsky-Khalatnikov-Lifshitz (BKL) singularity [7], is believed to be generic (below we shall further explain and discuss the concept of genericity). Since the BKL singularity is so far the *only* known type of generic singularity, in the last two decades it has been widely believed that the final state of a realistic gravitational collapse must be the strong, spacelike, oscillatory, BKL singularity.

Recently, there have been a variety of indications that a spacetime singularity of a completely different type actually forms inside realistic (rotating) black holes. In particular, this singularity is *null* and *weak*, rather than spacelike and strong. The first evidence for this new picture came from the mass-inflation model [8,3], a toy model in which the Kerr background is modeled by the spherically symmetric Reissner-Nordstrom solution, and the gravitational perturbations are modeled in terms of two crossflowing null fluids. Later more realistic analyses replaced the null fluids by a spherically symmetric scalar field [9]. More direct evidence came from a nonlinear perturbation analysis of the inner structure of rotating black holes [10]. Both the mass-inflation models and the nonlinear perturbation analysis of Kerr strongly suggest that a null, weak, scalar-curvature singularity develops at the inner horizon of the background geometry. (See also an earlier model by Hiscock [11].)

Despite the above compelling evidence, there still is a debate concerning the nature of generic black-hole singularities. It is sometimes argued that the Einstein equations, due to their nonlinearity, do not allow generic solutions with null curvature singularities [12]. According to this argument, the nonlinearity, combined with the diverging curvature, immediately catalyzes the transformation of the null curvature singularity into a strong spacelike one, presumably the BKL singularity [13]. A similar argument was also given, some time ago, by Chandrasekhar and Hartle [5]. According to this point of view, the results of the nonlinear perturbation analysis of Kerr are to be interpreted as an artifact of the perturbative approach used [15] (and the mass-inflation model is a toy model, after all). This objection clearly marks the need for a more rigorous, nonperturbative, mathematical analysis, to show that a generic null weak singularity is consistent with Einstein's equations.

Recently, Brady and Chambers showed that a null singularity could be consistent with the constraint section of Einstein's equations formulated on null hypersurfaces [16]. However, their result does not completely resolve the above issue. The hypothesis raised in Ref. [12], according to which nonlinear effects will immediately transform the singular initial data into a spacelike singularity, is not necessarily inconsistent with the analysis of Ref. [16]. It is possible that a spacelike singularity could form just at the intersection point of the two characteristic null hypersurfaces considered in Ref. [16]. It is primarily the *evolution* equations which will determine whether singular initial data will evolve into a null singularity or into a spacelike one.

The purpose of this paper is to present a new mathematical analysis which addresses the above question. Our analysis shows that (i) the vacuum Einstein equations (both the constraint and evolution equations) admit solutions with a null weak singularity and (ii) the class of such singular solutions is so large that it depends on the maximum possible number of independent functional degrees of freedom. We will call such classes of solutions *functionally generic* (see below). Therefore any attempt to argue, on local grounds, that a null weak singularity is necessarily inconsistent with the nonlinearities of Einstein's equations, must be false. In this Rapid Communication we outline this analysis and present the main results; a full account of this work is given in Ref. [17].

Let us first explain what we mean by "degrees of freedom" and "functionally generic." Suppose that ψ is some field on a $(3+1)$-dimensional spacetime, which may be a multicomponent field. Suppose that initial data for ψ are specified on some spacelike hypersurface S. We shall say that ψ has k "degrees of freedom" if k is the number of initial functions (i.e., functions of the three spacelike coordinates parametrizing S) which need to be specified on S in order to uniquely determine inside $D^+(S)$ the solution to the field equations satisfied by ψ [18]. The number k depends on the type of field, and also possibly on the gauge condition used if there is gauge freedom. For example, for a scalar field $k=2$, because one needs to specify both ψ and $\dot{\psi}$ on S. For the gravitational field, it is well known that there are $2\times 2=4$ *inherent* degrees of freedom. The *actual* number k, however, is 4 plus the number of unfixed gauge degrees of freedom, which depends on the specific gauge conditions used. In the gauge we use we find that $k=8$ (see below).

We shall say that a class of solutions to the field equations is *functionally generic*, if this class depends on k arbitrary functions of three independent variables [19]. This concept of genericity is basically the same as that used by BKL [7]. The motivation behind this definition is obvious: Suppose that a given particular solution admits some specific feature (e.g., a singularity of some type). Obviously, in order for this feature to be stable to small (but generic) perturbations in the initial data, it is necessary that the class of solutions satisfying this feature should depend on k arbitrary functions. Functional genericity is thus a necessary condition for stability, and is also necessary in order that there be an open set in the space of solutions with the desired feature, in any reasonable topology on the space of solutions [20].

As we mentioned above, our result is a mathematical demonstration of the existence of a functionally generic null weak singularity. More specifically, we prove that there exists a class of solutions (M,g) to the vacuum Einstein equations, which all admit a weak curvature singularity on a null hypersurface, and which depend on $k=8$ (see below) arbitrary analytic functions of three independent variables. (In Ref. [17] we shall give a more precise formulation of this statement.) The singularities may also be characterized by the fact that the manifold may be extended through the null surface to an analytic manifold (M',g') where the metric g' is analytic everywhere except on the null surface where it is only C^0. Our construction is *local* in the sense that the manifolds we construct are extendible (in directions away from the null singularity); roughly speaking they can be thought of as open regions in a more complete spacetime, part of whose boundary consists of the singular null hypersurface. We do *not* prove that null weak singularities arise in the maximal Cauchy evolution of any asymptotically flat, smooth initial data set. The spacetimes we construct are of the form $D^+(\Sigma)$, where Σ is an open region in an analytic initial data set. The curvature singularity is already present on the boundary of Σ in the initial data, in the sense that curvature invariants blow up along incomplete geodesics. We emphasize that we do *not* view Σ as a physically acceptable initial hypersurface; rather, the initial hypersurface Σ is merely a mathematical tool that we use to construct and parametrize the desired class of vacuum solutions.

We shall first demonstrate the main idea behind our mathematical construction by applying it to a simpler problem — a scalar field. Consider, as an example, a real scalar field ϕ in flat spacetime, satisfying the nonlinear field equation

$$\phi_{;\alpha}^{;\alpha}=V(\phi), \tag{1}$$

where $V(\phi)$ is some nonlinear analytic function. (We add this nonlinear piece in order to obtain a closer analogy with the nonlinear gravitational case.) In order to show that this field admits a functionally generic null singularity we proceed as follows: Let x,y,u,v be the standard, double-null, Minkowski coordinates (i.e., such that $ds^2=-4\,du\,dv+dx^2+dy^2$). Equation (1) reads

$$\phi_{,uv}=\phi_{,a}^{;a}-V(\phi) \tag{2}$$

where here and below the indices $a,b,\ldots$ run over the coordinates x and y. We now define

$$w\equiv v^{1/n} \tag{3}$$

for some odd integer $n\geqslant 3$. We also define

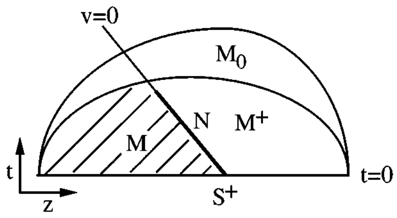

FIG. 1. Spacetime diagram in $z-t$ coordinates, illustrating the mathematical construction used. Our final spacetime (M,g) consists of the shaded region.

$$t \equiv w+u, \quad z \equiv w-u. \tag{4}$$

Reexpressing the field equation (2) in terms of t and z we obtain

$$\phi_{,tt}=\phi_{,zz}+n[(z+t)/2]^{n-1}[\phi^{;a}_{,a}-V(\phi)]. \tag{5}$$

Let M^0 denote some neighborhood of the origin $(x=y=z=t=0)$ with compact closure, and let S^+ be the intersection of the hypersurface t=0 with M^0 (see Fig. 1). Let $f_1(x,y,z)$ and $f_2(x,y,z)$ be two analytic functions of their arguments, defined on S^+. For any such pair of functions, there exists a neighborhood $M^+ \subseteq M^0$ of S^+ and a unique analytic solution $\phi(x,y,z,t)$ to the field equation (5) in M^+, such that on S^+, $\phi=f_1$ and $\phi_{,t}=f_2$. This follows directly from the Cauchy-Kowalewski theorem [21], in view of the form of Eq. (5). Let us denote the intersection of M^+ with the null hypersurface $v=0$ by N^+. Recall that N^+ includes a neighborhood of the origin in the hypersurface $v=0$.

Returning now to the original independent variables (u,v) we find that $\phi(x,y,u,v)$ is continuous throughout M^+. We now focus attention on the section $v<0$, $t \geqslant 0$ of M^+, which we denote by M. Since the transformation from (z,t) to (u,v) is analytic as long as $v \neq 0$, we find that $\phi(x,y,u,v)$ is analytic throughout M. However, ϕ will generally fail to be smooth at $v=0$: $\phi_{,v}=(1/n)v^{1/n-1}\phi_{,w}$ will diverge at $v=0$ as long as $\phi_{,w} \neq 0$ there. We assume that at the origin $\partial f_1/\partial z \neq \pm f_2$. This ensures that at least in some neighborhood of the origin, both $\phi_{,w}$ and $\phi_{,u}$ are nonzero. Let N be the intersection of that neighborhood with the section $t \geqslant 0$ of N^+. We find that $\phi_{,v}$ diverges on N. Moreover, the invariant $\phi_{,\alpha}\phi^{,\alpha}$ diverges on N too [it is dominated by $(1/n)v^{-1+1/n}\phi_{,u}\phi_{,w}$]. N is thus a singular null hypersurface.

We conclude that there exists a class of solutions to Eq. (1), which depends on two analytic functions of (x,y,z) (f_1 and f_2) that can be chosen arbitrarily (apart from the above inequality), and which contains a singularity on a null hypersurface. In other words, the scalar field admits a functionally generic null singularity. (Note that ϕ has a well-defined limit on the singular hypersurface; this is the scalar-field analogue of the notion of weak singularity.)

We turn now to generalize this construction to the gravitational field. As before, our coordinates are denoted (x,y,u,v). We adopt the gauge

$$g_{ux}=g_{uy}=g_{uu}=g_{vv}=0, \tag{6}$$

which in turn implies that $g^{vv}=0$. This ensures that the coordinate v is null (that is, the hypersurfaces v = const are null). There are six nontrivial metric functions, which we denote by g_i $(i=1,\dots,6)$, where here and below the indices $i,j,\dots$ represent the six pairs of spacetime coordinates (xx,xy,yy,vx,vy,uv).

In this gauge, the number k of arbitrary functions in a general solution is $k=8$. This can be seen as follows. Define the new variables $T \equiv v+u$, $Z \equiv v-u$. Then to determine a solution of the evolution equations, 12 initial functions need to be specified on the spacelike hypersurface $T=$const, namely, $g_i(x,y,Z)$ and $g_{i,T}(x,y,Z)$, $1 \leqslant i \leqslant 6$. However, these 12 functions must satisfy 4 constraint equations, as is always the case in general relativity, so that the number of independently specifiable functions is $k=8$. This conclusion can also be reached by adding the conventional number of intrinsic degrees of freedom of the vacuum gravitational field $(2\times 2=4)$ to the number of unfixed gauge degrees of freedom in the gauge (6), which we show in Ref. [17] to be 4.

We shall now outline the generalization of the above scalar-field construction to the gravitational field. First, one writes the Einstein equations $R_{\alpha\beta}=0$ in the gauge (6). These equations can be naturally divided into six evolution equations and four constraint equations. At this stage we focus attention on the evolution equations, which can be taken to be $R_i=0$. Next, we define w, t, and z as before [Eqs. (3) and (4)], and transform the field equations from the independent variables (u,v) to (z,t). [To avoid confusion we emphasize that what we are doing here is *not* a coordinate transformation: it is just a change of independent variables in the differential equations $R_{\alpha\beta}=0$; thus, the unknowns in Eq. (7) below are still the six metric functions g_i, which correspond to the coordinates (x,y,u,v).] By taking certain linear combinations of the equations $R_i=0$, it is possible to rewrite the evolution equations in the schematic form

$$g_{i,tt}=f_i(g_j,g_{j,t},g_{j,A},g_{j,AB},g_{j,At},z,t). \tag{7}$$

Here, the indices A,B run over the "spatial" variables x,y,z. If we impose certain inequalities on the initial data [which ensure that in the region of interest $\det(g) \cong -1$], then the functions f_i are analytic in all their arguments. [The gauge conditions (6) are crucial in deriving Eq. (7).]

We now consider the evolution of initial data under the system (7). As before, we take the initial hypersurface to be $t=0$. Equation (7) requires 12 initial functions to be specified on this hypersurface: the six functions $h_i(x,y,z) \equiv g_i(x,y,z;\ t=0)$ and the six functions $p_i(x,y,z) \equiv g_{i,t}(x,y,z;\ t=0)$. The form of Eq. (7) is suitable for an application of the Cauchy-Kowalewski theorem. Thus, defining S^+, M^+, and M as before, and following the arguments above, we arrive at the following conclusion: For any choice of the above 12 analytic functions $h_i(x,y,z)$ and $p_i(x,y,z)$ on the section S^+ of $t=0$ (subject to certain inequalities), there exists an analytic solution $g_i(x,y,z,t)$ to Eq. (7) in M. Again, returning from the variables (z,t) to the original independent variables (u,v), we find that the metric functions $g_i(x,y,u,v)$ are continuous throughout M^+ (and in particular at $v=0$) and, moreover, are analytic throughout M. However, at the hypersurface $v=0$, $g_{i,v}$ typically diverge like $v^{-1+1/n}$. As a consequence, the Riemann components

R_{avbv} generically diverge there [17]. Moreover, it can be shown that the scalar $K \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ also generically diverges at $v=0$ (like $v^{-2+1/n}$). However, it is easy to check directly that the singularity is weak. Thus, focusing attention on the physical region M, we find that the solutions constructed in that way are absolutely regular inside the region M, but develop a null, weak, scalar-curvature singularity on the portion $v=0$ of its boundary.

The 12 initial functions $h_i(x,y,z)$ and $p_i(x,y,z)$ are subject to four constraint equations. It is therefore natural to expect that 8 of these 12 initial functions can be chosen arbitrarily. This is not trivial to prove mathematically, however, especially because the constraint equations (expressed in the variables x,y,z) are somewhat pathological at $z=0$. After some effort we found a mathematical construction which proves the above statement. More specifically, in our mathematical scheme one is free to choose the six $h_i(x,y,z)$, $p_{xy}(x,y,z)$, and one other function $p(x,y,z)$. We can then show (using the Cauchy-Kowalewski theorem) the existence of a solution of the constraint equations (in a neighborhood of $z=0$). The remaining initial functions $p_i(x,y,z)$ are then determined from that solution. The above eight analytic functions can be chosen arbitrarily, up to some inequalities.

To summarize, our mathematical construction shows the existence of a class of (local) solutions to the vacuum Einstein equations, which contain a weak scalar-curvature singularity at the null hypersurface $v=0$, and which depends on $k=8$ analytic functions of (x,y,z). Our construction therefore demonstrates the existence of a functionally generic null, weak, scalar-curvature singularity.

The main limitation of our construction is its restriction to analytic initial functions. We believe that this is merely a technical limitation of the mathematical theorems used in our proof, and the same physical situation (a null weak singularity) will evolve even if the initial functions on S^+ are not analytic (provided they are sufficiently smooth for $v<0$). At any rate, it is worthwhile to compare the mathematical status of our generic null weak singularity to that of the BKL singularity. To the best of the authors' knowledge, the existence of even a single inhomogeneous singular vacuum solution of the BKL type has not yet been proved mathematically, let alone the generality of this class of singular solutions. On the other hand we have demonstrated rigorously the existence of a huge class of exact solutions containing null weak singularities.

Our results have a simple intuitive interpretation: In *linear* hyperbolic systems, it is well known that weak discontinuities of various types can freely propagate along characteristic lines. Our construction demonstrates that Einstein's equations, despite their nonlinearity, also behave in this way (at least with respect to the type of weak discontinuity considered here). This is perhaps contrary to what was sometimes thought in the past, but is not really surprising, because, after all, Einstein's equations are quasilinear. Thus, what we have shown is the *local* consistency of null weak singularities with Einstein's equations, despite the nonlinearity of the latter. The important issue of the *onset* of the singularity from regular, asymptotically flat, initial data (e.g., in gravitational collapse) still remains open; this issue is addressed (indirectly) by the nonlinear perturbation analysis of Ref. [10], but the onset still lacks a rigorous mathematical proof.

Finally, it should be pointed out that the inner-horizon singularity that is suggested by perturbation analyses in a realistic rotating or charged black hole (see, e.g., Ref. [10]), is qualitatively similar to the singularity constructed here, in that it is null and weak. There are some important differences, however. The main difference is that the structure of the inner-horizon singularity is analogous to what would have been obtained from our construction if we had set $w=1/\ln|v|$. Our method of proof does not generalize straightforwardly to this case, however, because v is no longer analytic in w at $w=0$ (though it is still C^∞) [22]. We hope to discuss the analytic features of this more realistic null weak singularity elsewhere.

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[18] This is a slight abuse of the more usual terminology, according

to which the number of degrees of freedom is said to be $k/2$, rather than k.

[19] Strictly speaking, in order for this definition to be useful, the nature of the dependence of the solutions on the k functions must be restricted in some (very natural) ways; see [17] for further discussion.

[20] See [17] for more detailed discussions of genericity, stability, and topologies on the space of solutions.

[21] See, e.g., Theorem 10.1.1. in R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[22] If the perturbation is not axially symmetric, the singularity structure is even more complicated, as the metric perturbation also oscillates in $\ln|v|$; see [10] (and recall that the coordinate analogous to our v is denoted V there).