Superfield description of the FRW universe

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An alternative procedure to construct local supersymmetric quantum cosmological models is presented. This is performed by introducing a superfield formulation and is applied here to the FRW model. It has the advantage of being more simple than models proposed based on full supergravity and gives, by means of this local symmetry procedure, in a direct manner, the corresponding fermionic partners. It also permits the inclusion of matter in a systematic way.

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In the generic case we have been unable to manage the canonical quantization of the full equations of general relativity; the minisuperspace approximation has been used to find results in the hope that they would illustrate the behavior of the general theory. The Bianchi cosmologies are the prime example. As is well known, the equation that governs the quantum behavior of these models is the Wheeler-DeWitt equation, which results in a quadratic Hamiltonian leading to an equation of the Klein-Gordon-type. The recent introduction of supersymmetric minisuperspace models has led to the definition and study of linear "square root" equations defining the quantum evolution of the Universe. To achieve these Dirac-type equations, one can make use of the fact that supergravity provides a natural square root of gravity [1–8] or supersymmetrize the models [9,10]. In the last procedure the Grassmann quantities are not in a clear manner the supersymmetric partners of the cosmological bosonic variable. In this paper we propose a different approach based on the superfield construction of the action which is invariant under $n=2$ ($n=1$ in the complex calculation) local supersymmetric transformations with a U(1) internal subgroup. This local symmetry procedure will provide in a systematic way the corresponding fermionic fields. This is here performed for the closed Friedmann-Robertson-Walker (FRW) models. Our approach is more simple than the use of full supergravity, and permits the inclusion of matter in a systematic way.

We begin by considering the homogeneous and isotropic metric defined by

$$ds^2=-N^2(t)dt^2+R^2(t)d\Omega_3^2, \tag{1}$$

where $d\Omega_3^2$ is the spatial FRW standard metric over three-space. The lapse function N and the scale factor R depend on the time parameter t.

Then, the FRW model for gravity, represented by this scale factor, is defined by the action

$$S=\int\left(-\frac{R(\dot{R})^2}{2N}+\frac{1}{2}kNR\right)dt, \tag{2}$$

where $k=1,0,-1$ stands for spherical, plane or hyperspherical three-space and $\dot{R}=dR/dt$. In the action (2) the cosmological term could be included. This one is invariant under reparametrization of $t[t\to t+a(t)]$, if the transformations of R,N are defined as $\delta R=a\dot{R}$, $\delta N=(aN)^{\cdot}$. The variations with respect to R and N lead to the classical equations of motion for the scale factor R and the constraint, which generates the local reparametrization of R and N. The constraint leads to the standard Wheeler-DeWitt equation in quantum cosmology.

Action (2) gives a simple one-dimensional model for the somehow interacting "matter" field $R(t)$ with the "gravity" field $N(t)$, defined by the one-dimensional metric $N^2(t)$. We will proceed with the superfield formulation of this action. For this purpose we need to generalize the local time transformations $t\to(t,\eta,\bar{\eta})$ in the following way:

$$\delta t=a(t)+i\eta\beta(t)+i\bar{\eta}\bar{\beta}(t),$$

$$\delta\eta=\bar{\beta}(t)+\left(\frac{\dot{a}(t)}{2}+ib(t)\right)\eta+i\dot{\bar{\beta}}\eta\bar{\eta}, \tag{3}$$

where η is a complex odd parameter (η odd "time" coordinates), $\beta(t)$ is the Grassmann complex parameter of local "small" $n=2$ SUSY transformation, and $b(t)$ is the parameter of local U(1) rotations of the complex η. It is well known that these supersymmetric transformations applied to the relativistic particle model lead to the description of spinning particles [11], superparticles [12], and spinning superparticles [13].

The superfield generalization of action (2), which is invariant under the transformation (3), has the form

$$S=\int\left[-\frac{1}{2}\mathbb{N}^{-1}\mathbb{R}\bar{D}_{\eta}\mathbb{R}D_{\eta}\mathbb{R}+\frac{\sqrt{k}}{2}\mathbb{R}^2\right]d\,t\,d\,\eta\,d\,\bar{\eta}, \tag{4}$$

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where $D_\eta=\partial/\partial\eta+i\bar{\eta}(\partial/\partial t)$ and $\bar{D}_\eta=-\partial/\partial\bar{\eta}-i\eta(\partial/\partial t)[\{D_\eta,\bar{D}_\eta\}=-2i(\partial/\partial t)]$ are the supercovariant derivatives of the global "small" supersymmetry of the generalized parameter corresponding to t. The local supercovariant derivatives have the form $\tilde{D}_\eta=\mathbb{N}^{-1/2}D_\eta$ and $\tilde{\bar{D}}_\eta=\mathbb{N}^{-1/2}\bar{D}_\eta$.

Note that the Ber$E_A^{\ B}$, as well as the super-Jacobian of the generalized time transformation (3), equal to Ber$E_B^{\ A}=(\mathbb{N}^{1/2}\mathbb{N}^{1/2})^{-1}\mathbb{N}=1$, is then omitted in the action (4).

In the action (4) $\mathbb{N}(t,\eta,\bar{\eta})$ is a real gravity superfield and has the form

$$\mathbb{N}(t,\eta,\bar{\eta})=N(t)+i\eta\bar{\psi}(t)+i\bar{\eta}\bar{\psi}(t)+\eta\bar{\eta}V(t), \tag{5}$$

and $\mathbb{R}(t,\eta,\bar{\eta})$ is a real "matter" superfield, which has the form

$$\mathbb{R}(t,\eta,\bar{\eta})=R(t)+i\eta\lambda(t)+i\bar{\eta}\bar{\lambda}(t)+\eta\bar{\eta}F(t). \tag{6}$$

The components of the superfield $\mathbb{N}(t,\eta,\eta)$ are gauge fields of the one-dimensional $n=2$ (real calculation) extended supergravity. $N(t)$ is einbein, $\psi(t),\psi(t)$ are the complex gravitino, and $V(t)$ is U(1) gauge field.

The supersymmetry transformation law for components of superfields $\mathbb{N}(t)$, $\mathbb{R}(t)$ follows from transformation laws (3). For example, the transformation of components N, ψ, and V of the supergravity multiplet has the form

$$\delta N=(aN)^{\cdot}, \quad \delta\psi=(a\psi)^{\cdot}, \quad \delta V=(aV)^{\cdot},$$

$$\delta N=i(\bar{\beta}\psi+\beta\bar{\psi}), \quad \delta\psi=\dot{\beta}+\bar{\beta}V, \quad \delta V=0,$$

$$\delta N=0, \quad \delta\psi=-ib\psi, \quad \delta V=\dot{b}.$$

The component $F(t)$ of the superfield $\mathbb{R}(t)$ is an auxiliary degree of freedom (nondynamical variable), and $\lambda(t),\lambda(t)$ are fermionic partners of the scale factor $R(t)$.

After integrating over the Grassmann coordinates $\eta,\bar{\eta}$ and making the field redefinitions $\psi\to\sqrt{N}\psi$, $\lambda\to\sqrt{(N/R)}\lambda$, and $V\to(1/N)(V+2\psi\psi/N)$, we obtain the action in its components:

$$S=\int\Bigg\{-\frac{1}{2}\Bigg[\frac{R(\dot{R})^2}{N}-2i\bar{\lambda}\dot{\lambda}+\frac{R}{N}F^2-\frac{\bar{\lambda}\lambda}{R}F-\frac{i\sqrt{R}}{N}$$
$$\times\dot{R}(\bar{\psi}\lambda+\psi\bar{\lambda})+\frac{\sqrt{R}}{N}(\bar{\psi}\lambda-\psi\bar{\lambda})F-\bar{\lambda}\lambda V\Bigg]$$
$$+\sqrt{k}\left(RF-\frac{N}{R}\bar{\lambda}\lambda\right)\Bigg\}dt. \tag{7}$$

The auxiliary field $F(t)$ may be determined from the appropriate equations of motion, and after substituting them again into the action we get

$$S'=\int\Bigg[-\frac{1}{2}\frac{R(\dot{R})^2}{N}+i\bar{\lambda}\dot{\lambda}+\frac{kN}{2}R-\frac{\sqrt{k}}{2}\frac{N}{R}\bar{\lambda}\lambda+\frac{1}{2}\bar{\lambda}\lambda V$$
$$-\frac{1}{4N}\bar{\psi}\psi\bar{\lambda}\lambda+\frac{i}{2}\frac{\sqrt{R}}{N}\dot{R}(\bar{\psi}\lambda+\psi\bar{\lambda})$$
$$-\frac{\sqrt{k}}{2}\sqrt{R}(\bar{\psi}\lambda-\psi\bar{\lambda})\Bigg]dt. \tag{8}$$

In this action $N(t),\psi(t),\bar{\psi}(t)$, and $V(t)$ are Lagrange multipliers.

Now we consider the Hamiltonian analysis of this system. The momentum π_R conjugate to R is given by

$$\pi_R=\frac{\partial L}{\partial\dot{R}}=\frac{i}{2}\frac{\sqrt{R}}{N}(\bar{\psi}\lambda+\psi\bar{\lambda})-\frac{R\dot{R}}{N}, \tag{9}$$

with respect to the canonical Poisson brackets $\{R,\pi_R\}=1$, and for the momenta conjugate to the Grassmann dynamical variables $\lambda,\bar{\lambda}$ we have

$$\pi_\lambda=\frac{\partial L}{\partial\dot{\lambda}}=-\frac{i}{2}\bar{\lambda}, \quad \pi_{\bar{\lambda}}=\frac{\partial L}{\partial\dot{\bar{\lambda}}}=-\frac{i}{2}\lambda. \tag{10}$$

Because of the definitions we have the odd second-class constraints:

$$\Pi_{\bar{\lambda}}\equiv\pi_{\bar{\lambda}}+\frac{i}{2}\lambda=0, \quad \Pi_\lambda\equiv\pi_\lambda+\frac{i}{2}\bar{\lambda}=0, \tag{11}$$

leading to the Dirac brackets for dynamical variables:

$$\{\lambda,\bar{\lambda}\}^*=-i,\{R,\pi_R\}^*=\{R,\pi_R\}=1. \tag{12}$$

After quantization of the Dirac brackets (12) we get the commutation-anticommutation relations

$$[R,\pi_R]=i,\{\lambda,\bar{\lambda}\}=1. \tag{13}$$

From the action (8) one derives the first-class constraints varying N, ψ, ψ, and V, respectively. We obtain

$$\tilde{H}\equiv H-\frac{1}{2N^2}\bar{\psi}\psi\bar{\lambda}\lambda=0, \quad \tilde{S}=0, \quad \tilde{\bar{S}}=0, \quad F=0, \tag{14}$$

where

$$H=-\frac{1}{2}\frac{\pi_R^2}{R}-\frac{kR}{2}+\frac{\sqrt{k}}{2R}\bar{\lambda}\lambda, \tag{15}$$

$$\tilde{S}=\left(\frac{\pi_R}{\sqrt{R}}-i\sqrt{k}\sqrt{R}\right)\lambda-\frac{i}{N}\psi\bar{\lambda}\lambda, \tag{16}$$

$$\tilde{\bar{S}}=\left(\frac{\pi_R}{\sqrt{R}}+i\sqrt{k}\sqrt{R}\right)\bar{\lambda}+\frac{i}{N}\bar{\psi}\bar{\lambda}\lambda, \tag{17}$$

$$F=\bar{\lambda}\lambda. \tag{18}$$

The classical constraints $\tilde{H},\tilde{\bar{S}},\tilde{S}$, and F form a closed algebra under Dirac brackets (12). The constraints (15), (16), (17), and (18) follow from the invariant action (8) under the "small" local supersymmetric (SUSY) transformation (3) [the parameters $a(t),\beta(t),\beta(t)$, and $b(t)$ are respectively generated by these first-class constraints].

The total Hamiltonian is $H_T=\dot{R}\pi_R+\dot{\bar{\lambda}}\pi_{\bar{\lambda}}+\dot{\lambda}\pi_\lambda-L$, and has the form

$$H_T=-\frac{1}{2}N\frac{\pi_R^2}{R}-\frac{kNR}{2}+\frac{\sqrt{k}}{2R}N\bar{\lambda}\lambda-\frac{1}{2}\bar{\lambda}\lambda V+\frac{1}{2N}\bar{\psi}\psi\bar{\lambda}\lambda$$
$$+\frac{i}{2}\frac{\pi_R}{\sqrt{R}}(\bar{\psi}\lambda+\psi\bar{\lambda})+\frac{\sqrt{k}}{2}\sqrt{R}(\bar{\psi}\lambda-\psi\bar{\lambda}), \tag{19}$$

and is given by

$$H_T=N\tilde{H}+\frac{i\bar{\psi}}{2}\tilde{S}+\frac{i\psi}{2}\bar{\tilde{S}}+F\left(-\frac{\bar{V}}{2}\right). \tag{20}$$

In order to give the canonical form of the theory, we replace $R(t)$ by $x(t)=\frac{2}{3}R^{3/2}(t)$; then the kinetic term $R(\dot{R})^2/2$ in the action (8) leads to $\dot{x}^2/2$.

In this case the Hamiltonian (15) takes the form

$$H=-\frac{\pi_x^2}{2}-\frac{1}{2}[W'(x)]^2+W''(x)\bar{\lambda}\lambda,$$

which is the well-known structure of the SUSY classical Hamiltonian in supersymmetric quantum mechanics with superpotential $W(x)=3^{4/3}\sqrt{k}x^{4/3}/2^{7/3}$. We observe that the minus signs in the first and second terms of the Hamiltonian arise as a consequence of pure gravity field, and the third term is a consequence of the SUSY and is positively defined.

In the quantum theory all the dynamical variables become operators with the commutation rule (13), and first-class constraints are imposed on states vectors. In our case $\lambda,\bar{\lambda}$ can be realized either by raising and lowering operators in accordance with their quantum anticommutation relation $\{\lambda,\bar{\lambda}\}=1$ or by matrices $\tau_{(+)}$ and $\tau_{(-)}$.

For the quantum generators $\tilde{H},\tilde{S},\bar{\tilde{S}},F$ we obtain the closed superalgebra

$$\{\tilde{S},\bar{\tilde{S}}\}=-2\tilde{H}-\frac{i}{N}\bar{\psi}\tilde{S}-\frac{i}{N}\psi\bar{\tilde{S}}+\frac{1}{N^2}\bar{\psi}\psi F,$$
$$\{\tilde{S},\tilde{S}\}=\frac{2i}{N}\psi\tilde{S},\quad \{\bar{\tilde{S}},\bar{\tilde{S}}\}=\frac{2i}{N}\bar{\psi}\bar{\tilde{S}},$$
$$[F,\tilde{S}]=-\tilde{S}-\frac{i}{2}\psi F,\quad [F,\bar{\tilde{S}}]=\bar{\tilde{S}}-\frac{i}{N}\bar{\psi}F, \tag{21}$$
$$[\tilde{H},\tilde{S}]=\frac{\bar{\psi}\psi}{2N^2}\tilde{S},\quad [\tilde{H},\bar{\tilde{S}}]=-\frac{\bar{\psi}\psi}{2N^2}\bar{\tilde{S}},$$
$$[F,\tilde{H}]=0.$$

We have then shown that the action (8) is invariant under the "small" SUSY transformations (3); then we can choose the gauge $\psi=0$, $N=1$. In this case the generators $\tilde{H},\tilde{S},\bar{\tilde{S}}$, and F lead to the form

$$H=-\frac{1}{2}\frac{\pi_R^2}{R}-\frac{kR}{2}+\frac{\sqrt{k}}{2R}\bar{\lambda}\lambda,$$
$$S=\left(\frac{\pi_R}{\sqrt{R}}-i\sqrt{k}\sqrt{R}\right)\lambda, \tag{22}$$
$$\bar{S}=\left(\frac{\pi_R}{R}+i\sqrt{k}\sqrt{R}\right)\bar{\lambda},$$

where H is the Hamiltonian, S is the single complex supersymmetric charge of $n=2$ supersymmetric quantum mechanics with the algebra

$$\{S,\bar{S}\}=-2H,\quad [S,H]=0,\quad [F,S]=-S,$$
$$S^2=\bar{S}^2=0,\quad [\bar{S},H]=0,\quad [F,\bar{S}]=\bar{S}, \tag{23}$$

which follows from (21) in the gauge $\psi=0$, and F is the fermion number operator. Generators (22) were found in the works (9) while looking for the hidden symmetry in cosmological models, but the superalgebra (23) of these generators corresponds to a global symmetry. The local supersymmetry of the subgroup of time reparametrization comes out from generators (15)–(18). So, the action (8) has the form of the localized version of $N=2$ supersymmetric quantum mechanics. The generators (22) satisfy the superalgebra (23). The $k=-1$ case deserves some comments. The third term in the Hamiltonian makes it non-Hermitian; this is the isotropic model corresponding to the Bianchi type V model. It is also well known [14] that, already in standard quantum cosmology, the quantization of the class B Bianchi models faces some problems. It is, however, at this stage not clear if the non-Hermiticity of the Hamiltonian (22) for $k=-1$ is connected with the difficulties found in the standard quantum cosmology. In any case this model deserves further study.

We suggest that the correct reduction from $d=4$ of different versions of $N=1$ supergravity must lead to local versions of the $n=2$ and $n=4$ supersymmetric quantum mechanics for cosmological models. Perhaps the consequent reduction, leading to supersymmetric quantum mechanics from $d=4$ supergravity, may be realized by using Lorentz harmonics.

Cosmological models interacting with matter may also be constructed by the procedure followed in this paper.

An interesting point is to try to find out if the "small" SUSY in the suggested cosmological models is a remnant of the "big" SUSY in supergravity theories, or if it is necessary to construct models possessing both types of local SUSY's in order to construct consequent quantum cosmology. In this case the problem of the spontaneous breaking of SUSY in the SUSY cosmological models and its relation with spontaneous breaking of SUSY in supergravity theories should also be studied.

The development of this approach in SUSY cosmological models will be discussed in the near future for all the Bianchi type models. In addition, we will discuss the $n=4$ construction in the supersymmetric cosmological models and its relation with $n=4$ Witten's mechanics.

On the one hand, our superfield approach provides in general an alternative way to construct supersymmetric models with a reduced number of degrees of freedom and is in particular more simple for quantum cosmology than to use the full $N=1$ supergravity applied to these reduced models. Moreover, the method proposed here provides a systematic way to include matter in any desired model. A generalization of this procedure to all Bianchi models and the inclusion of matter will be reported elsewhere.

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