Unstable states in QED of strong magnetic fields

M. Kachelriess
INFN, Laboratori Nazionali del Gran Sasso, I-67010 Assergi (AQ), Italy
and Theoretische Physik I, Ruhr-Universität Bochum, D-44780 Bochum, Germany
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We question the use of stable asymptotic scattering states in QED of strong magnetic fields. To correctly describe excited Landau states and photons above the pair creation threshold the asymptotic fields are chosen as generalized Licht fields. In this way the off-shell behavior of unstable particles is automatically taken into account, and the resonant divergences that occur in scattering cross sections in the presence of a strong external magnetic field are avoided. While in a limiting case the conventional electron propagator with Breit-Wigner form is obtained, in this formalism it is also possible to calculate S-matrix elements with external unstable particles.

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I. INTRODUCTION

The discovery of neutron stars with extremely strong magnetic fields B up to 10^{13} G has given the impetus to numerous calculations of QED processes in which the magnetic field is taken into account exactly. A remarkable feature of these magnetic field strengths is that the cyclotron energy becomes of the order of the electron rest energy, and, consequently, the quantization of the electron states into *discrete* Landau levels becomes important. These QED processes were recalculated using conventional perturbation theory in the Furry picture: the free electron propagator and the free wave function of an electron were replaced by the exact Green's function and the exact solution of the Dirac equation with a homogenous, static magnetic field. Then the same Feynman rules were applied. In this way, most first- and second-order processes have been recalculated in the last 15 years (for a review cf. Ref. [1]). However, the results often show an unsatisfactory behavior. Because of strict energy conservation, first-order processes such as cyclotron absorption have δ-function-like decay widths or, like cyclotron emission, become infinite averaging over "reasonable" distribution functions. This is remedied by accounting for the finite lifetime of the *external* states; i.e., one replaces the δ function expressing energy conservation by a Lorentz curve in the decay width. In the case of second- or higher-order processes singularities arise due to on-shell intermediate states. Here, one accounts for the finite lifetime of the *intermediate* Landau states and replaces in the electron propagator Feynman's $i\varepsilon$ with $\frac{1}{2}i\Gamma_{N,\tau}$, where $\Gamma_{N,\tau}$ is the decay width of the electron state with Landau quantum number N and polarization τ.

Despite the use of the electron propagator with a complex mass, some processes still lead to divergent cross sections. The most prominent example is magnetic Compton scattering with an initial photon which is above the pair creation threshold. If the intermediate electron is in the stable Landau ground state, there is no decay width associated with it, and, consequently, the total cross section is divergent everywhere above the pair creation threshold [2]. Another problem arises if more than one particle is unstable. Then it is not obvious how the decay widths should be defined. Usually, for first-order processes the total decay width is assumed to be additive, i.e., to be the sum of the decay widths of the individual particles. In contrast, for second-order processes the decay width of every single virtual particle is chosen as its individual on-shell decay width. This seems to be arbitrary and shows the absence of a comprehensive strategy to treat the instability of electrons and photons in magnetic fields. Therefore, it is the purpose of this paper to formulate a well-defined perturbation theory for QED of strong magnetic fields where the finite lifetime of excited Landau states and photons above the pair creation threshold is automatically incorporated. To this end, we give up the concept of stable scattering states and instead introduce generalized Licht fields for the unstable particles. The energy of particles described by Licht fields is not fixed by an on-shell condition but is given by some spectral function. We do not attempt to calculate these spectral functions from first principles because they are well approximated for practical calculations by Lorentz curves [3]. The advantage of using Licht fields is that within this formalism it is possible to take into account consistently the instability of intermediate *and* external particles.

As an application we show, for the generic example of magnetic Compton scattering in which the initial photon is above the pair creation threshold, how the use of Licht fields eliminates resonant divergences of QED with strong magnetic fields.

II. UNSTABLE STATES AND PROPAGATORS

To see the underlying reasons for the unsatisfactory behavior of cross sections of QED with strong magnetic fields, we remind the reader of two failures of perturbation theory in the Furry picture for $B > 0$ using stable particle states and propagators.

(i) It is a highly distinctive feature of vacuum theory that the Hilbert space structure does not change in going from the free to the interacting theory. This remains true for $B > 0$ only if the electron self-energy and the vacuum polarization vanish on shell. But this is not the case due to the imaginary part of the self-energy indicating the decay of Landau states for $N > 0$ and of the vacuum polarization indicating the decay of a photon in an e^-e^+-pair above the pair creation threshold [4,5].

(ii) Fields that describe unstable states have a vanishing Lehmann-Symanzik-Zimmermann (LSZ) limit [6,7].

For us these results are the starting point for identifying the correct states and propagators for decaying electrons and photons in strong magnetic fields. There are two ways of introducing unstable particles in quantum field theory. Usually, the notation of a complex mass shell, $p_\mu p^\mu = m^2 - i\Gamma$, is used. This approach is easily applied for propagators yielding the typical Breit-Wigner shape for resonances in cross sections [8], but was recently also generalized to external lines [9]. Another method is the use of generalized Licht fields. Here, one replaces the on-shell condition $p_\mu p^\mu = m^2$ by an off-shell mass spectral density. The abandoning of the on-shell condition is justified by the time-energy uncertainty which forbids an unstable particle to have a fixed energy. The use of Licht fields is additionally motivated by the following two reasons: First, generalized Licht fields have nonvanishing LSZ limits. Second, in the case of propagators, the Licht field approach is the more general one and contains the complex mass shell method as a special case. Therefore, we follow the second approach in this work and, in the spirit of Refs. [6,7,10], introduce suitable generalized Licht fields for the unstable particles. However, the two different methods result in a different treatment of external lines. These differences will be discussed at the end of Sec. III.

For the electron we define the Licht field by

$$\Psi(x) = \sum_a \int_0^\infty dE[Z_{+,n}^{1/2}(E) b_a(E) \psi_a^{(+)}(\mathbf{x}) e^{-iEt} + Z_{-,n}^{1/2}(E) d_a^\dagger(E) \psi_a^{(-)}(\mathbf{x}) e^{+iEt}] \tag{1}$$

and $\bar\Psi = \Psi^\dagger \gamma^0$, where the anticommutation relations are the usual ones for $N = 0$; for $N > 0$,

$$\left\{b_a(E), b_{a'}^\dagger(E')\right\} = \left\{d_a(E), d_{a'}^\dagger(E')\right\} = \delta_{a,a'}\delta(E - E') \tag{2}$$

and the other anticommutators are zero. Here, $\psi_a^{(\lambda)}$ are the energy solutions of the Dirac equation in the presence of the external magnetic field $\mathbf{B} = B\mathbf{e}_z$, $a = \{N, \tau, p_y, p_z\}$ denotes the set of quantum numbers needed in order to completely characterize the solutions, and $\lambda = \pm$ distinguishes positive and negative energy solutions [11,12]. The energy E of the particle is smeared around the on-shell value $E_n = \sqrt{m^2 + 2NeB + p_z^2}$ due to the integration over the spectral functions $Z_{\lambda,n}^{1/2}(E)$. These functions are generalizations of the wave function renormalization constant Z of a stable field and labeled by that subset of quantum numbers $n = \{N, \tau\}$ which enters in the decay width of the unstable states. In the following, we will suppress the other quantum numbers p_y and p_z [13]. The dependence of $Z_{\lambda,n}^{1/2}$ on λ reflects the different time evolution of positive and negative energy solutions. Therefore, the difference between $Z_{+,n}^{1/2}$ and $Z_{-,n}^{1/2}$ should show up only as some kind of boundary condition. In order to get a charge symmetric theory, the condition $Z_{+,n}^{1/2*} = e^{i\phi} Z_{-,n}^{1/2}$, where ϕ is an arbitrary phase, follows.

A physical interpretation of Eq. (1) is that Ψ describes n-times different particles, i.e., every Landau state with distinct N and τ would be identified as a different particle. Excited states with $N > 0$ are unstable because of the interaction with the photon field. These particles can decay and $Z_{\lambda,n}^{1/2}(E)$ weights the contributions of creating an unstable "(N, τ) particle" with energy E and polarization τ. Only the ground state $n = (N = 0, \tau = -1)$ remains stable. To recover the usual on-shell energy relation and wave function of a stable electron, one has to set $Z_{\lambda,0,-1}^{1/2} = \delta(E - E_0)$. Then, the ground state has the usual LSZ limit $\psi_{0,-1} \to Z_2^{1/2}\psi_{0,-1}^{\text{out}}$ for $t \to \infty$, where Z_2 is the normal electron wave function renormalization *constant* and the limit, as in all of the following, should be understood in the weak operator topology.

In the case of the photon field with its continuous energy spectrum we adopt, in an analogous way, the Licht field

$$A_\mu(x) = \int_0^\infty ds' [Z_{+,r}^{1/2}(s') A_\mu^{(+)}(x, s') + Z_{-,r}^{1/2}(s') A_\mu^{(-)}(x, s')] , \tag{3}$$

where $A^{(+)}$ and $A^{(-)}$ are the positive and negative energy photon fields, respectively. Similarly to the case of the electron field, the functions $Z_{\lambda,r}^{1/2}(s')$ are labeled besides by λ by those quantum numbers on which the one-γ-pair production probability $\Gamma_r(s)$ depends: the energy perpendicular to the magnetic field $s = \omega \sin\theta$, and the polarization r of the photon [14]. The part of the photon field $A_\mu^{(\lambda)}(x)$ with energy below the pair creation threshold $\omega = 2m/\sin\theta$ is stable. Therefore, for $s < 2m$, the functions $Z_{\lambda,r}^{1/2}(s') = \delta(s - s')$ and the field has the usual LSZ limit $A_\mu^{(\lambda)}(x, s) \to Z_3^{1/2} A_\mu^{\text{out}(\lambda)}(x, s)$ for $t \to \infty$.

In contrast with the fields Ψ and A^μ describing unstable particles, the component fields $A^{(\lambda)}(x, s')$ and

$$\psi_n^{(+)}(x, E) = b_n(E) \psi_n^{(+)}(\mathbf{x}) e^{-iEt}, \tag{4}$$

$$\psi_n^{(-)}(x, E) = d_n^\dagger(E) \psi_n^{(-)}(\mathbf{x}) e^{+iEt}, \tag{5}$$

do have a nonvanishing LSZ limit [6,7]:

$$A_\mu^{(\lambda)}(x, s') \to A_\mu^{(\lambda)\text{out}}(x, s') \quad \text{for } t \to \infty \tag{6}$$

and

$$\psi_n^{(\lambda)}(x, E) \to \psi_n^{(\lambda)\text{out}}(x, E) \quad \text{for } t \to \infty . \tag{7}$$

Therefore, one is able to compute Green's functions with the Gell-Mann–Low or the LSZ reduction formula using the decomposed out fields. Expressed in terms of these, the interaction Hamiltonian H_I reads

$$H_I(t) = -e\int d^3x \sum_{n_1,n_2}\sum_{\lambda_1,\lambda_2,\lambda'}\int_0^\infty dE_1\int_0^\infty dE_2\int_0^\infty ds' Z^{1/2}_{\lambda_1,n_1}(E_1)\, Z^{1/2}_{\lambda_2,n_2}(E_2)\, Z^{1/2}_{\lambda',r}(s')$$
$$\times\, \overline{\psi}^{(\lambda_1)\mathrm{out}}_{n_1}(x,E_1)\gamma^\mu \psi^{(\lambda_2)\mathrm{out}}_{n_2}(x,E_2) A^{(\lambda')\mathrm{out}}_\mu(x,s')\,, \tag{8}$$

where we omit all counterterms. Since here we are only interested in first- and second-order processes, we do not take care of renormalization. But we want to mention that, since physical parameters should be chosen as directly observable quantities, the bare parameter m_0 should be expressed by the physical mass m only for the Landau ground state. In this way, the usual electron wave function renormalization constant Z_2 is fixed. However, for decaying states one should choose as a physical parameter instead of m some characteristic parameter of an unstable state, e.g., the decay width Γ_n.

Now, we are ready to derive diagrammatic perturbation theory. As a first step we compute the components of the electron propagator (omitting the index "out" from now on) for $N > 0$,

$$iS_F(x_1,E_1,n_1;x_2,E_2,n_2) = \left\langle 0\left|T\left(\psi^{(\lambda)}_{n_1}(x_1,E_1)\overline{\psi}^{(\lambda)}_{n_2}(x_2,E_2)\right)\right|0\right\rangle$$
$$= \delta(E_1-E_2)\delta_{n_1,n_2}\int_{-\infty}^{\infty}\frac{ds}{2\pi}\frac{1}{s-\lambda\,(E_1-i\varepsilon)}\,\psi^{(\lambda)}_{n_1}(\mathbf{x}_1)\overline{\psi}^{(\lambda)}_{n_2}(\mathbf{x}_2)\,e^{-is(t_1-t_2)}\,, \tag{9}$$

and read off the vertex as

$$ie\gamma^\mu Z^{1/2}_{\lambda_1,n_1}(E_1)Z^{1/2}_{\lambda_2,n_2}(E_2)Z^{1/2}_{\lambda',r}(s')\,. \tag{10}$$

Using the decomposed propagator, one has not only to integrate over all not fixed momenta, but also over the variables of the functions Z. However, we are mainly interested in Green's functions of Ψ. The total propagator is sandwiched between two vertices. One of the two integrations over E breaks down due to the δ function and one obtains

$$iS_F(x_1,x_2) = \langle 0|T\left(\Psi(x_1)\overline{\Psi}(x_2)\right)|0\rangle$$
$$= \sum_{n,\lambda}\int_0^\infty dE\, Z_n(E)\int_{-\infty}^{\infty}\frac{ds}{2\pi}\frac{1}{s-\lambda\,(E-i\varepsilon)}\,\psi^{(\lambda)}_n(\mathbf{x}_1)\overline{\psi}^{(\lambda)}_n(\mathbf{x}_2)\,e^{-is(t_1-t_2)}\,. \tag{11}$$

Here, we set the ill-defined $Z_{0,-1}(E)$ equal to $\delta(E-E_0)$ to obtain a compact expression and the vertex becomes the usual $ie\gamma^\mu$. The functions $Z_n(E)$ are abbreviations for $|Z^{1/2}_{\lambda,n}|^2$. Therefore, they are real and independent from λ. One should remember that the propagator obtained, although similar to the spectral representation of the full propagator in vacuum theory, is a bare one.

The derivation of the photon propagator is similar. However, there arises the additional difficulty that the usual spin projection operators do not work for off-shell states: a propagator for spin-s particles will generally contain particles with lower spin values $(s-1, s-2, \ldots, 0)$. But since Feynman diagrams with virtual photons do not play a prominent role in the astrophysical applications, we omit the derivation here [15].

III. SPECTRAL FUNCTIONS

In order to make the whole treatment consistent, the spectral functions $Z_n(E)$ and $Z_r(s)$ have to be—at least in principle—computable. A hard way is to use the fact that the $Z^{1/2}_{\lambda,n}(E)$ completely determine through the Eqs. (1) and (7) the normalization of the components of the field Ψ. The latter is fixed by the canonical anticommutation relations. Therefore one can use perturbation theory in the Heisenberg picture to calculate $Z^{1/2}_{\lambda,n}(E)$, and, similarly, $Z^{1/2}_{\lambda,r}(s)$ [7,10].

In practice, these calculations are nearly intractable and one will use a guess. The *Ansätze* according to conventional wisdom are Lorentzians:

$$Z^L_n(E) = \frac{\Gamma_n}{\pi\left((E-E_n)^2+\frac{1}{4}\Gamma_n^2\right)} \tag{12}$$

for the electron and

$$Z^L_r(s') = \frac{\Gamma_r(s)}{\pi\left((s-s')^2+\frac{1}{4}\Gamma_r^2(s')\right)} \tag{13}$$

for the photon, where we choose Γ_n and $\Gamma_r(s)$ to be the total decay width of the Landau level (N,τ) [12] and the one-γ-pair production probability $\Gamma_r(s)$ [14], respectively, calculated in conventional perturbation theory.

This ansatz reproduces the electron propagator with the normally used Breit-Wigner prescription $i\varepsilon \to \frac{1}{2}i\Gamma_n$ for $\Gamma_n \ll E_n$. In this case, after expanding the poles, the lower limit of integration can be extended from 0 to $-\infty$ producing two poles at $s-\lambda(E_n-\frac{1}{2}i\Gamma_n)$. According to Ref. [16], the Breit-Wigner approximation yields a result that is always consistent within the perturbation theoretical order of calculation. In particular, the author showed that for all B and N the deviations from the Breit-Wigner line shape are small. From our derivation follows the usual restriction $\Gamma_n \ll E_n$ for the validity of the approximation. Using the approximative formulas of Ref. [17] for Γ_n, one sees that the condition $\Gamma_n \ll E_n$ is indeed always satisfied.

However, from a more fundamental point of view, it is clear that the extension of the integration to negative E violates the spectral condition and is at the root of the violation of unitarity and causality. Although therefore the consistency of the approach is lost, it seems to us worthwhile to explore the consequences of this *Ansatz.* In contrast with the usual derivation of the propagator with Breit-Wigner shape, the derivation presented here gives a complete scheme that describes unstable particles as external as well as intermediate states. Furthermore, one can treat processes with more than one unstable particle without ambiguities. Therefore, in this section we do not attempt to calculate the spectral functions Z from first principles but restrict ourselves to the simpler task of investigating the consequences of the Breit-Wigner approximation in the Licht field approach.

Denoting the electron wave functions obtained in this approximation by $\psi^{(\lambda)}_{L,n}(x)$, we obtain

$$\psi^{(\lambda)}_{L,n}(x) = N\,\psi^{(\lambda)}_n(\mathbf{x})\,e^{-i\lambda E_n t - \frac{1}{2}\Gamma_n t}\theta(t) \tag{14}$$

and $\bar{\psi}^{(\lambda)}_{L,n} = \psi^{(\lambda)\dagger}_{L,n}\gamma^0$. Here, N is a normalization constant and we chose the signs in

$$Z^{1/2}_{\lambda,n} = \sqrt{\frac{\Gamma_n}{\pi}}\frac{i\lambda}{E - E_n + \frac{1}{2}i\lambda\Gamma_n} \tag{15}$$

according to the following two requirements: first, we demand that the wave functions do not vanish for $t > 0$ and, second, in the limit $\Gamma_n \to 0$ the phase of $\psi^{(\lambda)}_{L,n}$ has to coincide with the phase of $\psi^{(\lambda)}_n$. The first requirement leads automatically to decaying states, for both positive and negative energy solutions. The choice of nonvanishing wave functions for $t < 0$ results in states whose norm grows in time. Therefore, the choice between nonvanishing wave functions for $t > 0$ or $t < 0$ corresponds to the choice of the direction of the time arrow and has to be made by hand. As anticipated, the different form of $Z^{1/2}_{\lambda,n}$ for $\lambda = \pm$ is necessary to obtain the correct boundary condition for decaying states. Similarly, we obtain, for the photon wave functions,

$$A^{(+,r)}_{L,\mu}(x) = N\,(2\omega V)^{-1/2}\,\varepsilon^{(r)}_\mu\,e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}e^{-\frac{1}{2}\Gamma_r(\omega)t}\theta(t) \tag{16}$$

and $A^{(-,r)}_{L,\mu} = A^{(+,r)*}_{L,\mu}$.

Since the norm of the states varies with time, the correct normalization is not obvious. A reasonable prescription is the requirement that cross sections calculated with decaying states coincide in the limit $\Gamma \to 0$ with the same cross sections calculated in the usual formalism.

Formally, S-matrix elements with external unstable particles will be calculated in the usual way. But because of the θ functions, the time integration over vertices with external unstable particle goes effectively only from 0 to ∞. Thereby, no divergent time integrals will be caused by the real part of the exponentials of decaying states.

Finally, we want to compare our approach with that of a recent paper [9]. The authors propose the use of propagators with complexified energies and—this is the main difference—the use of external states

$$\psi^{(\lambda)}_{L,n}(x) = \psi^{(\lambda)}_n(\mathbf{x})\,e^{-i\lambda E_n t - \frac{1}{2}\lambda\Gamma_n t}, \tag{17}$$

$$\bar{\psi}^{(\lambda)}_{L,n}(x) = \bar{\psi}^{(\lambda)}_n(\mathbf{x})\,e^{+i\lambda E_n t + \frac{1}{2}\lambda\Gamma_n t} \tag{18}$$

for the electrons and similar ones for photons. In contrast to our wave functions, the norm of the negative energy solutions grows in time. Furthermore, these wave functions are valid for $t < 0$ as well as for $t > 0$. Therefore, the real part of the exponentials will lead to divergent time integrals. Consequently, no scattering amplitudes in the normal sense (i.e., for transitions between $t_{i,f} \to \pm\infty$) with external unstable states can be calculated in this approach. Instead, one calculates matrix elements of the time evolution operator $U(t_f, t_i)$ which are dependent on the time lapse $t_f - t_i$ between preparation of the initial state and measurement of the final state. The authors of Ref. [9] claim that this approach, which is the direct transcription of the Wigner-Weisskopf method of nonrelativistic quantum mechanics, eliminates all divergences of QED with strong magnetic fields. However, the main object of field theory, the S matrix, is not generally computable in this formalism. Moreover, the wave function $\bar{\psi}$ is not the Dirac conjugate spinor of ψ since the real part of the exponent changes sign. Consequently, charge symmetry is lost and the S-matrix elements lack crossing symmetry. Finally, a more practical objection seems to be important. Since the main application of QED of strong magnetic field is astrophysics, the usefulness of cross sections which are dependent on the time lapse between "preparation" and "measurement" of the states is restricted.

IV. APPLICATIONS

In this section, we want to illustrate some basic consequences of this formalism.

First, we consider a generic first-order process. The S-matrix element is given by

$$S_{fi} = \int dE_1 dE_2 ds'\, Z^{1/2}_1(E_1) Z^{1/2}_2(E_2) Z^{1/2}_r(s') S^{\text{conv}}_{fi}\,, \tag{19}$$

where S^{conv}_{fi} is the conventional S-matrix element but with off-shell energies. Choosing the spectral functions to be Lorentz curves, there is no difference between the approach presented in this work and conventional perturbation theory where the delta function expressing energy conservation is replaced *ad hoc* by a Lorentzian. Since the wave functions depend on Γ only through an exponential factor, the simple assumption that the total decay width is additive, is valid, i.e. (as used, e.g., in Ref. [12]),

$$\Gamma_{\text{tot}} = \Gamma_{n_1} + \Gamma_{n_2} + \Gamma_r(s)\,. \tag{20}$$

Hence, quantum correlations between different decaying states exist only for non-Lorentzian spectral functions.

Second, let us consider magnetic Compton scattering as a typical second-order process. The electron propagator coincides with the conventional one with complexified energy for $Z_n(E) = Z_n^L(E)$. Therefore, in the simplest case of $N_i = 0 \to N_f = 0$ Compton scattering where the energy of the photons is below $2m/\sin\theta$, we obtain the old, well-known result [2,18]. Otherwise, the instability of the initial and final particles is also incorporated in the S-matrix element. Assuming the functions $Z^{1/2}$ are well behaved, the integration over the off-shell energies will remove the remaining singularity of the electron propagator when the virtual electron is in the stable Landau ground state. In particular, the S-matrix element of magnetic Compton scattering is now finite in the case where the initial photon is above the pair creation threshold.

Now we want to make our argument quantitative. The S-matrix element of $N_i = 0 \to N_f = 0$ Compton scattering is given by

$$S_{fi} = (ie)^2 \int d^4x d^4x' \bar{\psi}_f^{(+)}(x)\gamma_\mu i S_F(x,x')\gamma_\nu \psi_i^{(+)}(x') \int ds_i' ds_f' Z_{+,r_i}^{1/2}(s_i') Z_{-,r_f}^{1/2}(s_f') \times \left(A_f^{\mu *}(x,s_f') A_i^\nu(x',s_i') + A_i^\mu(x,s_i') A_f^{\nu *}(x',s_f') \right) = S^{(1)} + S^{(2)} , \tag{21}$$

where i and f refer to initial and final states, while the the quantum numbers of the virtual electron will be marked by the subscript a. Using stable fields, the divergence for $s_i > 2m$ occurs in the exchange diagram $S^{(2)}$ when the virtual electron propagates as a positron in the Landau ground state, i.e., has the quantum numbers $N_a = 0$ and $\lambda_a = -1$ [2]. Since the space integrals remain unchanged and are finite, we only have to consider the time integrals of $S^{(2)}$ for $N_a = 0$ and $\lambda_a = 0$:

$$S^{(2)} \propto \int dt dt' e^{iE_f t} \left(e^{iE_a(t-t')} \int \frac{dz}{2\pi} e^{iz(t-t')} \right) e^{-iE_i t'} \times \int ds_i' ds_f' Z_{+,r_i}^{1/2}(s_i') Z_{-,r_f}^{1/2}(s_f') e^{-i\omega_i' t'} e^{i\omega_f' t} = S_t^{(2)} . \tag{22}$$

Here, E denotes the energy of the electrons and $\omega' = s' \sin\theta$ the (off-shell) energy of the photons. Performing the two time integrations and the integration over z, which comes from the θ function of the electron propagator, results in

$$S_t^{(2)} = \int ds_i' ds_f' \frac{Z_{+,r_i}^{1/2}(s_i') Z_{-,r_f}^{1/2}(s_f')}{-E_f - E_a + \omega_i' + i\varepsilon} 2\pi\delta(E_f + \omega_f' - E_i - \omega_i') . \tag{23}$$

The result of the integration over s_i' depends on the energy perpendicular to the magnetic field s_f' of the final photon:

$$S_t^{(2)} = \frac{2\pi Z_{+,r_i}^{1/2}(s_{i,1}')}{-E_i - E_a + \omega_f + i\varepsilon} \sin\theta_f \Big|_{s_f < 2m} + \int ds_f' \frac{2\pi Z_{+,r_i}^{1/2}(s_{i,2}') Z_{-,r_f}^{1/2}(s_f')}{-E_i - E_a + \omega_f' + i\varepsilon} \sin\theta_i \Big|_{s_f' \geq 2m} , \tag{24}$$

where we introduce $s_{i,1}' = (E_f + \omega_f - E_i)\sin\theta_i$, $s_{i,2}' = (E_f + \omega_f' - E_i)\sin\theta_i$ and ω_f denotes the on-shell energy of the final photon. In the second term of $S^{(2)}$, the integration over s_f' gives a finite result as long as a principal value integral of the integrand can be defined. Therefore, the only remaining dangerous part of $S^{(2)}$ is the first term. But for $s_f < 2m$ the denominator can never become zero because $E_i + E_a = m + \sqrt{m^2 + \omega_f^2 \cos^2\theta_f}$ is always greater than ω_f. Here, we assumed $p_{i,z} = 0$ without loss of generality and used momentum conservation parallel to the magnetic field.

In the more general case of Compton scattering when N_i and N_f are not restricted to be zero, this result remains valid. In this case, there are additionally integrations over the off-shell energies of the unstable electrons. As above, the necessary condition for a finite S-matrix element is that a principal value integral of the integrand can be defined.

This example illustrates well the connection between the instability of external particles and singularities of intermediate states: as soon as the energy of the initial photon is above the pair creation threshold, the virtual electron can become real, producing a divergent cross section. At the same time, however, the photons also become unstable. Taking this instability into account properly, one obtains well-behaved cross sections.

Finally, we want to comment on the behavior of magnetic Compton scattering if all external particles are stable. Then there is no decay width which could cure the resonance if the virtual electron is in the Landau ground state $N_a = 0$. Formally, the resonance energies $\omega_{\rm res}^{N_a}$ are given for all $N_a = 0, 1, \ldots$, by [2]

$$\omega_{\rm res}^{N_a} = \left[\left(m^2 + 2N_a eB\sin^2\theta_i\right)^{1/2} - m\right] / \sin^2\theta_i\,, \quad (25)$$

i.e., $\omega_{\rm res}^{N_a=0} = 0$. In the case of $N_i \neq 0 \to N_f = 0$ Compton scattering, the limit $\omega_i \to 0$ results in divergent cross sections [19]. In Ref. [20], these divergences were interpreted not as resonances but as infrared divergences. By contrast, in the case of $N_i = 0 \to N_f = 0$ scattering, the S-matrix element diverges like ω_i^{-1} while the cross section goes to a finite, constant value in the limit $\omega_i \to 0$ [19].

V. SUMMARY

We have presented a consistent method to describe the instability of excited Landau states and photons above the pair creation threshold in QED of strong magnetic fields. This approach consists in using Licht fields for unstable states and introduces additionally integrations over the off-shell energies of the unstable particles. We have shown for the generic example of Compton scattering where the energy of the photon is above the pair creation threshold how in this way the resonant divergences of S-matrix elements of QED of strong magnetic field are avoided.

In the Breit-Wigner approximation the Licht states are exponentially decaying or growing in time. Since the divergent part of the wave functions is cut off by θ functions, no divergent time integrals will be caused by the real part of the exponentials. Therefore, in this formalism it is possible to calculate S-matrix elements with external unstable particles.

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