Critical behavior of dimensionally continued black holes

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The critical behavior of black holes in even and odd dimensional spacetimes is studied based on Bañados-Teitelboim-Zanelli dimensionally continued black holes. In even dimensions it is found that asymptotically flat and anti–de Sitter Reissner-Nordström black holes present up to two second-order phase transitions. The case of asymptotically anti–de Sitter–Schwarzschild black holes presents only one critical transition and a minimum of temperature, which occurs at the transition. Finally, it is shown that phase transitions are absent in odd dimensions.

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I. INTRODUCTION

The possibility of critical behavior and scaling of classical objects such as black holes in general relativity is an interesting and open question. Scaling behavior was discovered by Choptuik [1] in connection with the numerical study of gravitational collapse of massless scalar fields. In that paper, a universal behavior of the black-hole mass described by a critical exponent $\beta \sim 0.37$ independent of the initial shape of the collapsing scalar field was found. Since then, critical behavior and scaling in other collapsing systems have been reported [2].

To study the thermodynamics of black holes, and in particular their heat capacity and critical behavior, it is assumed that there is an existing analogy between the laws of thermodynamics and the laws that govern black-hole mechanics derived from general relativity. This was first established by Bardeen, Carter, and Hawking [3]. To guarantee this analogy one needs to make the formal replacements $E \to M$, $T \to c\kappa$, and $S \to A/8\pi c$, where A is the area, κ the surface gravity, and c is a constant [4]. With these substitutions, the four laws of black-hole thermodynamics can be enunciated and the study of critical behavior seems to be a plausible natural extension of these ideas. Two early contributions to the study of critical behavior in gravitational systems are those of Davies [5], and Hut [6], who discussed phase transitions in Kerr-Newman and Reissner-Nordström black holes in four spacetime dimensions. In the same direction, Lousto [7] has argued in favor of the validity of the scaling laws in gravitational systems. He has calculated the critical exponents of black holes in four dimensions and has shown the validity of the scaling laws in those transitions previously found by Davies [5]. However, the relationship between the results found in [1,2] and [7] are yet to be understood.

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Bañados, Teitelboim, and Zanelli have recently reported Schwarzschild and Reissner-Nordström anti–de Sitter black-hole solutions for even and odd dimensional spacetime as a particular dimensional continuation of general relativity with nonvanishing cosmological constant Λ [8]. By a suitable choice of coefficients in the Lovelock action they obtain a unique solution for the metric with dressed singularity, although only for positive masses. The entropy becomes a monotonically increasing function r_+, and therefore the second law of thermodynamics for black holes remains valid.

It is our purpose to analytically study the scaling behavior in gravitational systems and provide further results to compare with numerical studies in this subject. In this sense, Bañados-Teitelboim-Zanelli (BTZ) black holes seem to be an interesting and relatively accessible arena in which to test these ideas. Also the fact that these objects are defined in general spacetime dimensions, seems to be a distinctive feature that might help clarify whether the *universality hypothesis* of scaling behavior is true for gravitational systems; that is, whether the critical exponents depend only on the dimensionality of the system, on the dimensionality of some order parameter and on the range of the gravitational force. In addition to this, results on the thermodynamics and critical behavior of Reissner-Nordström anti–de Sitter black-hole solutions are scarce in the literature.

In this paper we study the occurrence of phase transitions in dimensionally continued BTZ black holes. We review some of the results found in [8], and particularize them for the different cases where the charge Q and Λ are zero or nonvanishing. In our study, as in the BTZ paper, we only consider nonrotating ($J = 0$) black holes. The scaling behavior associated with these transitions needs further study, and we shall report on that elsewhere [9]. In Sec. II we briefly introduce the Lovelock action and the particular choice of coefficients from where the Schwarzschild and Reissner-Nordström anti–de Sitter black holes are derived. In Secs. III and IV we study the critical behavior of these black holes in even and odd dimensions, respectively, by evaluating the full thermal capacity at constant Q. We find that phase transitions are possible in even dimensions, except for the case of

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Schwarzschild black holes with zero cosmological constant. We also obtain that odd dimensional scenarios do not present transitions. Here, we also study possible discontinuities in the derivatives of the thermal capacity to assure that there are no phase transitions of any odd order. Section V is dedicated to give our conclusions.

II. DIMENSIONALLY CONTINUED BLACK HOLES

The Lovelock action [11] in D dimensions, which is made by the sum of the dimensionally continued Euler characteristics of dimensions less than D [12], is considered to be the most general extension of Einstein's gravity that keeps the field equations for the metric to second order.[1]

The action is written as

$$S = \kappa \sum_{p=0}^{n} \alpha_p S_p \ , \tag{1}$$

where

$$S_p = \int \epsilon_{a_1 \cdots a_D} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \cdots \wedge e^{a_D} \ . \tag{2}$$

Here $R^a_b = d\omega^a_b + \omega^a_c \omega^c_b$ is the curvature two-form, ω^a_b is the spin connection, and e_a is the local frame one-form. The action is a local Lorentz-invariant D-form and is made of e_a, ω^a_b and their exterior derivatives. However, these conditions do not restrict the values of the α_p coefficients. To obtain α_p Bañados, Teitelboim, and Zanelli [8] consider the embedding of the Lorentz group SO$(D-1,1)$ into the anti–de Sitter group SO$(D-1,2)$. We consider the choice of coefficients α_p that appears in [8]

$$\alpha_p = \begin{cases} \binom{n}{p} l^{-D+2p} & \text{if } D = 2n \ , \\ \dfrac{1}{D-2p} \binom{n-1}{p} l^{-D+2p} & \text{if } D = 2n-1 \ , \end{cases} \tag{3}$$

where l is a length related to the cosmological constant by $l = -a^2/\Lambda (a > 0)$. These coefficients are constants with dimensions $[\text{mass}]^{D-2p}$, κ has units of action (dimensionless, if $\hbar = 1$) and $a_i = 0, \ldots, D-1$.

The authors of [8] restricted their analysis to the cases of $D = 4k$ and $D = 4k - 1 (k \in \mathbf{Z})$ in order to avoid naked singularities with positive mass in the BTZ model. However, black holes with regular horizons exist in the remaining dimensions provided one takes only one of the two possible branches of real solutions, namely that corresponding to the positive root of $(2M/r)^{1/(n-1)}$ for $D = 2n$, or the positive root of $(M+1)^{1/(n-1)}$ for $D = 2n-1$, and similarly for the charged solutions.

III. EVEN DIMENSIONAL BLACK HOLES

In even dimensions the action (1) is of the form

$$\mathcal{L}_{\text{even}} = \kappa (R^{a_1 a_2} + l^{-2} e^{a_1} \wedge e^{a_2}) \wedge \cdots \wedge (R^{a_{D-1} a_D} + l^{-2} e^{a_{D-1}} \wedge e^{a_D}) \epsilon_{a_1 \cdots a_D} \ , \tag{4}$$

and the equations of motion are given by

$$(R^{a_1 a_2} + l^{-2} e^{a_1} \wedge e^{a_2}) \wedge \cdots \wedge (R^{a_{D-3} a_{D-2}} + l^{-2} e^{a_{D-3}} \wedge e^{a_{D-2}}) \wedge e^{a_D} \epsilon_{a_1 \cdots a_{D-1}} = 0 \ . \tag{5}$$

The factorized form of these equations, due to the particular choice of the coefficients, leads to a much simpler study of the physical properties of its solutions [8].

To study spherically symmetric solutions of black holes we start from the metric

$$ds^2 = -g^2(r) dt^2 + g^{-2}(r) dr^2 + r^2 d\Omega^2 \ , \tag{6}$$

where the coefficients $g^2(r)$ can be expressed as a function of r, l, M, and Q as

$$g^2(r) = 1 + \frac{r^2}{l^2} - \left(\frac{2M}{r} - \frac{Q^2}{(D-3) r^{D-2}} \right)^{2/(D-2)} \ . \tag{7}$$

Spherically symmetric solutions for the Einstein-

[1] Other models have been considered in the literature, which include string theory based black-hole solutions and two-dimensional black holes with a dilaton field [13], but we will not go into those approaches here.

Lovelock equations have been studied in the literature by several authors [14]; we consider here only static and spherically symmetric metrics.

The solution of $g^2(r) = 0$ gives us the value of the even horizon $r = r_+$:

$$\frac{Q^2}{2M(D-3)} + \frac{r_+^{D-2}}{2M}\left(1+\frac{r_+^2}{l^2}\right)^{(D-2)/2} = r_+^{D-3}\ , \quad (8)$$

which is the constraint relation we will use in following expressions to write the relevant thermodynamical quantities. The above equation has zero, one, or two real solutions, depending on the values of M, Q, and l. We will illustrate in some detail how the roots can be obtained with a reasoning that will be used in the rest of the paper.

The left-hand side (LHS) in (8) is a polynomial in r_+ of degree $2D-4$ with strictly positive coefficients. The RHS is a polynomial of order $D-3$, with a positive coefficient also. Both sides are monotonically increasing functions of r_+. The LHS is dominant for small (provided $Q^2/M \neq 0$) and large values of r_+. The RHS may dominate in the intermediate region, depending on the values of Q and M. Then, there will be two, one, or zero solutions to (8). For very large values of Q^2/M the LHS is always dominant, and there is no root, which means that we have a naked singularity. If we decrease the ratio Q^2/M, it will reach a value for which the LHS and RHS curves will be tangent to each other at one point. In that case we have only one root and therefore one event horizon. Below that value of Q^2/M the two curves intersect in two points, and thus two horizons arise. We take the greater root of the two as the black-hole horizon. We will use this kind of reasoning later on to get a feeling of how the system behaves in the general case where explicit solutions will not be available.

For $l \to \infty$, that is $\Lambda = 0$, and $Q = 0$, Eq. (8) gives us $r_+ = 2M$ for the case of Schwarzschild black holes, whereas the finite cosmological constant situation, gives $r_+ < 2M$. Therefore the largest Schwarzschild horizon happens for the $\Lambda = 0$ scenario.

From the standard expression for the temperature

$$T = \frac{1}{4\pi}\left(\frac{dg^2(r)}{dr}\right)_{r=r_+} \quad (9)$$

and relation (8), we find

$$T = \frac{r_+^{1-D}(1+{r_+}^2/l^2)^{(4-D)/2}}{2\pi(D-2)}\left[r_+^{D-2}\left(1+(D-1)\frac{{r_+}^2}{l^2}\right)\left(1+\frac{{r_+}^2}{l^2}\right)^{(D-4)/2} - Q^2\right]\ . \quad (10)$$

In the asymptotically anti–de Sitter Schwarzschild case ($Q = 0$ and negative Λ), it is easy to see that the above expression for the temperature reduces to the result obtained in [8]

$$T_{Q=0} = \frac{1+(D-1)(r_+/l)^2}{2\pi(D-2)r_+}\ . \quad (11)$$

In the asymptotically flat Reissner-Nordström solution in arbitrary spacetime dimension ($\Lambda = 0$ and $Q \neq 0$), the expression for the temperature reduces to

$$T_{\Lambda=0} = \frac{1}{2\pi(D-2)}\left(\frac{1}{r_+} - \frac{Q^2}{r_+^{D-1}}\right) \quad (12)$$

while for the asymptotically flat Schwarzschild case is

$$T_{\Lambda=Q=0} = \frac{1}{2\pi(D-2)r_+} = \frac{1}{4\pi(D-2)M}\ . \quad (13)$$

For $D = 4$ these results reproduce the standard relations found in the literature.

The expression for the entropy of the black hole in even dimensions can be computed in closed form, obtaining [8]

$$S(r_+) = \pi l^2\left[\left(1+\frac{{r_+}^2}{l^2}\right)^{(D-2)/2} - 1\right]\ , \quad (14)$$

which for the $l \to \infty$ case reduces to

$$S_{\Lambda=0}(r_+) = \frac{\pi}{2}(D-2)r_+^2\ . \quad (15)$$

A. Phase transitions

For the study of phase transitions we need to assume that the system is held in equilibrium at some temperature T with a surrounding heat bath. In $D = 4$ and $\Lambda = 0$, this condition was proved to be true only for supermassive black holes ($M \geq 10^5 M_\odot$) [10]. Above this limit, there is not enough energy for the emission of nonzero rest mass particles and the discharge of the black hole due to Hawking evaporation is negligible. Hence, the assumption of reversible transfer of energy at constant charge will be true.

We will consider a small reversible transfer of energy between the hole and its environment in such a way that the electric charge Q remains unchanged. The heat capacity we calculate is related to this transfer of energy.

By using expressions (10) and (14) we obtain

$$\mathcal{C}_Q = T\frac{\partial S}{\partial T}\bigg|_Q = \frac{T}{\Delta}\left[2\pi^2(D-2)^2\left(1+\frac{{r_+}^2}{l^2}\right)^{D-3} r_+^{D+1}\right] , \quad (16)$$

where

$$\Delta = \left(\frac{D-1}{l^2}r_+^D - r_+^{D-2}\right)\left(1+\frac{{r_+}^2}{l^2}\right)^{(D-2)/2} + Q^2\left(\frac{2D-5}{l^2}r_+^2 + D - 1\right) . \quad (17)$$

To study the critical behavior in these black holes, we look for solutions of $r_+ = r_+^c$ that make the denominator Δ vanish in the above expression. We shall divide our study in different cases.

(1) Asymptotically flat Schwarzschild solution in D dimensions. This case does not present transitions, since there are no values of r_+ that make $\Delta = 0$. Taking the limit $l \to \infty$ in Eq. (16) we obtain $\mathcal{C}_Q = -\pi(D-2)r_+^2$, which is always negative for any value of D. In $D = 4$, $r_+ = 2M = 1/(4\pi T)$ leading to $\mathcal{C}_Q = -M/T$. The negative heat capacity implies that a slight drop in black-hole temperature will cause a further drop as energy is adsorbed. The process will continue indefinitely, with the black-hole feeding on the surrounding heat bath. The fact that the temperature of Schwarzschild black holes increases as they radiate energy [5,6] is also realized for BTZ black holes, independently of the dimensionality D.

(2) Asymptotically flat Reissner-Nordström solutions. This case corresponds physically to retaining only the highest-order dimensionally continued Euler density in the action (4), for example, the Einstein-Hilbert term for $D = 4$. Making $\Delta = 0$ in (17) we find

$$r_+^c = [Q^{c2}(D-1)]^{1/(D-2)} . \quad (18)$$

As the value of D goes to infinity, it is easy to see that $r_+^c \to 1$ independent of the charge of the black hole. For any dimension equal or greater than four we find that the value of the critical event horizon becomes smaller as the dimension of spacetime increases. However, this critical horizon will never become zero. Substituting (18) into Eq. (12) we find the value for the critical temperature to be

$$T^c = \frac{1}{2\pi(D-1)r_+^c} . \quad (19)$$

From Eq. (8) we have

$$M^c = \frac{r_+^c}{2}\left(1 + \frac{1}{(D-1)(D-3)}\right) , \quad (20)$$

so we can write the critical temperature in terms of the critical mass as

$$T^c = \frac{1}{4\pi M^c}\left(\frac{1}{D-1} + \frac{1}{(D-1)^2(D-3)}\right) . \quad (21)$$

The entropy is

$$S^c = \frac{\pi}{2}(D-2)[Q^{c2}(D-1)]^{2/(D-2)} . \quad (22)$$

For the case of $D = 4$ black holes, the critical values reduce to $r_+^c = Q^c\sqrt{3}$, $T^c = 1/(6\pi r_+^c) = 1/(9\pi M^c)$ (since $M^c = 2r_+^c/3$), and $S^c = \pi r_+^{c2} = 3\pi Q^{c2}$. These results coincide with those of [7] where

$$T^c = (2\pi M[3 + \sqrt{3 - q_J}])^{-1} \quad (23)$$

and q_J satisfies the constraint

$$j_J^2 + 6j_J + 4q_J = 3 .$$

Since we are only concerned with the case of nonrotating black holes, $J = 0$ and then $q = \frac{3}{4}$. Hence, one finds $T^c = (9\pi M^c)^{-1}$.

We should note here that T, S, and M all remain finite through the transition. Since

$$\Delta = Q^2(D-1) - r_+^{D-2} , \quad (24)$$

the heat capacity $\mathcal{C}_Q$ presents two branches, going from positive to negative values through an infinite discontinuity which we can classify as of second order.

As we mentioned before, we need to check whether these values lie in the thermodynamical region, or if superradiant discharge modes become important. Following Gibbons and Carter [10], for $D = 4$ we see that $Q^c/M^c = \sqrt{3}/2 \gg m_e/e \approx 10^{-21}$ and therefore the critical point lies within the region for which emission of charged particles is energetically favored. However, the condition for the superradiant modes to be negligible, $Q^c/M^{c2} \leq m_e^2/e$ implies $M^c \geq 10^{48} \approx 10^5 M_\odot$. This imposes a lower limit for the value of the black-hole mass for spontaneous discharge through superradiant modes to be negligible and the previous expressions to give a valid critical transition. Then for $D = 4$ it is possible to have critical transitions in Reissner-Nordström black holes provided $M^c \geq 10^5 M_\odot$. We expect that the situation would be similar for the more general BTZ black holes with $\Lambda \neq 0$ and general D. However, it is still necessary to show that the Coulomb barrier arguments presented in [10] will still hold for any D.

In the study of the temperature and mass as functions of the event horizon, the maximum value of the temperature is reached at the critical point T^c and the minimum happens at $T = 0$, which corresponds to $r_+ = Q^{2/(D-2)}$. This value of the horizon takes place for $Q^{2/(D-2)} = 2M(D-3)/(D-2)$, which is the limiting case of a degenerate horizon. If Q exceeds this value there is a naked singularity.

In (22) we encounter a limiting value for the zero-point entropy

$$S^{(0)} = \frac{\pi}{2}(D-2)Q^{4/(D-2)} , \quad (25)$$

or $S_{D=4}^{(0)} = \pi Q^2$. We thus expect the ground state of these BTZ black holes to be degenerate.

As the black hole horizon goes over r_+^c given by Eq. (18), the black-hole temperature decreases, reaching the zero value for infinite horizon. The mass behaves as a monotonically increasing function of r_+ for those values that make the temperature positive. The region where the mass is monotonically decreasing corresponds to negative values of T, which lacks of physical meaning. The heat capacity as a function of the temperature presents two branches, depending on the value of the event horizon being greater or smaller than r_+^c. C_Q is positive for $r_+ < r_+^c$ and negative otherwise.

(3) Let us now study asymptotically anti–de Sitter Schwarzschild black holes in arbitrary dimensions. The critical value we obtain for the Schwarzschild horizon is

$$r_+^c = \frac{l}{\sqrt{D-1}} . \tag{26}$$

From relation (8) the critical mass is

$$M^c = \frac{l}{2} \frac{D^{(D-2)/2}}{(D-1)^{(D-1)/2}} , \tag{27}$$

and Eq. (11) gives us

$$\begin{aligned} T_{Q=0}^c &= \frac{1}{\pi(D-2)r_+^c} \\ &= \frac{\sqrt{D-1}}{\pi(D-2)l} \\ &= \frac{1}{2\pi M^c(D-2)} \left(\frac{D}{D-1}\right)^{(D-2)/2} . \end{aligned} \tag{28}$$

The critical entropy at the horizon is given by

$$S^c(r_+) = \pi l^2 \left[\left(\frac{D}{D-1}\right)^{(D-2)/2} - 1\right] . \tag{29}$$

The critical entropy is a monotonically increasing function of D, which reaches its maximum value when the dimensionality approaches infinity:

$$S^c(r_+) = \pi l^2 [\sqrt{e} - 1] .$$

The sign of the thermal capacity is determined by the sign of Δ:

$$\Delta = \frac{D-1}{l^2} r_+^{D-2} \left(r_+^2 - \frac{l^2}{D-1}\right) \left(1 + \frac{r_+^2}{l^2}\right)^{(D-2)/2} . \tag{30}$$

We see that there is only one transition, which takes place at r_+^c given by (26), and the heat capacity goes from negative to positive values as r_+ increases. As in the asymptotically Reissner-Nordström flat case, C_Q is infinite and the rest of the thermodynamical variables remain finite during the transition. Thus we can characterize it as a second-order phase transition.

Following Hawking and Page [15] we use the Helmholtz free energy to characterize the equilibrium of the system. In anti–de Sitter space, the gravitational potential causes confinement of particles and one can consider the system formed by the black hole and radiation as confined in a box. In addition, we are taking the superradiant effects to be negligible, and the Schwarzschild to be in equilibrium with the thermal bath. The free energy can be written as

$$F = M - TS = \frac{l^2}{2(D-2)r_+} \left[1 + (D-1)\frac{r_+^2}{l^2} - \left(1 + \frac{r_+^2}{l^2}\right)^{D/2}\right] , \tag{31}$$

where we have used the expressions of M, T, and S coming from (8), (11), and (14), respectively.

The expression for the $D = 4$ free energy is

$$F_{D=4} = \frac{r_+}{4} \left(1 - \frac{{r_+}^2}{l^2}\right) , \tag{32}$$

with $r_+^{F=0} = l$ the only strictly positive root.

For the particular case of four-dimensional spacetime, we find the following critical results: $r_+^c = l/\sqrt{3}$, $M^c = 2l/3\sqrt{3}$, $T^c = \sqrt{3}/(2\pi l) = 1/(3\pi M^c)$, and $S^c(r_+) = \pi l^2/3$. Since $\Lambda < 0$ we can write $l = \sqrt{a/|\Lambda|}$, $r_+^c = \sqrt{a/3|\Lambda|}$, $T^c = \sqrt{3|\Lambda|/4\pi^2 a}$, and $S^c(r_+) = a\pi/3|\Lambda|$, which agrees with the results obtained in [15], provided we take $a = 3$ and the mass and entropy of the BTZ black hole in units of m_p^2, where m_p is the Planck mass.

Some interesting results in D dimensions can now be obtained. First of all, there is a minimum temperature a Schwarzschild anti–de Sitter black hole can have in any dimension. This can easily be seen from Eq. (11) upon derivation. The minimum value of the temperature turns out to be at r_+^c, and thus is given by relation (28). Above T^c, there are two possible black-hole radii for each temperature, respectively, smaller and larger than r_+^c. The black hole with smaller horizon lies in the region $C_Q < 0$ and therefore is unstable decaying into pure thermal radiation or to a larger black-hole state. The larger black hole has $C_Q > 0$ and hence is locally stable, although we need to study the free energy to determine its behavior.

The roots and sign of Eq. (31) are determined by

$$1 + (D-1)\frac{r_+^2}{l^2} = \left(1 + \frac{r_+^2}{l^2}\right)^{D/2} . \tag{33}$$

We can show that the above equation has only one root $r_+^{F=0}$ because both sides are monotonically increasing functions of r_+, and the LHS is greater than the RHS for small r_+, and smaller for large r_+. Then, F is positive for $r_+ < r_+^{F=0}$ and negative for $r_+ > r_+^{F=0}$. By evaluating the free energy at r_+^c we obtain $F(r_+^c) > 0$ for all D, and therefore $r_+^c < r_+^{F=0}$. For general D, if $T(r_+^c) \leq T \leq T(r_+^{F=0})$ the heat capacity and free energy are positive. For a given temperature the largest black-hole solution tends to reduce its free energy by evaporating completely. For $T > T(r_+^{F=0})$, the free energy of the black hole will be less than that of the pure radiation, which will then tend to tunnel to the black-hole state. Similar effects have been previously noticed in the literature for $D = 4$ [15].

(4) Asymptotically anti–de Sitter Reissner-Nordström

in D dimensions. The roots of the denominator of Eq. (16) are given by

$$\left(\frac{D-1}{l^2}r_+^{c\,D} - r_+^{c\,D-2}\right)\left(1+\frac{r_+^{c\,2}}{l^2}\right)^{(D-2)/2} + Q^2\left(\frac{2D-5}{l^2}r_+^{c\,2} + D - 1\right) = 0 \quad (34)$$

or, equivalently, by

$$f(r_+^c) = r_+^{c\,(D-2)/2} , \quad (35)$$

where

$$f(r_+) = Q^2\left(D-1+\frac{2D-5}{l^2}r_+^2\right) + \sum_{k=1}^{2D}\binom{D/2}{k}(2k-1)\left(\frac{r_+}{l^2}\right)^k r_+^{(D-2)/2} . \quad (36)$$

This is a polynomial with strictly positive coefficients and there will be then two solutions for values of Q such that $0 \le Q < Q_{\rm crit}$, and one solution for $Q = Q_{\rm crit}$, where $Q_{\rm crit}$ is the value of the charge for which $f(r_+^c) = r_+^{c\,(D-2)/2}$ and $f'(r_+^c) = (D-2)r_+^{c\,(D-4)/2}/2$. For $Q > Q_{\rm crit}$ there will be no transitions.

As in previous cases, the full thermal capacity is infinite and it corresponds to second order phase transitions.

In Fig. 1 we plot the full thermal capacity C_Q as a function of r_+ for the case $D = 8$. The cases of two, one, or zero phase transitions are shown.

It is possible to show for sufficiently large values of Q^2/M that a naked singularity arises. The condition for this to occur turns out to be $Q^2/M > (Q^2/M)_{\rm max}$, where

$$(Q^2/M)_{\rm max} = \frac{2(D-3)}{D-2} r_{+\,\rm max}^{D-3} \frac{1+(D-1)r_{+\,\rm max}^2/l^2}{1+2r_{+\,\rm max}^2/l^2} \quad (37)$$

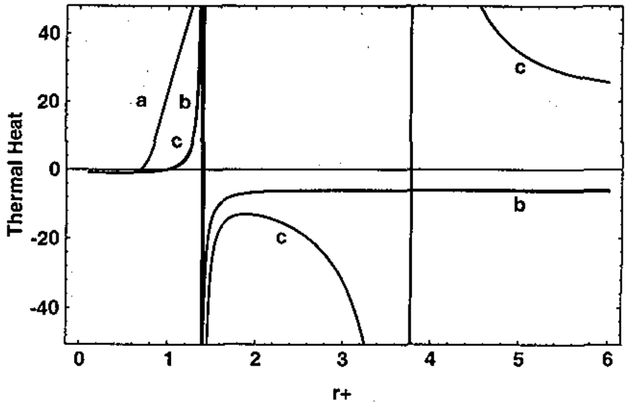

FIG. 1. Thermal heat for $D = 8$ and $Q = 1$. Branches (a), (b), and (c) correspond respectively to zero, one, or two phase transitions, for three different values of Λ.

and $r_{+\,\rm max}$ is the radius for $(Q^2/M)_{\rm max}$, where the inner and outer horizons are degenerate and $T = 0$. We also found, as in the asymptotically flat case, a limiting value of zero-point entropy, so the black-hole ground state is expected to be degenerate.

IV. ODD DIMENSIONAL BLACK HOLES

In odd dimensions $D = 2n - 1$ the action (1) is constructed by considering the Euler density (basically the product of n Ricci tensors) in $D + 1$ dimensions. This density ε_{D+1} is an exact form and thus can be written as the exterior derivative of a D form:

$$\varepsilon_{D+1} = \kappa\epsilon_{a_1\cdots a_{D+1}} R^{a_1a_2}\wedge\cdots\wedge R^{a_Da_{D+1}} = d\mathcal{L}_D , \quad (38)$$

with

$$\mathcal{L}_D^{\rm odd} = \kappa\sum_{p=0}^{n-1}\alpha_p\epsilon_{a_1\cdots a_D}R^{a_1a_2}\wedge\cdots\wedge R^{a_{2p-1}a_{2p}}\wedge e^{2p+1}\wedge\cdots\wedge e^D . \quad (39)$$

The Lovelock coefficients are those given in (3). The equations of motion are given by

$$(R^{a_1a_2}+l^{-2}e^{a_1}\wedge e^{a_2})\wedge\cdots\wedge(R^{a_{D-2}a_{D-1}}+l^{-2}e^{a_{D-2}}\wedge e^{a_{D-1}})\wedge e^{a_D}\epsilon_{a_1\cdots a_{D+1}} = 0 . \quad (40)$$

As in the previous section, to study odd dimensional black holes we start from the expression of the metric (6), where

$$g^2(r) = 1+\frac{r^2}{l^2} - \left(M+1-\frac{Q^2}{2(D-3)r^{D-3}}\right)^{2/(D-1)} , \quad (41)$$

with D odd. The value of the horizon r_+ is obtained from

$$1+\frac{{r_+}^2}{l^2} = \left(M+1-\frac{Q^2}{2(D-3)r_+^{D-3}}\right)^{2/(D-1)} . \quad (42)$$

The relation above tells us that there are no Schwarzschild black holes in odd dimensions when we restrict ourselves to the case of zero cosmological constant, the reason being that M becomes zero. However, for the case of l finite, there is the possibility for such black holes, since in this case M takes positive values.

As we did in Sec. III, we will write the general expressions for the temperature and entropy of the black hole as a function of the dimensionality. From (9) and (41), the temperature is found to be

$$T = \frac{1}{4\pi}\left[\frac{2r_+}{l^2} - \frac{Q^2}{(D-1)r_+^{D-2}}\left(1+\frac{{r_+}^2}{l^2}\right)^{(3-D)/2}\right] , \tag{43}$$

where we have used again the implicit approach coming from the constraint equation (42). Here one can see that for $l \to \infty$ and $Q \neq 0$ we arrive at a negative value of the temperature, for any value of D.

For the case of Schwarzschild black holes, the relation found in [8] is recovered,

$$T_{Q=0} = \frac{r_+}{4\pi l^2} . \tag{44}$$

In odd dimensions the expression for the entropy of the black hole given in [8] does not have a close form, being

$$S = 2\pi(D-1)\int_0^{r_+} ds\left(1+\frac{s^2}{l^2}\right)^{(D-3)/2} . \tag{45}$$

For the case of $D=4$ the previous expression reduces to the usual $S = 4\pi r_+$.

A. Phase transitions

The full thermal capacity at constant Q is given by

$$\mathcal{C}_Q = \frac{T}{\hat{\Delta}}\left[8\pi^2(D-1)\left(1+\frac{{r_+}^2}{l^2}\right)^{D-2} r_+^{D-1}\right] , \tag{46}$$

where

$$\hat{\Delta} = \frac{2}{l^2}\left(1+\frac{{r_+}^2}{l^2}\right)^{(D-1)/2} r_+^{D-1} + \frac{Q^2}{D-1}\left(\frac{2D-5}{l^2}r_+^2 + D - 2\right) . \tag{47}$$

We are interested in the following cases.

(1) Asymptotically anti–de Sitter Schwarzschild black holes. $\hat{\Delta}$ reduces to a polynomial expression in T, which is finite and nonzero for all $T \neq 0$. This implies that $\mathcal{C}_Q$ is finite and regular for any temperature. Therefore there are no critical transitions.

(2) The general case, l finite and $Q \neq 0$, does not present any transitions either. It is easy to see, by inspection of (46) that there is no transition $\mathcal{C}_Q$ divergent ($\hat{\Delta} \neq 0$) for any value of T, Q, or l. For finite values of the heat capacity there is no transition either, as the derivatives will be regular to any order. This can be seen from the fact that one can write

$$\frac{\partial \mathcal{C}_Q}{\partial T} = \frac{\partial S}{\partial T} + T\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial T}\right) . \tag{48}$$

The first term on the right-hand side is equal to $\mathcal{C}_Q/T$ and it is finite, since it is the ratio of two polynomials in r_+ with positive coefficients. The second term reduces to

$$\frac{1}{\hat{\Delta}}\frac{\partial}{\partial r_+}\left(\frac{\mathcal{C}_Q}{T}\right) ,$$

where the ratio $\mathcal{C}_Q/T$ and its derivative with respect to r_+ are both finite. Because $\hat{\Delta}$ never vanishes in odd dimensions, the second term in (48) is also finite. Doing a similar analysis for the following derivatives of $\mathcal{C}_Q$ we find them all to be finite. Therefore in odd dimensionally continued BTZ black holes there are no critical transitions; the full thermal capacity and all its derivatives remain finite for all values of T and Q. This result is general for any dimension.

V. CONCLUSIONS

In this paper we studied the possibility of critical transitions in Bañados, Teitelboim, and Zanelli dimensionally continued black holes. In even dimensions there exist transitions depending on the value of the charge and the cosmological constant. Asymptotically flat Schwarzschild black holes do not present phase transitions with the specific heat being always negative. In this case $T = 0$ is asymptotically reachable as the horizon approaches infinity. For the Schwarzschild anti–de Sitter case there is one critical transition, with $\mathcal{C}_Q$ being negative (unstable) or positive depending on the horizon being respectively smaller or larger than the critical horizon. These objects present a minimum temperature, which is different from zero.

For Reissner-Nordström black holes with zero cosmological constant there is only one second-order transition, while the anti–de Sitter case presents two. A characteristic feature of the latter case is that there could exist up to three black-hole radii for a given temperature.

Odd dimensional BTZ black holes do not present critical behavior. The full thermal capacity remains finite with finite derivatives for any value of the temperature.

Further study is still needed to fully understand the physical meaning of these different transitions and the scaling behavior associated with them [9].

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