Searching for nonminimally coupled scalar hairs

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We study the asymptotically flat, static, and spherically symmetric black-hole solutions of the theory described by the action $S = \int d^n x \sqrt{-g}\{(1-\xi\phi^2)R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\}$, with $n>3$ and arbitrary ξ. We demonstrate the absence of scalar hairs for $\xi<0$. For $\xi>\xi_c = (n-2)/4(n-1)$, we show that there is no scalar hair obeying $|\phi(r)|<1/\sqrt{\xi}$ or $|\phi(r)|>1/\sqrt{\xi}$. For $0<\xi<\xi_c$, we prove the absence of scalar hairs such that $|\phi(r)|<1/\sqrt{\xi}$ or $1/\xi<\phi^2(r)<\xi_c/\xi(\xi_c-\xi)$. [S0556-2821(96)02512-X]

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The Bekenstein black-hole solution [1] for Einstein gravity conformally coupled to a scalar field in four dimensions has had a prominent role in gravitational physics. It has an extremal Reissner-Nordström geometry and a $1/r$-type scalar field, and it was one of the first counterexamples to the ''no-hair'' conjecture [2]. The scalar field diverges in the horizon, and such a divergence is crucial to the violation of the no-hair theorems, as recent works have revealed [3,4]. The Bekenstein solution is an asymptotically flat, static, and spherically symmetric solution for the theory described by the action

$$S[g,\phi] = \int d^n x \sqrt{-g}\{(1-\xi\phi^2)R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\} \quad (1)$$

with $\xi=1/6$ and $n=4$. The coupling defined by such values is called conformal because with them the action 1 is invariant under the map defined by the conformal transformation $g_{\mu\nu}=\Omega^2\bar{g}_{\mu\nu}$, $\Omega^2>0$, and by the field transformation $\phi=\Omega^{-1}\bar{\phi}$. This map can be easily extended for n-dimensional space-times; with the coupling given by $\xi_c = (n-2)/4(n-1)$, the action 1 is conformal invariant, with the field redefinition given by $\phi=\Omega^{[(2-d)/2]}\bar{\phi}$.

Although we know that scalar fields are not elementary fields in nature, they commonly arise in effective actions. In fact, some scalar actions have been considered recently in astrophysical contexts, see for instance [5]. However, with the conformally coupled case as the only exception [3,6–8], only minimally coupled scalar fields have been examined. In [4] a new theorem is presented which rules out a multicomponent scalar hair with a non-quadratic Lagrangian, but with minimal coupling to gravity. As is stressed in [4], scalar field effective actions are obtained by integrating the functional integral of the elementary fields in nature over some of the fields, and more complicated actions involving nonminimally coupling should arise.

In this line, the first natural question that can arise in the analysis of 1 is if there exists another nonvacuum (ϕ not constant) black-hole for some ξ or n. Such an investigation has already been started, and the up-to-now data are the following. Long standing results state that for the minimally coupled case, $\xi=0$, there is no other black-hole solution than the vacuum Schwarzschild one for $n>3$ [9–12]. For the conformally coupled case, $\xi=\xi_c$, recent works have established

that the only static and spherically symmetric nonvacuum black-hole solution of 1 with $n>3$ is the four-dimensional Bekenstein one [6–8]. The present work points toward the conclusion of this investigation by showing that the only black-hole solutions of 1 are the Schwarzschild ones for $n>3$ and for very large ranges of ξ and ϕ. Hereafter, we will use the term black-hole solution to denote an asymptotically flat, static, and spherically symmetric black-hole solution. We show that the Schwarzschild black-hole solution is the only one for $\xi<0$. For $\xi>\xi_c$, we prove the absence of scalar hairs obeying $|\phi(r)|<1/\sqrt{\xi}$ or $|\phi(r)|>1/\sqrt{\xi}$. We also demonstrate the absence of scalar hairs obeying $1/\xi<\phi^2(r)<\xi_c/\xi(\xi_c-\xi)$ or $|\phi(r)|<1/\sqrt{\xi}$ for $0<\xi<\xi_c$. These results are in agreement with the recent results about the uniqueness of the four-dimensional Bekenstein black-hole solution [3,6–8].

The demonstration of our results centers in a covariant method for generating solutions for 1 starting from the well-known solutions of the minimally coupled case. It generalizes the method for generating solutions for the conformally coupled case in $n>3$ dimensions presented in [7]. For $n=4$, our method reproduces the method used in [13] for generating solutions for arbitrary ξ in four dimensions. Such methods are based in conformal transformations and ϕ redefinitions and they have a long history. Reference [14], for instance, presents a good set of references on the subject. A method of this type was early presented by Bekenstein [1] and used by him for generating solutions for the conformally coupled case starting from the minimally coupled one in four dimensions; this was the way that he obtained his black-hole solution with conformal hair. We notice also that Maeda in [15] has used very close machinery to show that the action given by $\int d^4 x \sqrt{-g}\{F(\phi,R) - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\}$ is equivalent to an Einstein-Hilbert action plus minimally coupled self-interacting scalar fields, equivalent in the sense that there is a conformal transformation and ϕ-redefinition connecting them.

The method will provide us with a general asymptotically flat, static, and spherically symmetric solution of 1. Such a general solution will be given by a two-parameter (λ, r_0) family of solutions, and we will systematically search for values of (λ, r_0) such that the solution corresponds to a black hole. To this end, one needs to be capable to identify a black-

hole solution. Because of our experience with the Bekenstein solution we will not pay attention to possible divergences of the scalar field ϕ. We recall that the general static and spherically symmetric n-dimensional metric has the following form in isotropic coordinates:

$$ds^2 = -e^f dt^2 + e^{-h} dr^2 + e^{-h} r^2 d\Omega^2, \tag{2}$$

where $d\Omega^2$ denotes for the metric of the unitary $(n-2)$ sphere. The metric 2 describes a black hole if it has a regular event horizon, say the hypersurface $r=r_0$. The necessary and sufficient conditions in order for this hypersurface to be a regular horizon are (i) e^f vanishes at $r=r_0$, so that the hypersurface is of the null type, and (ii) the invariants of the metric are finite in $r=r_0$. If condition (ii) is not verified the hypersurface $r=r_0$ is said to be a naked singularity. The invariant of the metric that we will use in our case is the scalar of curvature R, which can be obtained from the Euler-Lagrange equations of 1:

$$R = \frac{1-\xi/\xi_c}{1-\xi(1-\xi/\xi_c)\phi^2} g^{rr} \left(\frac{d\phi}{dr} \right)^2. \tag{3}$$

We will see that for all candidate solutions to be black hole R will be singular for $r=r_0$ or such a hypersurface will be not of the null type, with the only exception of the Schwarzschild solution.

To present the method for generate solutions let us suppose first that $1-\xi\phi^2(r)>0$. We can check that the transformations

$$g_{\mu\nu} = (1-\xi\phi^2)^{-2/(n-2)} \bar{g}_{\mu\nu}, \tag{4}$$

$$\bar{\phi}(\phi) = \int_a^\phi d\chi \frac{\sqrt{1+\xi[(\xi/\xi_c)-1]\chi^2}}{1-\xi\chi^2}, \tag{5}$$

on the action 1 leads to $S[\Omega^2 \bar{g}, \phi(\bar{\phi})] = \bar{S}[\bar{g}, \bar{\phi}]$, where

$$\bar{S}[\bar{g}, \bar{\phi}] = \int d^4x \sqrt{-\bar{g}} \{\bar{R} - \bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi}\} \tag{6}$$

is the minimally coupled action. Because of the assumption $1-\xi\phi^2(r)>0$, the right-handed side of 5 is a monotonically increasing function of ϕ, which guarantees the existence of the inverse $\phi(\bar{\phi})$. The constant a is to be determined by the boundary conditions of ϕ and $\bar{\phi}$. The conformal transformation 4 is valid in general only locally. It is the spherical symmetric in our case that guarantees that a unique conformal transformation can be used for the whole black-hole exterior.

The transformation given by Eqs. 4 and 5, therefore, maps a solution $(g_{\mu\nu}, \phi)$ of 1 to a solution $(\bar{g}_{\mu\nu}, \bar{\phi})$ of 6. The transformation is independent of any assumption of symmetries, and in this sense is covariant. Also, we can easily infer that the transformation is one-to-one in general, in the sense that any solution of 1 is mapped in a unique solution of 6. The transformation preserves symmetries, what that means is that if $\bar{g}_{\mu\nu}$ admits a Killing vector ξ such that $\pounds_\xi \bar{\phi} = 0$, then ξ is also a Killing vector of $g_{\mu\nu}$ and $\pounds_\xi \phi = 0$. From this, one concludes if we know all solutions $(\bar{g}_{\mu\nu}, \bar{\phi})$ with a given symmetry we automatically know all $(g_{\mu\nu}, \phi)$ with the same

symmetry and obeying $1-\xi\phi^2(r)>0$. We will obtain the general asymptotically flat, static, and spherically symmetric solution of 1 in this way.

The general asymptotically flat, static, and spherically symmetric solution for $n>3$ dimensions $(\bar{g}_{\mu\nu}, \bar{\phi})$ of the minimally coupled case 6 was derived in [12]. It is given in isotropic coordinates by the two-parameter (λ, r_0) family of solutions

$$\bar{\phi} = -\sqrt{\frac{n-2}{n-3}(1-\lambda^2)} \ln \mathcal{R}_n,$$

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\mathcal{R}_n^{2\lambda} dt^2 + \left(1 - \frac{r_0^{2n-6}}{r^{2n-6}}\right)^{2/(n-3)}$$
$$\times \mathcal{R}_n^{-2\lambda/(n-3)} (dr^2 + r^2 d\Omega^2), \tag{7}$$

where $\mathcal{R}_n = (r^{n-3} - r_0^{n-3})/(r^{n-3} + r_0^{n-3})$. The parameter λ can take values in $[-1,1]$ in principle, although the negative range corresponds to solutions with negative Arnowitt-Deser-Misner (ADM) mass [12]. For $\lambda=1$, the solution is the usual exterior n-dimensional vacuum Schwarzschild solution with the horizon at $r_0'=4r_0$, as one can check by using the coordinate transformation $r'=r(1+r_0/r)^2$. For $0 \le \lambda < 1$, 7 does not represent a black-hole because the surface $r=r_0$ is not a horizon, i.e., a regular null surface, but it is instead a naked singularity, as we can check by calculating the scalar of curvature

$$\bar{R} = \frac{4(n-2)(n-3)r^{2n-8}r_0^{2n-6}}{(r^{n-3}+r_0^{n-3})^{2(n-2+\lambda)/n-3}}$$
$$\times \frac{1-\lambda^2}{(r^{n-3}-r_0^{n-3})^{2(n-2-\lambda)/n-3}}. \tag{8}$$

Note that the solution 7 for general λ describes only the exterior region $(r>r_0)$ of the space-time. However, it is still valid for the interior region for some values of λ and n. In these cases, the solution for the scalar field is $\bar{\phi} = -\sqrt{[(n-2)/(n-3)](1-\lambda^2)} \ln|\mathcal{R}_n|$, and the signature for the interior region could eventually change to $(+,-,\cdots,-)$. Note that cases like $\lambda=0$ and $n=5$ do not correspond to acceptable interior solutions because the signature in the interior region would be $(-,-,\cdots,-)$.

For the range $\xi<0$ the assumption of $1-\xi\phi^2(r)>0$ is automatically verified and we can use the transformation 4 and 5 for generating the solutions $(g_{\mu\nu}, \phi)$ starting from 7. From the regularity of the integrand we have for this case $\lim_{\bar{\phi}\to\infty} \phi(\bar{\phi}) = \infty$. The situation is the same if $\xi>0$ and $1-\xi\phi^2(r)>0$, but we will have $\lim_{\bar{\phi}\to\infty} \phi = 1/\sqrt{\xi}$. If $\xi\phi^2(r)-1>0$ we can also apply the same formulation with minor modifications. It is easy to verify that the transformation given by

$$g_{\mu\nu} = (\xi\phi^2-1)^{-2/(n-2)} \bar{g}_{\mu\nu}, \tag{9}$$

$$\bar{\phi}(\phi) = \int_a^\phi d\chi \frac{\sqrt{1+\xi(\xi/\xi_c - 1)\chi^2}}{\xi\chi^2 - 1}, \tag{10}$$

maps also a solution $(g_{\mu\nu}, \phi)$ of $\underline{1}$ to a solution $(\bar{g}_{\mu\nu}, \bar{\phi})$ of $\underline{6}$. However, we see from $\underline{10}$ that in this case one needs also that $\xi \geqslant \xi_c$. The integrals $\underline{5}$ and $\underline{10}$ can be explicitly solved, see [13] for instance, but the final expressions are rather cumbersome and in fact we will need only some asymptotic expansions. To summarize, we are able to generate all solutions of $\underline{1}$ for $\xi < 0$. For $\xi > \xi_c$, we can generate all solutions with $|\phi(r)| < 1/\sqrt{\xi}$ or with $|\phi(r)| > 1/\sqrt{\xi}$. Finally, for $0 < \xi < \xi_c$ we can generate the solutions such that $1/\xi < \phi^2(r) < \xi_c/\xi(\xi_c - \xi)$ or $|\phi(r)| < 1/\sqrt{\xi}$. Now we will examine each one of these special ranges of ξ.

For $\xi < 0$, the transformations $\underline{4}$ and $\underline{5}$ can be used for generating solutions with ϕ of any range. From $\underline{4}$, we see that the only candidate to null hypersurface for the metric $g_{\mu\nu}$ is that one for which $r = r_0$. To search for black-hole solutions one needs to search for values of λ, ξ, and n such that the hypersurface $r = r_0$ is a regular one of null type, and to this end we will evaluate the scalar of curvature R $\underline{3}$ and g_{tt} for $r = r_0$. Such a work can be simplified considerably if one uses an asymptotic expansion for large ϕ. From $\underline{5}$ we have that $\bar{\phi} \approx \sqrt{1/\xi_c - 1/\xi} \ln\phi$ for large ϕ, and from this we get that $\phi(r) \approx \mathcal{R}_n^{-\alpha}$ for $r = r_0 + \varepsilon$, where

$$\alpha = \sqrt{\frac{[(n-2)/(n-3)](1-\lambda^2)}{1/\xi_c - 1/\xi}}. \tag{11}$$

Using that $g_{rr} \approx (1 + r_0^{n-3}/r^{n-3})^{4/(n-3)} \mathcal{R}_n^{2(\beta - \alpha)}$, where

$$\beta = \frac{1-\lambda}{n-3}, \tag{12}$$

and inserting the asymptotic expansion for ϕ in $\underline{3}$ we get that, for $r = r_0 + \varepsilon$,

$$R \approx \frac{4\alpha^2(n-3)^2}{\xi^{n/(n-2)}} \frac{r^{2n-4} r_0^{2n-6}}{(r^{n-3} + r_0^{n-3})^{(4n-8)/(n-3)}}$$
$$\times \mathcal{R}_n^{-2[\alpha/(n-2) + \beta + 1]}. \tag{13}$$

Expression $\underline{13}$ has a nonremovable singularity in $r = r_0$ for any $n > 3$, $\xi < 0$, and for any $\lambda \neq \pm 1$. Thus, for such a hypersurface $r = r_0$ is a naked singularity and this excludes the possibility that some solution does represent a black hole. The only possibility of a black hole corresponds to the choice $\lambda = 1$, as we see from the expression of g_{tt}:

$$g_{tt} = -\mathcal{R}_n^{2[\alpha/(n-2) + \lambda]}, \tag{14}$$

what leads to $\phi = a$, and this solution is the usual n-dimensional Schwarzschild one. For completeness, let us analyze the solution generated by the interior solution of $\underline{7}$. Again, the unique null hypersurface is $r = r_0$. We can check that the same asymptotic expansions $\underline{13}$ and $\underline{14}$ are valid for $r = r_0 - \varepsilon$, and from this we conclude also these solutions cannot give new black holes.

We conclude that for $\xi < 0$ and $n > 3$ there is no other black-hole solution for the action $\underline{1}$ than the Schwarzschild one.

For $\xi > \xi_c$ we are able to generate solutions with $|\phi(r)| < 1/\sqrt{\xi}$ and with $|\phi(r)| > 1/\sqrt{\xi}$ by using the transformations $\underline{4}$ and $\underline{5}$ and $\underline{9}$ and $\underline{10}$, respectively. We begin by the first case. We can see from $\underline{4}$ that the possible hypersurface of null type corresponds to that one for which $r = r_0$. We also will examine the scalar of curvature to search for horizons, but in this case we will use an asymptotic expansion for $\underline{5}$ valid for ϕ very close to $1/\sqrt{\xi}$. For small $1 - \xi\phi^2$ we obtain $\bar{\phi}(\phi) \approx -\sqrt{(n-1)/(n-2)} \ln(1 - \xi\phi^2)$, what leads to $1 - \xi\phi^2 \approx \mathcal{R}_n^\delta$ for $r = r_0 + \varepsilon$, where

$$\delta = 2\sqrt{\xi_c \frac{n-2}{n-3}(1-\lambda^2)}. \tag{15}$$

The derivative $d\phi/dr$ present in $\underline{3}$ can be evaluated for $r = r_0 + \varepsilon$ by using $d\phi/dr = (d\phi/d\bar{\phi})(d\bar{\phi}/dr)$ and calculating $d\phi/d\bar{\phi}$ from $(\underline{5})$ for small $1 - \xi\phi^2$. One gets

$$\frac{d\phi}{dr} \approx -2\sqrt{\frac{\xi_c(n-3)(n-2)(1-\lambda^2)}{\xi}}$$
$$\times \frac{r^{n-4} r_0^{n-3}}{(r^{n-3} + r_0^{n-3})^2} \mathcal{R}_n^{\delta - 1}. \tag{16}$$

Using that $g_{rr} \approx (1 + r_0^{n-3}/r^{n-3})^{4/(n-3)} \mathcal{R}_n^{2\beta - [2/(n-2)]\delta}$ we finally get

$$R \approx \frac{4\xi_c(\xi_c - \xi)(n-3)(n-2)(1-\lambda^2)}{\xi^2}$$
$$\times \frac{r^{2n-4} r_0^{2n-6}}{(r^{n-3} + r_0^{n-3})^{(4n-8)/(n-3)}} \mathcal{R}_n^{2[(n-1)/(n-2)\delta - \beta - 1]}. \tag{17}$$

Due to $[(n-1)/(n-2)]\delta - \beta - 1 < 0$ we see that $\underline{17}$ is divergent for any $n > 3$, $\xi > \xi_c$, and $\lambda \neq \pm 1$. For this case we also have

$$g_{tt} = -\mathcal{R}_n^{2(\lambda - 2\delta)}, \tag{18}$$

discarding the possibility of $\lambda = -1$. Again the only nonsingular situation is the usual vacuum solution.

The solutions with $|\phi(r)| > 1/\sqrt{\xi}$ are generated by using the transformations $\underline{9}$ and $\underline{10}$. The expressions for R and g_{tt} valid for $r = r_0 \pm \varepsilon$ are also given, up to signs, by $\underline{13}$ and $\underline{14}$, what excludes any black-hole solution for $|\phi| > 1/\sqrt{\xi}$ and $\lambda \neq 1$.

We have that for $\xi > \xi_c$ and $n > 3$ there is no other black-hole solution with the scalar field obeying $|\phi(r)| < 1/\sqrt{\xi}$ or $|\phi(r)| > 1/\sqrt{\xi}$ than the Schwarzschild one.

Finally we have the case $0 < \xi < \xi_c$. Solutions for which $|\phi(r)| < 1/\sqrt{\xi}$ are generated by $\underline{4}$ and $\underline{5}$, and the asymptotic expressions for R and g_{tt} are $\underline{17}$ and $\underline{18}$, respectively. For the range $1/\xi < \phi^2(r) < \xi_c/\xi(\xi_c - \xi)$ the asymptotic expressions for R and g_{tt} are also given by $\underline{13}$ and $\underline{14}$. Thus, we conclude again that for $0 < \xi < \xi_c$ and $n > 3$ there is no other black-hole solution with the scalar field obeying $|\phi(r)| < 1/\sqrt{\xi}$ or $1/\xi < \phi^2(r) < \xi_c/\xi(\xi_c - \xi)$ than the Schwarzschild one.

We finish by saying that our ''no-go'' results for scalar hairs with arbitrary coupling buttresses the recent conclu-

sions that the the four-dimensional Bekenstein black-hole so-
lution is truly exclusive and outstanding [3,6–8]. If we re-
member that the Bekenstein solution has the same number of
free parameters as the Reissner-Nordström solution and not

one more [6], we can say that the essence of the no-hair
conjecture is not compromised by the conformal scalar hair.

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