Koebe 1/4 theorem and inequalities in $N=2$ supersymmetric QCD

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The critical curve $\mathscr{C}$ on which $\mathrm{Im}\,\hat{\tau}=0$, $\hat{\tau}=a_D/a$, determines hyperbolic domains whose Poincaré metric can be constructed in terms of a_D and a. We describe $\mathscr{C}$ in a parametric form related to a Schwarzian equation and prove new relations for $N=2$ supersymmetric SU(2) Yang-Mills theory. In particular, using the Koebe 1/4 theorem and Schwarz's lemma, we obtain inequalities involving u, a_D, and a, which seem related to the renormalization group. Furthermore, we obtain a closed form for the prepotential as a function of a. Finally, we show that $\partial_{\hat{\tau}}\langle \mathrm{tr}\phi^2\rangle_{\hat{\tau}}=1/8\pi i b_1\langle\phi\rangle_{\hat{\tau}}^2$, where b_1 is the one-loop coefficient of the beta function. [S0556-2821(96)06810-5]

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The effective action of the low-energy limit of $N=2$ supersymmetric Yang-Mills theory, solved exactly in [1], is described in terms of the prepotential $\mathscr{F}$ [2],

$$S_{\mathrm{eff}}=\frac{1}{4\pi}\mathrm{Im}\left(\int d^4x d^2\theta d^2\bar{\theta}\,\overline{\Phi}_D^i\overline{\Phi}_i+\frac{1}{2}\int d^4x d^2\theta\,\tau^{ij}W_iW_j\right), \tag{1}$$

where $\Phi_D^i\equiv\partial\mathscr{F}/\partial\Phi_i$ and $\tau^{ij}\equiv\partial^2\mathscr{F}/\partial\Phi_i\partial\Phi_j$. Let us denote by $a_i\equiv\langle\phi^i\rangle$ and $a_D^i\equiv\langle\phi_D^i\rangle$ the vacuum expectation values (VEV's) of the scalar component of the chiral superfield. For the gauge group SU(2) the moduli space of quantum vacua, parametrized by $u\equiv\langle\mathrm{tr}\phi^2\rangle$, is $\Sigma_3=\mathbf{C}\backslash\{-\Lambda^2,\Lambda^2\}$, the Riemann sphere $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$ with punctures at $\pm\Lambda^2$ and ∞, where Λ is the dynamically generated scale. It turns out that [1] (we set $\Lambda=1$)

$$a_D=\partial_a\mathscr{F}=\frac{\sqrt{2}}{\pi}\int_1^u\frac{dx\sqrt{x-u}}{\sqrt{x^2-1}},\qquad a=\frac{\sqrt{2}}{\pi}\int_{-1}^1\frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}. \tag{2}$$

A crucial property of a_D and a is that they satisfy the equation [3] (see also [4])

$$[4(u^2-1)\partial_u^2+1]a_D=0=[4(u^2-1)\partial_u^2+1]a. \tag{3}$$

This equation is the "reduction" of the uniformizing equation for Σ_3 [5,3,4]

$$[4(1-u^2)^2\partial_u^2+u^2+3]\psi=0, \tag{4}$$

which is satisfied by $\sqrt{1-u^2}\partial_u a_D$ and $\sqrt{1-u^2}\partial_u a$.

Let us summarize the main results in [4]. First of all it has been shown that

$$u=\pi i(\mathscr{F}-a\partial_a\mathscr{F}/2), \tag{5}$$

that is

$$\mathscr{F}(\langle\phi\rangle)=\frac{1}{\pi i}\langle\mathrm{tr}\phi^2\rangle+\frac{1}{2}\langle\phi\rangle\langle\phi_D\rangle. \tag{6}$$

In order to specify the functional dependence of u we set $u=\mathscr{G}_1(a)$, $u=\mathscr{G}_2(\hat{\tau})$, and $u=\mathscr{G}_3(\tau)$, where $\hat{\tau}=a_D/a$ and $\tau=\partial_a^2\mathscr{F}$. By Eq. (3) we have

$$(1-\mathscr{G}_1^2)\partial_a^2\mathscr{G}_1+\frac{a}{4}(\partial_a\mathscr{G}_1)^3=0, \tag{7}$$

that by (5) provides recursion relations for the instanton contribution and implies

$$\partial_a^3\mathscr{F}=\frac{\pi^2(a\partial_a^2\mathscr{F}-\partial_a\mathscr{F})^3}{16[1+\pi^2(\mathscr{F}-a\partial_a\mathscr{F}/2)^2]}. \tag{8}$$

By (2) we have $a(u=-1)=-i4/\pi$ and $a(u=1)=4/\pi$ so that the initial conditions for the second-order equation (7) are $\mathscr{G}_1(-i4/\pi)=-1$ and $\mathscr{G}_1(4/\pi)=1$.

In this paper we will investigate some consequences of the relation (5). This relation allows us to find the differential equation satisfied by the functions $\mathscr{G}_k$ and implies a new relation which involves theta functions and the prepotential $\mathscr{F}$. Furthermore, we investigate the structure of $\mathscr{F}$ as function of a that, although its explicit expression is still unknown, we give in a closed form in Eq. (22).

In our investigation the Seiberg-Witten critical curve $\mathscr{C}$, on which $\mathrm{Im}\,\hat{\tau}=0$, plays a crucial role. This curve determines hyperbolic domains. In particular, using uniformization theory, we will construct the natural (Poincaré) metrics on the quantum moduli space and on the hyperbolic domain inside $\mathscr{C}$. Such metrics should be also useful in finding the building blocks for the nonholomorphic part (higher order derivatives) of the effective action. General theorems concerning univalent functions, such as Schwarz's lemma and the Koebe 1/4 theorem, imply inequalities that for the case at hand take a simple form in terms of the VEV's of ϕ, ϕ_D, and $\mathrm{tr}\phi^2$. We will suggest that such inequalities are related to the renormalization group and will discuss a possible connection with the uncertainty principle. In obtaining such inequalities we introduce the Euclidean distance between points in the quantum moduli spaces. The structure of this distance is investigated making use of Eq. (5) which also implies the relation $\partial_{\hat{\tau}}\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}=1/8\pi i b_1\langle\phi\rangle_{\hat{\tau}}^2$, where b_1 is the one-loop coefficient of the beta function.

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We note that some properties of the critical curve have been investigated in [1,6,7] whereas related physical aspects have been discussed in [1,8].

The critical curve $\mathscr{C}$ can be seen as the curve on which the torus with modular parameter $\hat{\tau}$ degenerates. Let us consider the mass of a dyon hypermultiplet

$$M_{n_m n_e}=\sqrt{2}|n_m a_D+n_e a|=\sqrt{2}|a||n_m\hat{\tau}+n_e|, \qquad (9)$$

where n_e and n_m are the electric and magnetic charges, respectively. $M^2_{n_m n_e}$ is related to the eigenvalues of the Laplacian on the $\hat{\tau}$ torus. More precisely we have the Schrödinger equation

$$\Delta\psi_{n_m n_e}=E_{n_m n_e}\psi_{n_m n_e}, \qquad (10)$$

where $\Delta=-2\partial_{\bar{z}}\partial_z$ is the Laplacian on the torus and $\psi_{n_m n_e}=\sqrt{2}\cos 2\pi(n_m x-n_e y)$, $z=x+\hat{\tau}y$. One has $E_{n_m n_e}=2\pi^2|n_m\hat{\tau}+n_e|^2/(\mathrm{Im}\hat{\tau})^2$, so that

$$E_{n_m n_e}=\frac{\pi^2 M^2_{n_m n_e}}{|a|^2(\mathrm{Im}\hat{\tau})^2}. \qquad (11)$$

Notice that (10) admits the following interpretation. One can consider the theory $N=1$ in $D=6$. Compactifying two dimensions on the torus $\hat{\tau}$, one has $Z\sim P_5+iP_6$ so that in the massless sector $P^2=0$

$$\Box_6\psi=0\Rightarrow\Box_4\psi=\Delta_2\psi=|Z|^2\psi, \quad \Delta_2=\frac{2\pi^2}{|a|^2(\mathrm{Im}\hat{\tau})^2}\Delta. \qquad (12)$$

In crossing the curve $\mathscr{C}$ a Bogomol'ni-Prasad-Sommerfield- (BPS-)saturated particle of given charges can appear or disappear. Equations (10),(11) show that the tori $\hat{\tau}\in\mathscr{C}$ correspond to critical points for the structure of the energy eigenvalues. In [6,7] it has been shown that inside $\mathscr{C}$, that is in the domain A containing the point $u=0$ and such that $\mathscr{C}=\partial A$, one has $\mathrm{Im}\hat{\tau}<0$. We will show that $\mathscr{C}$ can be parametrically described by the solution of a Schwarzian equation. Eventually we will obtain inequalities which resemble a sort of uncertainty relations for quantum field theory (QFT) and where the critical curve plays a crucial role. In order to find the differential equation associated to $\mathscr{C}$ we first recall that τ corresponds to the inverse of the map which uniformizes Σ_3 [4]. An important point is that both τ and $\hat{\tau}$ have $\Gamma(2)$ monodromy implying that the structure of the associated fundamental domains D_1 and D_2 differ for the value of the opening angle at the cusps [7] (0 and 2π, respectively). To describe the critical curve, we first note that by definition (see also [7])

$$\mathscr{C}=\{u=\mathscr{G}_2(\hat{\tau})|\hat{\tau}\in[-1,1]\}. \qquad (13)$$

On the other hand, the dependence of u on $\hat{\tau}$ can be determined by solving a differential equation which follows from (3). Using the following property of the Schwarzian derivative

$$\{y(x),x\}=\frac{y'''}{y'}-\frac{3}{2}\left(\frac{y''}{y'}\right)^2=-y'^2\{x(y),y\}, \qquad (14)$$

one has $\{\mathscr{G}_2,\hat{\tau}\}=-(\partial_{\hat{\tau}}\mathscr{G}_2)^2\{\hat{\tau},\mathscr{G}_2\}$. To obtain the differential equation satisfied by $\mathscr{G}_2$ we recall that if ψ_1,ψ_2 are linearly independent solutions of $\partial_x^2\psi(x)+P(x)\psi(x)=0$, then $\{\psi_1/\psi_2,x\}=2P$, which follows by $f^{1/2}\partial_x f^{-1}\partial_x f^{1/2}=\psi_k^{-1}\partial_x\psi_k^2\partial_x\psi_k^{-1}=\partial_x^2-\psi_k''/\psi_k=\partial_x^2+P$, $k=1,2$, $f=\psi_1/\psi_2$ [note that $\psi_2(x)=A\psi_1(x)+B\psi_1(x)\int^x\psi_1^{-2}$, $B\neq 0$]. Thus, since $\hat{\tau}$ is the ratio of two solutions of (3), we have

$$2(1-\mathscr{G}_2^2)\{\mathscr{G}_2,\hat{\tau}\}=(\partial_{\hat{\tau}}\mathscr{G}_2)^2. \qquad (15)$$

An explicit computation gives $\hat{\tau}(u=-1)=\pm 1$, $\hat{\tau}(u=1)=0$ [observe that -1 and $+1$ are identified after factorizing the image of the $\hat{\tau}$ map by $\Gamma(2)$], so that

$$\mathscr{G}_2(-1)=\mathscr{G}_2(1)=-1, \mathscr{G}_2(0)=1. \qquad (16)$$

Therefore the critical curve is given by (13) with $\mathscr{G}_2$ solution of (15) and initial conditions (16). The solutions of Eqs. (7),(15) should be related to the $\wp$ function. To show this we use (14) again so that $\{\mathscr{G}_3,\tau\}=-(\partial_\tau\mathscr{G}_3)^2\{\tau,\mathscr{G}_3\}$ and by (4)

$$2(1-\mathscr{G}_3^2)^2\{\mathscr{G}_3,\tau\}=-(3+\mathscr{G}_3^2)(\partial_\tau\mathscr{G}_3)^2. \qquad (17)$$

Since τ and $\hat{\tau}$ have the same monodromy, it follows by (16) that (see also [7])

$$\mathscr{G}_3(-1)=\mathscr{G}_3(1)=-1, \quad \mathscr{G}_3(0)=1, \qquad (18)$$

which also follows by an explicit computation [use (2) and recall that $\tau=a_D'/a'$]. A way to find the solution of (17) with initial conditions (18) is to consider u as the uniformizing map. In the case of Σ_3 we have [9]

$$u=\mathscr{G}_3(\tau)=\frac{2\wp\left(\frac{1+\tau}{2}\right)-\wp\left(\frac{\tau}{2}\right)-\wp\left(\frac{1}{2}\right)}{\wp\left(\frac{\tau}{2}\right)-\wp\left(\frac{1}{2}\right)}$$
$$=1-2\left[\frac{\Theta_2(0|\tau)}{\Theta_3(0|\tau)}\right]^4, \qquad (19)$$

that by the ''inversion formula'' (5) implies the new relation

$$\pi i\left(\mathscr{F}-\frac{a}{2}\partial_a\mathscr{F}\right)=1-2\left[\frac{\Theta_2(0|\partial_a^2\mathscr{F})}{\Theta_3(0|\partial_a^2\mathscr{F})}\right]^4, \qquad (20)$$

showing that such a combination of theta-functions acts on $\partial_a^2\mathscr{F}$ as integral operators.

By (19) the problem of finding the explicit solutions of Eqs. (7),(15) is equivalent to the problem of finding the explicit dependence of τ as function of a and $\hat{\tau}$, respectively. In this context we note that once Eq. (7) is solved, we can use Eq. (5) to obtain the explicit dependence of $\mathscr{F}$ on a, namely

$$\mathscr{F}(a)=\frac{2i}{\pi}a^2\int_{a_0}^a dx\,\mathscr{G}_1(x)x^{-3}-\frac{iu_0}{\pi a_0^2}a^2+\frac{a_{D0}}{2a_0}a^2, \quad (21)$$

where u_0 is an arbitrary point on the compactified moduli space $\Sigma_3=\hat{\mathbf{C}}$, and $a_0\equiv a(u_0)$, $a_{D0}\equiv a_D(u_0)$. Choosing $u_0=1$ we have

$$\mathcal{F}(a)=\frac{2i}{\pi}a^2\int_{4/\pi}^{a}dx\,\mathcal{G}_1(x)x^{-3}-\frac{i\pi}{16}a^2. \tag{22}$$

In [1] it has been emphasized that the properties of the metric[1]

$$ds^2=\mathrm{Im}\left(\frac{\partial^2\mathcal{F}}{\partial a^2}\right)|da|^2=\frac{e^{-\varphi/2}}{2\pi|1-u^2|}|du|^2, \tag{23}$$

are at heart of the physics. Actually, the natural framework to investigate these properties is uniformization theory. An interesting aspect of the results in [1] is that the classical moduli space of the theory is the Riemann sphere with a puncture whereas in the quantum case one has the Riemann sphere with three punctures. Thus, since by Gauss-Bonnet formula for n-punctured Riemann spheres one has $\int_{\Sigma_n}\sqrt{g}R_g=2\pi(2-n)$, there is a ''transition'' from positively (classical moduli) to negatively (quantum moduli) curved spaces. This transition makes it evident that quantum aspects are related to deep aspects concerning uniformization theory. In particular, one can apply basic inequalities, such as the Koebe 1/4 theorem and Schwarz's lemma, which are at heart of the theory of univalent functions (i.e., uniformization, Teichmüller spaces, etc.).

We now use the prepotential $\mathcal{F}$ to construct the positive definite metric

$$ds_P^2=\frac{|\partial^3\mathcal{F}/\partial a^3|^2}{(\mathrm{Im}\partial^2\mathcal{F}/\partial a^2)^2}|da|^2=\frac{|\partial^3\mathcal{F}/\partial u\partial a\partial a|^2}{(\mathrm{Im}\partial^2\mathcal{F}/\partial a^2)^2}|du|^2$$
$$=e^{\varphi}|du|^2. \tag{24}$$

Let $H=\{w|\mathrm{Im}w>0\}$ be the upper half plane endowed with the Poincaré metric $ds_P^2=(\mathrm{Im}w)^{-2}|dw|^2$. Since τ is the inverse of the uniformizing map $J_H:H\rightarrow\Sigma_3$, it follows that e^{φ} is the Poincaré metric on Σ_3 so that φ satisfies the Liouville equation $\varphi_{u\bar{u}}=e^{\varphi}/2$.

To prove the last equality in (23) observe that $\partial_u\partial_a^2\mathcal{F}=(a'a_D''-a_Da'')/a'^2$ where $'\equiv\partial_u$. By (3) and using [4]

$$aa_D'-a_Da'=\frac{2i}{\pi}, \tag{25}$$

it follows that

$$\frac{\partial^3\mathcal{F}}{\partial u\partial a\partial a}=\frac{1}{2\pi i a'^2(1-u^2)}, \tag{26}$$

which is equivalent to (8). The last equality in (23), that is $e^{-\varphi/2}=2\pi|a'|^2|1-u^2|\mathrm{Im}(\partial_a^2\mathcal{F})$, follows by the definition of e^{φ} given in (24).

[1]We will show that e^{φ} is the Poincaré metric on Σ_3 so that $e^{-\varphi/2}$ is a ''nonchiral'' solution of the uniformizing equation (4) (see [10]).

We now observe that the A domain can be seen as a distortion by a regular (note that $\infty\notin A$) univalent function of the Poincaré disk $\Delta=\{z||z|<1\}$. This remark suggests to apply distortion theorems for hyperbolic domains. A basic point in our construction is that the Poincaré metric on A is easily identified in terms of the VEVS of ϕ and ϕ_D. In particular, it can be explicitly expressed in terms of the function $\hat{\tau}(u)$ which maps Σ_3 to D_2.

Setting $w=\hat{\tau}(u)$ in $ds_P^2=(\mathrm{Im}w)^{-2}|dw|^2$ (which is divergent for $w\in\mathbf{R}$), we obtain the Poincaré metric on the hyperbolic domain A

$$ds_P^2=-4\frac{|aa_D'-a_Da'|^2}{(a_D\bar{a}-\bar{a}_Da)^2}|du|^2=-\frac{16}{\pi^2}\frac{1}{(a_D\bar{a}-\bar{a}_Da)^2}|du|^2$$
$$=e^{\varphi_A}|du|^2, \tag{27}$$

where we used (25). We now apply the general construction for hyperbolic domains (see for example [11]) to show that[2]

$$1\leqslant e^{\varphi_A(u,\bar{u})}(\Delta_A u)^2\leqslant 4, \tag{28}$$

where $\Delta_A u$ denotes the Euclidean distance from $u\in A$ to the critical curve $\mathcal{C}=\partial A$.

We note that a similar geometrical uncertainty relation appears in the description of the cutoff $(\Delta z_{\mathrm{min}})^2$ in two-dimensional quantum gravity [10]. In particular, it has been shown that in a hyperbolic domain D one has

$$(\Delta z)^2\geqslant\frac{\epsilon}{4}e^{-\sigma_D}(\Delta_D z)^2, \tag{29}$$

where $\epsilon=(\Delta s_{\mathrm{min}})^2=e^{\varphi_D+\sigma_D}(\Delta z_{\mathrm{min}})^2$ is the minimal invariant length, e^{φ_D} is the Poincaré metric on D and σ_D is the Liouville field. Equation (29) provides the relation between the minimal length in configuration space z, the structure of the boundary ∂D, and the Liouville field.

To prove (28) we need Schwarz's lemma and the Koebe 1/4 theorem (see for example [12]). Let $f(z)$ be an analytic and regular function in Δ vanishing in zero. Schwarz's lemma states that if $|f(z)|\leqslant 1$, $z\in\Delta$, then $|f(z)/z|\leqslant 1$, $z\in\Delta$, where equality can hold only if f and z differ by a phase. Setting $z=0$ in the expansion $f(z)=f'(0)z+f''(0)z^2/2!+\cdots$, we obtain

$$|f'(0)|\leqslant 1. \tag{30}$$

Let u be a point in A and F a conformal map of A onto Δ. Since the Poincaré metric on Δ is $ds_P^2=4(1-|z|^2)^{-2}|dz|^2$, we have

$$e^{\varphi_A(u,\bar{u})}=4|F'(u)|^2. \tag{31}$$

Setting $f(z)=F(u+z\Delta_A u)$, by (30) we have $|F'(u)|\Delta_A u\leqslant 1$, so that (31) implies $e^{\varphi_A(u,\bar{u})}(\Delta_A u)^2\leqslant 4$. The other inequality in (28) is an application of the Koebe 1/4 theorem, a consequence of the area theorem. It states that if f is a regular and univalent function in Δ with normalization

[2]We stress that similar results hold in the complementary domain $\hat{\mathbf{C}}\backslash A$.

$f(0)=0$, $f'(0)=1$ (functions with such properties constitute the $\mathscr{S}$ class), then $f(\Delta)$ contains the open disk of radius 1/4 center 0. Let g be a conformal mapping of Δ onto A with $g(0)=u$; we have

$$e^{\varphi_A(u,\bar{u})}=4|g'(0)|^{-2}. \quad (32)$$

We now set $G(z)=[g(z)-g(0)]/g'(0)$, so that $G\in\mathscr{S}$, and observe that since by construction $g(\Delta)$ contains the open disk of radius $\Delta_A u$ center u, it follows that $G(\Delta)$ contains the open disk of radius $\Delta_A u/|g'(0)|$ center 0. On the other hand, by the Koebe 1/4 theorem, $G(\Delta)$ contains the open disk of radius 1/4 with center 0, so that $\Delta_A u/|g'(0)|\geqslant 1/4$ and (32) implies $e^{\varphi_A(u,\bar{u})}(\Delta_A u)^2\geqslant 1$.

Up to now we considered unit where $\hbar=1$. However, the structure of (28) suggests performing a dimensional analysis setting the Euclidean distance dimensionless. Since $\mathscr{F}$ has the dimensions of $\hbar$ and being $a_D=\partial_a\mathscr{F}$, by (28) we have

$$\hbar\Delta_A\langle\mathrm{tr}\phi^2\rangle\leqslant\pi\mathrm{Im}(\langle\phi\rangle\overline{\langle\phi_D\rangle})\leqslant 2\hbar\Delta_A\langle\mathrm{tr}\phi^2\rangle, \quad (33)$$

where we used the fact that $\mathrm{Im}a_D/a<0$. Let us denote by $\mathscr{C}_d$ the curve in A on which $\Delta_A\langle\mathrm{tr}\phi^2\rangle=d$. On $\mathscr{C}_{1/2}$ Eq. (33) has the structure

$$\frac{\hbar}{2}\leqslant\pi\mathrm{Im}(\langle\phi\rangle\overline{\langle\phi_D\rangle})\leqslant\hbar. \quad (34)$$

In order to investigate the physical meaning of (33) we should first discuss two aspects. The first one concerns the structure of the Euclidean distance $\Delta_A u$. Let us denote by v the points in $\mathscr{C}$ and by x the values of $\hat{\tau}$ such that $v=\mathscr{G}_2(x)$, so that by (13) $\mathrm{Im}x=0$ and $x\in[-1,1]$. Note that by definition $\Delta_A u=|u-v_0|$, where $u\in A$ and v_0 is the minimum of $|u-v|$. In order to determine v_0 one should first solve the equation

$$\partial_x|\mathscr{G}_2(\hat{\tau})-\mathscr{G}_2(x)|=0, \quad (35)$$

which can be also seen as an equation for x, so that $v_0=\mathscr{G}_2(x_0)$, $x_0=x_0(\hat{\tau})$.

The second point concerns the parametrization of quantum vacua. In this context we stress that since $\hat{\tau}(u)$ is a univalent function, that is the equality $\hat{\tau}(u_1)=\hat{\tau}(u_2)$ implies $u_1=u_2$, it follows that the moduli space of quantum vacua can be equivalently identified with the $\hat{\tau}$ space (or its fundamental domain D_2). Therefore there is the correspondence

$$u-\text{moduli space}\Leftrightarrow\hat{\tau}-\text{moduli space}.$$

In the following we will use the subscript $\hat{\tau}$ to emphasize the $\hat{\tau}$ parametrization of the vacuum states. The above remarks suggest writing Eq. (33) in the form (here $\hat{\tau}_0\equiv x_0$)

$$\hbar|\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}-\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}_0}|\leqslant\pi\mathrm{Im}(\langle\phi\rangle_{\hat{\tau}}\overline{\langle\phi_D\rangle_{\hat{\tau}}})\leqslant 2\hbar|\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}} -\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}_0}|. \quad (36)$$

The fact that $\tau=\partial_a^2\mathscr{F}$ is dimensionless implies that $(2-a\partial_a)\mathscr{F}=\Lambda\partial_\Lambda\mathscr{F}$, so that Eq. (5) is equivalent to $\Lambda\partial_\Lambda\mathscr{F}=-8\pi i b_1\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}$ where, as stressed in [13,14], $b_1=1/4\pi^2$ is the one-loop coefficient of the beta function. Thus Eq. (36) implies inequalities involving $\Lambda\partial_\Lambda\mathscr{F}$. In [13] it has been suggested that the relation $\Lambda\partial_\Lambda\mathscr{F}=-8\pi i b_1\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}$ should be understood in terms of renormalization group ideas (see [13,14] for relevant generalizations of this formula and [15] for other interesting consequences). In a forthcoming paper [16] we will argue that Eq. (36) is related to the beta function $\Lambda\partial_\Lambda\tau$ whose structure can be investigated in the framework of the relation (5) and Eq. (8). We note that $\mathscr{C}$ can be seen as the curve which separates the local and asymptotic regions. Once the nonperturbative beta function is constructed the ray of convergence of the local expansions should be related to the structure of the A domain and its boundary $\mathscr{C}$. In particular, Eq. (36) should also play a role in investigating Borel summability.

There is another aspect which should be mentioned in discussing Eq. (33). In ordinary quantum mechanics one has $\sqrt{\langle x^2\rangle_\psi-\langle x\rangle_\psi^2}\sqrt{\langle p^2\rangle_\psi-\langle p\rangle_\psi^2}\geqslant\hbar/2$ (note that the square roots may be seen as Euclidean distances). Fields ϕ and ϕ_D play the role of x and p, respectively, and $\mathscr{F}$ is the analog of the action. Similarly, one should consider the correspondence $\langle x^2\rangle_\psi\rightarrow\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}$ and investigate whether $|\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}-\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}_0}|\sim|\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}-\langle\hat{\Theta}\rangle_{\hat{\tau}}|$, with $\hat{\Theta}$ some field operator [recall that by (35) $\hat{\tau}_0$ is $\hat{\tau}$ dependent]. To have a deeper analogy with the uncertainty relation one should identify $\langle\,\mathrm{tr}\phi_D^2\rangle_{\hat{\tau}}$ and investigate the structure of the relation between $\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}_0}$, $\langle\,\mathrm{tr}\phi^2\rangle_{\hat{\tau}}$, $\langle\phi\rangle_{\hat{\tau}_0}$, $\langle\phi\rangle_{\hat{\tau}}$ and their duals. The asymptotic behavior $\langle\,\mathrm{tr}\phi^2\rangle_{\hat{\tau}}\sim\langle\phi\rangle_{\hat{\tau}}^2/2$ suggests that a similar relation should exist. In this context it is interesting to note that $\langle\phi\rangle_{\hat{\tau}_0}^2$ appears in evaluating the Euclidean distance $|\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}-\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}_0}|$. In particular, deriving Eq. (5) with respect to $\hat{\tau}$ we have

$$\partial_{\hat{\tau}}\langle\mathrm{tr}\phi^2\rangle_{\hat{\tau}}=\frac{1}{8\pi i b_1}\langle\phi\rangle_{\hat{\tau}}^2, \quad (37)$$

so that Eq. (35) becomes

$$a^2(v)[\overline{\mathscr{G}_2(\hat{\tau})}-\overline{\mathscr{G}_2(x)}]=\overline{a^2(v)}[\mathscr{G}_2(\hat{\tau})-\mathscr{G}_2(x)]. \quad (38)$$

Finding the solution of Eq. (38) is an interesting open problem which should be possible to solve once the solution of (15) is known. As we said, solving Eq. (15) is equivalent to finding the inverse of the map $\hat{\tau}=\mathscr{H}(\tau)$ whose explicit expression is given by (2),(19). All these aspects show that the uncertainty relations (33) are described by the Schwarzian equation (15).

Another aspect concerns the geometrical origin of the lower and upper bounds in (33). These seem to be a consequence of the good infrared and ultraviolet properties of negatively curved spaces. These aspects have been investigated in a different context by Callan and Wilczek [17]. Here we have seen that the Koebe 1/4 theorem and Schwarz's lemma applied to hyperbolic geometry explain the origin of such geometrical regulators. We also observe that the role of the Koebe 1/4 theorem in hyperbolic geometry seems related to the crucial mechanism which arises in the compactification of moduli spaces of Riemann surfaces. In particular, the fact that in the Deligne-Knudsen-Mumford (DKM) compactification of moduli spaces of punctured Riemann spheres (configuration space of anyons) ''punctures never collide'' can be seen as a request for the Weyl-Petersson Hamiltonian to remain self-adjoint at the DKM boundary. This implies the mass gap and the exclusion principle for anyons (punctures) [18].

There is another physical application of our results. Namely a is a nowhere vanishing function of u so that one should investigate the role of the lower bound for $|a|$. This is relevant in order to recover the structure of the a-moduli space and the properties of $M_{n_m n_e}$ (such as the structure of the mass gap). Similar arguments should be also useful to better understand some aspects concerning confinement. In this context we recall that according to Nehari theorem [19] a sufficient condition for the univalence of a function g defined on Δ is

$$(1-|z|^2)^2|\{g,z\}|\leqslant 2, \tag{39}$$

whereas the necessary condition is

$$(1-|z|^2)^2|\{g,z\}|\leqslant 6. \tag{40}$$

It can be shown that the constant 2 in (39) cannot be replaced by any larger one. An interesting question is to find the sharp inequality for the case at hand.

In conclusion we note that our investigation is related with the theory of quasidisks. They have interesting structures. For example a generic quasidisk has a fractal boundary [20]. Quasidisks also appear in some nonperturbative aspects of string theory [21].

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