

Interpretation of time-reparametrization-invariant quantum mechanics: An exactly soluble model

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The classical and quantum dynamics of simple time-reparametrization-invariant models containing two degrees of freedom are studied in detail. Elimination of one “clock” variable through the Hamiltonian constraint leads to a description of time evolution for the remaining variable which is essentially equivalent to the standard quantum mechanics of an unconstrained system. In contrast with a similar proposal of Rovelli, evolution is with respect to a geometrical proper time, and the Heisenberg equation of motion is exact. The physical phase space contains no coordinate associated with the eliminated “clock” variable. Therefore, time evolution is *not* with respect to the observable readings of a physical clock. Rather, the eliminated variable can be construed as intrinsic to a particular observer, and, hence, as unobservable, in principle, by this observer. The possibility of a “test clock,” which would reveal time evolution while contributing negligibly to the Hamiltonian constraint, is examined, and found to be viable in the semiclassical limit of large quantum numbers. [S0556-2821(96)05112-0]

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I. INTRODUCTION

Among the many problems which attend the canonical quantization of general relativity is the fact that all gauge-invariant quantities, which might qualify as observables, necessarily commute with the Hamiltonian, and are, therefore, constants of motion. This intriguing problem has received wide attention, which has been comprehensively reviewed, for example, in [1,2] (see also [3,4]). It seems possible to obtain some insight into the “problem of time” from simple quantum-mechanical models which share with general relativity the essential feature of time-reparametrization invariance, and the purpose of the present work is to explore in some detail the interpretation of time evolution in one such model.

It is widely thought that the time evolution of physical quantities which is apparent to actual observers in our Universe should be understood, in some sense, as evolution relative to the state of a physical clock, which provides an observer’s local definition of time. A particular implementation of this idea has been proposed by Rovelli [5–7], who defines “evolving constants of motion” by eliminating the arbitrary time coordinate t in favor of the value of a physical clock variable. Thus, in [5], he considers a model of two harmonic oscillators having the same frequency, whose equations of motion can be solved for two functions, say, $x_1(t)$ and $x_2(t)$. Neither of these functions gives a gauge-invariant description of time evolution, but, by eliminating t , Rovelli arrives at a new function $x_1(x_2)$ which is gauge invariant (or almost so — see Sec. IV below) provided that x_2 is regarded as a real parameter, rather than as a dynamical variable. Classically, at least, this defines a family of observables $x_1(s)$

which can be interpreted as the value of x_1 when $x_2=s$. However, the quantum-mechanical operator corresponding to $x_1(s)$ is approximately self-adjoint (and approximately obeys a standard Heisenberg equation of motion) only when restricted to a subspace of the physical Hilbert space which Rovelli calls the “Schrödinger regime.” Since this is a large-quantum-number regime, he suggests that the notion of time evolution does not exist at a fundamental level, and emerges only as a semiclassical property of macroscopic objects.

In this paper, we study a model whose dynamical variables are an oscillator and a free particle, the latter serving as a clock whose reading is (classically) a measure of geometrical proper time. The classical and quantum dynamics of this model are discussed in Sec. II where, in order to obtain a well-defined quantum theory, the free particle is treated as the low-frequency limit of a second oscillator. We find it possible to define a parametrized family of observables $x(\tau)$ evolving with a parameter τ which classically coincides with the geometrical proper time. The linearity of the clock variable turns out to be inessential for this purpose. Moreover, this evolution is governed by an exact Heisenberg equation of motion, and is not restricted to a macroscopic regime. The introduction of a clock variable permits time evolution by relaxing the energy constraint on the oscillator, and indeed the energy of the clock can saturate this constraint. In Sec. III, we investigate the possibility of a “test clock” by restricting the state of the clock to a narrow range, which approximately reproduces the constraint on the oscillator alone. This restriction imposes a limit on the resolution with which the value of $x(\tau)$ can be determined, but there is a semiclassical limit in which both the energy of the clock and the eigenvalues of $x(\tau)$ are fairly sharply defined. The interpretation of time is discussed in detail in Sec. IV where, in particular, we point out that the clock cannot itself be

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regarded as a physically observed object, but should rather be taken as representing the observer's intrinsic sense of time.

II. CLASSICAL AND QUANTUM DYNAMICS OF AN OSCILLATOR AND A CLOCK

Consider the Lagrangian

$$L = \frac{1}{2\lambda}(\dot{x}^2 + \dot{q}^2) - \frac{\lambda}{2}(\omega^2 x^2 - 2E_0), \quad (1)$$

defined on a $(0+1)$ -dimensional manifold with "time" coordinate t . The only geometrical variable is the lapse function $\lambda(t)$, which we take to be strictly positive. The action $\int L dt$ is easily seen to be invariant under a change of time coordinate, with $x'(t') = x(t)$, $q'(t') = q(t)$, and $\lambda'(t') = (dt/dt')\lambda(t)$, where t' is any increasing function of t . The equations of motion

$$\frac{1}{\lambda} \frac{\partial}{\partial t} \left(\frac{1}{\lambda} \frac{\partial x}{\partial t} \right) = -\omega^2 x, \quad (2)$$

$$\frac{1}{\lambda} \frac{\partial}{\partial t} \left(\frac{1}{\lambda} \frac{\partial q}{\partial t} \right) = 0, \quad (3)$$

obtained by varying x and q , have the general solution

$$x(t) = a_1 \cos \psi(t) + a_2 \sin \psi(t), \quad (4)$$

$$q(t) = c_1 + c_2 \psi(t), \quad (5)$$

so that the clock variable q is linear in the proper time $\omega^{-1} \psi(t) = \int_0^t \lambda(t') dt'$. The constraint

$$\left(\frac{\dot{x}}{\lambda} \right)^2 + \left(\frac{\dot{q}}{\lambda} \right)^2 + \omega^2 x^2 = 2E_0, \quad (6)$$

obtained by varying λ , implies the relation

$$C \equiv a_1^2 + a_2^2 + c_2^2 = \rho_0^2, \quad (7)$$

where $\rho_0^2 = 2E_0/\omega^2$.

For this model, the solution space [ignoring the factor Λ of gauge functions $\lambda(t)$, all of whose points are gauge equivalent] is $S^2 \times R$. With the parametrization

$$a_1 = \rho_0 \cos \chi \cos \gamma, \quad (8)$$

$$a_2 = -\rho_0 \cos \chi \sin \gamma, \quad (9)$$

$$c_2 = \rho_0 \sin \chi, \quad (10)$$

the angles χ and γ ($-\pi/2 \leq \chi \leq \pi/2$, $0 \leq \gamma < 2\pi$) provide coordinates on S^2 , while c_1 is the coordinate on R . The general solutions (4) and (5) can be written as

$$x(t) = \rho_0 \cos \chi \cos[\psi(t) + \gamma], \quad (11)$$

$$q(t) = c_1 + \rho_0 \sin \chi \psi(t), \quad (12)$$

and the presymplectic form on the solution space is

$$\Omega = \omega(da_1 \wedge da_2 + dc_1 \wedge dc_2)$$

$$= \frac{1}{2} \omega \rho_0^2 \sin(2\chi) d\chi \wedge d\gamma - \omega \rho_0 \cos \chi d\chi \wedge dc_1. \quad (13)$$

As can be seen from (11) and (12), time evolution is generated by the Hamiltonian vector field

$$X_C = \frac{\partial}{\partial \gamma} + \rho_0 \sin \chi \frac{\partial}{\partial c_1}, \quad (14)$$

associated with the constraint C . As expected, we find that X_C annihilates Ω , $i_{X_C} \Omega = 0$, so the reduced phase space is found by identifying all points along each gauge orbit, which is an integral curve of X_C . Equivalently, points of the reduced phase space correspond to distinct gauge orbits. The angle χ is constant along each gauge orbit. For each value of χ (except $\chi = 0, \pm \pi/2$), the orbits are helical curves on the cylinder $S^1 \times R$ coordinatized by γ and c_1 , with a period along R of $2\pi \rho_0 \sin \chi$. Distinct orbits can therefore be labeled by the values of c_1 in an interval of length $2\pi \rho_0 \sin \chi$, the end points of which are identified, since they belong to the same orbit. The set of distinct orbits is thus topologically S^1 . For $\chi = 0$, the gauge orbits are circles on the cylinder, and there is a distinct orbit for each $c_1 \in R$. For $\chi = \pm \pi/2$, the amplitude of oscillation in (11) vanishes, so all values of γ are equivalent. In this case, the S^1 of gauge orbits collapses to a point.

We see that the reduced phase space is the union of three sets.

(i) The open disk D_+ , which is the direct product of the interval $0 < \chi \leq \pi/2$ with the circle S^1 , the circle at $\chi = \pi/2$ being contracted to a point. Physically, states in this region are those in which the clock runs forward. The gauge-invariant observables which distinguish the states are χ , which set the amplitude of the x oscillation and also (via the constraint) the rate of the clock, and c_1 , which characterizes the phase of oscillation at which the clock reads zero.

(ii) The open disk D_- , which is similar to D_+ , but corresponds to the interval $-\pi/2 \leq \chi < 0$ and contains those states in which the clock runs backwards.

(iii) The line R_0 , with $c_1 \in (-\infty, +\infty)$ and $\chi = 0$. In this region, the oscillation has its maximum amplitude, while the clock has a constant reading c_1 , which is the gauge-invariant observable distinguishing the states.

This reduced phase space is *not* a manifold — a fact which presents serious difficulties for the method of geometrical quantization, and we have not succeeded in obtaining a fully consistent quantum version of this model. A quantum theory of a related model will be described below. Within the classical theory, we can construct gauge-invariant observables which evolve with proper time, as measured by the clock, by restricting attention to the subspace $D_+ \cup D_-$. Consider that region of the solution space where $\chi \neq 0$ and identify the points (χ, γ, c_1) and $(\chi, \gamma, c_1 + 2\pi \rho_0 \sin \chi)$. On the resulting space, we replace the coordinates c_1 and γ with

$$c = \frac{c_1}{\rho_0 \sin \chi}, \quad (15)$$

$$\alpha = \left(\gamma - \frac{c_1}{\rho_0 \sin \chi} \right) \bmod 2\pi. \tag{16}$$

In terms of these coordinates, the generator (14) of gauge orbits is $X_c = \partial/\partial c$ and the presymplectic form (13) becomes $\Omega = (\omega \rho_0^2/2) \sin(2\chi) d\chi \wedge d\alpha$. As expected, this Ω is independent of c , and is also the symplectic form on the region $D_+ \cup D_-$ of the reduced phase space, now coordinatized by χ and α .

With the coordinates (χ, α, c) , the solutions to the equations of motion read

$$x(t) = \rho_0 \cos \chi \cos[\psi(t) + \alpha + c], \tag{17}$$

$$q(t) = \rho_0 \sin \chi [\psi(t) + c]. \tag{18}$$

The quantity $\tau = [\psi(t) + c]/\omega$ can be identified as the proper time which has elapsed since the clock read zero and (with an obvious economy of notation) the value of x at this time is given by

$$x(\tau) = \rho_0 \cos \chi \cos(\omega\tau + \alpha) = X \cos(\omega\tau) + \omega^{-1} P \sin(\omega\tau), \tag{19}$$

where

$$X = \rho_0 \cos \chi \cos \alpha, \tag{20}$$

$$P = -\omega \rho_0 \cos \chi \sin \alpha. \tag{21}$$

Since they do not depend on c , the quantities X and P are gauge invariant. Thus, if τ is regarded as a parameter, rather than as standing for the expression $[\psi(t) + c]/\omega$, then $x(\tau)$ is a gauge-invariant function of τ . Classically, this function can be interpreted as ‘‘the value of x at proper time τ ,’’ given that $\tau=0$ is located at the event $q=0$. It is similar to the ‘‘evolving constants of the motion’’ defined by Rovelli [5,6] but with an important difference: whereas Rovelli’s evolution is with respect to the actual value of a physical clock variable, τ refers (at the classical level) to the geometrical proper time. The symplectic form can be expressed in terms of X and P , with the satisfactory result $\Omega = dX \wedge dP$, so X and P have the canonical Poisson bracket algebra. Moreover, the evolution of $x(\tau)$ is easily seen to be governed by the usual equation of motion

$$\frac{dx(\tau)}{d\tau} = \{x(\tau), H_0\}, \tag{22}$$

where

$$H_0 = \frac{1}{2} (P^2 + \omega^2 X^2) = \frac{1}{2} \left[\left(\frac{dx(\tau)}{d\tau} \right)^2 + \omega^2 x(\tau)^2 \right] \tag{23}$$

is the Hamiltonian for the oscillator alone.

To obtain a quantum version of this model in a controlled manner, we consider the Lagrangian

$$L = \frac{1}{2\lambda} (\dot{x}^2 + \dot{q}^2) - \frac{\lambda}{2} (\omega^2 x^2 + N^{-2} \omega^2 q^2 - 2E_0), \tag{24}$$

where N is an integer, with a view to recovering the model (1) in the limit $N \rightarrow \infty$. Somewhat surprisingly, the results are

essentially independent of N . The classical analysis is similar to that given above. If the solutions to the equations of motion are written as

$$x(t) = a_1 \cos \psi(t) + a_2 \sin \psi(t), \tag{25}$$

$$q(t) = b_1 \cos[N^{-1} \psi(t)] + b_2 \sin[N^{-1} \psi(t)], \tag{26}$$

then the quantities

$$a = \sqrt{\frac{\omega}{2}} (a_1 + i a_2), \tag{27}$$

$$b = \sqrt{\frac{\omega}{2N}} (b_2 - i b_1) \tag{28}$$

have Poisson brackets $\{a, a^*\} = \{b, b^*\} = -i$, which will shortly be promoted to commutators. The constraint $a_1^2 + a_2^2 + N^{-2} (b_1^2 + b_2^2) = \rho_0^2$ can be solved in terms of the coordinates χ , α , and c , replacing (17) and (18) with

$$x(t) = \rho_0 \cos \chi \cos[\psi(t) + \alpha + c], \tag{29}$$

$$q(t) = N \rho_0 \sin \chi \sin\{N^{-1} [\psi(t) + c]\}. \tag{30}$$

For any finite value of N , the reduced phase space of this model is S^2 , the angles χ and α ($0 \leq \chi \leq \pi/2$, $0 \leq \alpha \leq 2\pi$) providing coordinates such that the points $\chi=0, \pi/2$ are opposite poles. In particular, for $\chi=0$, the angle α in (29) is indistinguishable from c , so all values of α are gauge equivalent. Each point of this phase space, of course, corresponds to a periodic oscillation, for which the values $\psi(t) = 0, 2N\pi$ may be identified. The limit $N \rightarrow \infty$ is actually singular in the following sense. To achieve a finite limit in (30), with $\chi \neq 0$, we may assume that $[\psi(t) + c]$ either has a finite value, or differs by a finite amount from $N\pi$. In the limit, therefore, each point of the phase space S^2 represents two distinct solutions, corresponding to the half-periods of oscillation in which q is an increasing or a decreasing function of $\psi(t)$. Clearly, the two limits of (30) differ by a sign, which can be regarded as the sign of χ , and correspond to the two regions D_{\pm} of the phase space described above. To reproduce the region R_0 , we must first take $N \rightarrow \infty$ with $\chi \neq 0$ and then take $\chi \rightarrow 0$, with c given by (15).

For finite values of N , the reduced phase space S^2 is a manifold, and the model can be quantized straightforwardly. However, since this manifold includes the point $\chi=0$, we may anticipate some difficulty in interpreting the variables X and P [Eqs. (20) and (21)] as operators on the physical Hilbert space. Since all values of α are gauge equivalent at $\chi=0$, X and P [and, hence, also $x(\tau)$] are not gauge-invariant functions on the reduced phase space, although they *are* gauge-invariant functions on any region which excludes the point $\chi=0$. In any such region, the above expressions for the symplectic form are still valid, so X and P still have the canonical Poisson algebra.

Models of this kind can be quantized in several, more or less equivalent, ways (see, for example [5,8–10]). The essential result is that one can identify gauge-invariant functions $(s_1, s_2, s_3) = (\omega \rho_0^2/4) [\sin(2\chi) \sin \alpha, \cos(2\chi), \sin(2\chi) \cos \alpha]$ on the reduced phase space, whose Poisson algebra,

$\{s_i, s_j\} = \epsilon_{ijk} s_k$, is the Lie algebra of $SU(2)$, and the Casimir-invariant $s_1^2 + s_2^2 + s_3^2$ is proportional to the constraint \mathcal{C} . The physical Hilbert space is therefore an irreducible representation of $SU(2)$ whose dimension is set (up to operator ordering ambiguities) by the value of E_0 . For our purposes, it is convenient to construct this Hilbert space according to the Dirac prescription, taking a and b as the basic variables (a similar route is followed in [10]).

On promoting a and b to quantum ladder operators, with the usual commutators $[a, a^\dagger] = [b, b^\dagger] = 1$, $[a, b] = [a, b^\dagger] = 0$, we obtain an unconstrained Hilbert space, spanned by the vectors $|m, n\rangle$, where m and n label the energy levels of the a and b oscillators respectively.

The physical Hilbert space is the subspace of vectors satisfying the constraint

$$(a^\dagger a + N^{-1} b^\dagger b) |\psi\rangle = \bar{\nu} |\psi\rangle, \quad (31)$$

where $\bar{\nu} = \omega \rho_0^2 / 2 = E_0 / \omega$. Here, it is assumed that any constant arising from factor ordering in the constraint operator has been absorbed into E_0 . Quantization clearly requires that $N\bar{\nu}$ be an integer, and we define

$$\bar{\nu} = \nu + \nu' / N, \quad (32)$$

where ν and ν' are integers, with $0 \leq \nu' < N$. This physical Hilbert space is spanned by the vectors

$$|m\rangle\rangle = |m, N(\nu - m) + \nu'\rangle, \quad 0 \leq m \leq \nu. \quad (33)$$

We would now like to realize X and P as gauge-invariant operators, acting in the physical Hilbert space, but, as anticipated, this is not quite straightforward. Classically, we can define variables A and A^* , with the Poisson brackets $\{A, A^*\} = -i$, by

$$A = \sqrt{\frac{\omega}{2}} \left(X + \frac{i}{\omega} P \right) = a \left(\frac{b^*}{\sqrt{b^* b}} \right)^N. \quad (34)$$

Quantum mechanically, the operator ordering

$$A = a [(b^\dagger b)^{-1/2} b^\dagger]^N, \quad (35)$$

$$A^\dagger = a^\dagger [b (b^\dagger b)^{-1/2}]^N \quad (36)$$

ensures that these operators have the expected properties $A|m\rangle\rangle = \sqrt{m}|m-1\rangle\rangle$ and $A^\dagger|m\rangle\rangle = \sqrt{m+1}|m+1\rangle\rangle$, except that the action of A^\dagger on the maximal state $|\nu\rangle\rangle$ is not well defined. To proceed, we introduce the regularized operators

$$A_\epsilon = a [(b^\dagger b + \epsilon)^{-1/2} b^\dagger]^N, \quad (37)$$

$$A_\epsilon^\dagger = a^\dagger [b (b^\dagger b + \epsilon)^{-1/2}]^N. \quad (38)$$

These are well-defined, gauge-invariant operators, and we find, in particular, that $A_\epsilon^\dagger |\nu\rangle\rangle = 0$. We can now define operators A and A^\dagger by

$$A|m\rangle\rangle = \lim_{\epsilon \rightarrow 0} A_\epsilon|m\rangle\rangle = \sqrt{m}|m-1\rangle\rangle, \quad (39)$$

$$A^\dagger|m\rangle\rangle = \lim_{\epsilon \rightarrow 0} A_\epsilon^\dagger|m\rangle\rangle = \sqrt{m+1}(1 - \delta_{m,\nu})|m+1\rangle\rangle. \quad (40)$$

These operators are also well defined and gauge invariant, but they have the anomalous commutator $[A, A^\dagger] = 1 - \theta$, where θ projects onto the maximal state: $\theta|m\rangle\rangle = (1 + \nu) \delta_{m,\nu} |m\rangle\rangle$. The operators

$$X = \left(\frac{1}{2\omega} \right)^{1/2} (A + A^\dagger), \quad (41)$$

$$P = -i \left(\frac{\omega}{2} \right)^{1/2} (A - A^\dagger) \quad (42)$$

are well defined, gauge invariant, and self-adjoint, but they have the anomalous commutator $[X, P] = i[A, A^\dagger] = i(1 - \theta)$. The anomalous term is nonzero only when acting on the maximal state $|\nu\rangle\rangle$, where the clock has the smallest energy allowed by the constraint (31). Classically, the interpretation of X and P as the position and momentum of the oscillator ‘‘at time $\tau=0$ ’’ is ambiguous in the state $\chi=0$, where the clock permanently reads zero, and therefore does not distinguish this instant of time, and the anomaly can be understood as reflecting this fact in the quantum theory. Formally, the classical versions of (37) and (38) correspond to regularized variables

$$X_\epsilon = \rho_0 \cos \chi \cos \alpha \left[1 + \frac{2\epsilon}{N\omega\rho_0^2 \sin^2 \chi} \right]^{-N/2} \quad (43)$$

and $P_\epsilon = -\omega\rho_0 \tan \alpha X_\epsilon$. These are truly gauge-invariant quantities, having the unique values $X_\epsilon = P_\epsilon = 0$ at $\chi=0$, but do not have the canonical Poisson algebra. However, they differ significantly from the original X and P only where $\sin^2 \chi$ is not much greater than $\epsilon / \omega\rho_0^2$, and when ϵ is sufficiently small, this is a small neighborhood of the point $\chi=0$. It is straightforward to show that the Heisenberg equation of motion

$$\frac{dx(\tau)}{d\tau} = i[H_0, x(\tau)] \quad (44)$$

reproduces (19) in the expected way, provided that the Hamiltonian has the factor ordering $H_0 = \omega A^\dagger A$.

At this point, we have a formalism which, taken at face value, is equivalent to ordinary time-dependent quantum mechanics. There is a physical Hilbert space, spanned by the vectors $|m\rangle\rangle$, a set of operators (X, P) on this space and a Hamiltonian H_0 which generates time evolution through the standard equation of motion (44). Whether this formalism should be taken at face value is, of course, another matter, and the interpretation of the model will be discussed in Sec. IV. Apparently, one can ask, and answer, questions such as ‘‘given that x was determined to have the value x_1 at time τ_1 , what is the probability that it has the value x_2 at time τ_2 ?.’’ According to standard quantum mechanics, this question is legitimate only if x_1 and x_2 are eigenvalues of $x(\tau_1)$ and $x(\tau_2)$. Because the classical range of x is restricted by the constraint and, correspondingly, the physical Hilbert space is finite-dimensional, these eigenvalues form a finite, discrete spectrum, which is easily found. Using (41), we can express $x(\tau)$ as $x(\tau) = (2\omega)^{-1/2} (A e^{-i\omega\tau} + A^\dagger e^{i\omega\tau})$. The properties (39) and (40) then imply that

$$|x_j, \tau\rangle\rangle = N_j \sum_{m=0}^{\nu} (2^m m!)^{-1/2} H_m(\omega^{1/2} x_j) e^{im\omega\tau} |m\rangle\rangle \quad (45)$$

is an eigenvector of $x(\tau)$ with eigenvalue x_j , where H_m is the Hermite polynomial, N_j is a normalizing constant and, in order to truncate the series at $m = \nu$, $\omega^{1/2} x_j$ must be a zero of $H_{\nu+1}$. Since there are $\nu+1$ of these zeros, the states (45) span the $(\nu+1)$ -dimensional Hilbert space. Also, since $p(\tau) = \dot{x}(\tau) = \omega x(\tau + \pi/2\omega)$, the eigenvalues of $p(\tau)$ are ωx_j .

The quantum dynamics of this model can be illustrated by considering the coherent state

$$|z\rangle\rangle = C(z) e^{zA^\dagger} |0\rangle\rangle = C(z) \sum_{m=0}^{\nu} \frac{z^m}{\sqrt{m!}} |m\rangle\rangle, \quad (46)$$

where $C(z)$ is a normalizing factor. The time-dependent wave function for this state is

$$\begin{aligned} \psi(x_j, \tau; z) &= \langle\langle x_j, \tau | z \rangle\rangle \\ &= N_j C(z) \sum_{m=0}^{\nu} \frac{1}{m!} H_m(\omega^{1/2} x_j) \left(\frac{z(\tau)}{\sqrt{2}} \right)^m, \end{aligned} \quad (47)$$

where $z(\tau) = e^{-i\omega\tau} z$. If $z = \sqrt{\omega/2}[x_0 + (i/\omega)p_0]$, then $z(\tau) = \sqrt{\omega/2}[x_0(\tau) + (i/\omega)p_0(\tau)]$, where $(x_0(\tau), p_0(\tau))$ is the classical trajectory passing through (x_0, p_0) . For an unconstrained oscillator (corresponding to the limit $\nu = \infty$), the coherent state is, of course, a Gaussian wave packet, with $|\psi|^2 \propto \exp[-\omega(x - \sqrt{2/\omega} \text{Re} z)^2]$, whose peak follows the classical trajectory. The wave function (47) is defined only at the discrete values x_j , and is not simply a function of $x_j - z(\tau)$. Nevertheless, if x_0 is well within the range defined by the largest and smallest x_j then, even for quite small values of ν , these discrete values follow the Gaussian packet rather closely, as illustrated in Fig. 1.

III. TEST CLOCKS

In classical general relativity, one can assess the physical characteristics of a spacetime by examining the trajectories of “test particles” which follow timelike geodesics, but do not contribute to the stress tensor. Here, we consider the possibility of a “test clock,” which might be used to reveal the time evolution of a quantum universe, without itself contributing significantly to the quantum dynamics.

In the clockless model, obtained from (1) or (24) by deleting the variable q , there seems classically to be a sense in which remaining oscillator evolves with proper time as $x(\tau) = \sqrt{2E_0/\omega^2} \cos(\omega\tau)$, even though $x(\tau)$ cannot be regarded as a gauge-independent observable. By adding the clock, we are able to obtain a genuine observable $x(\tau)$, but the energy constraint which applied to the original oscillator is relaxed by the presence of the clock, which can itself saturate the constraint. The idea now is to see whether the apparent time dependence of the clockless model can be revealed by restricting attention to those states in which the clock has only small energies, so that the energy constraint of the clockless model is approximately realized. It may be anticipated that this restriction will impose a limit on the

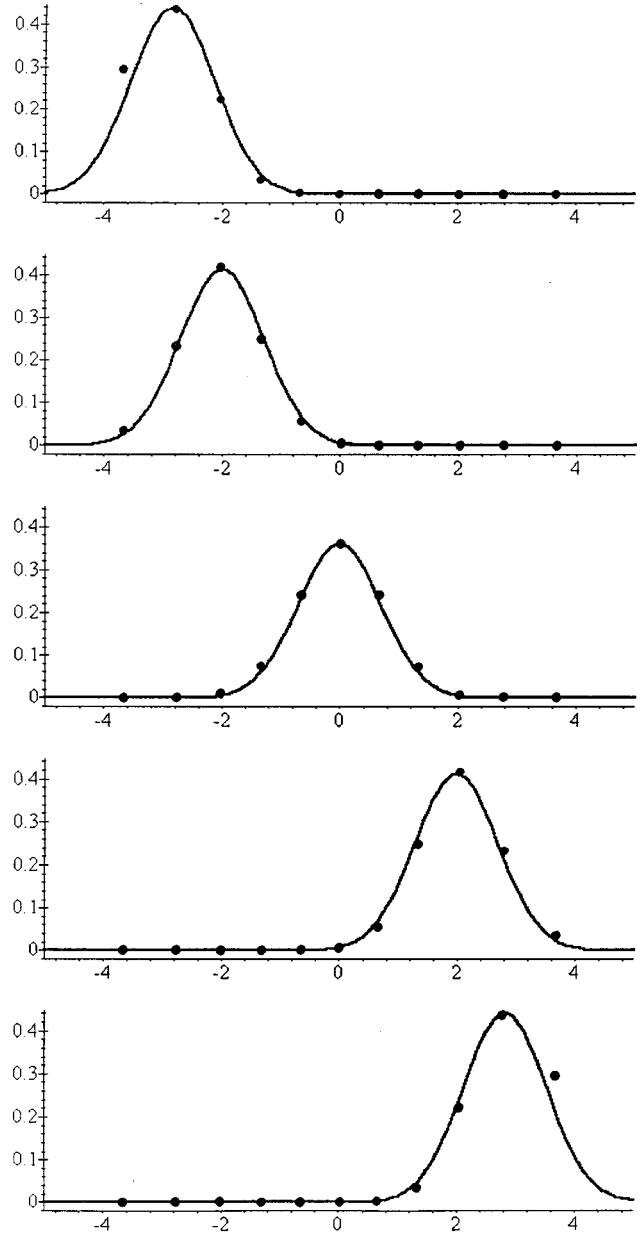


FIG. 1. The squared magnitude of the wave function (47) (circles) for a coherent state of the constrained oscillator, which is defined only at the discrete values x_j , compared with the corresponding Gaussian wave packet (solid curve) for the standard unconstrained oscillator. The normalization of the Gaussian packet has been adjusted so that the peaks of both wave functions have the same height. Time evolution is depicted over a half cycle of oscillation, with amplitude slightly smaller than the largest eigenvalue x_j . In this case, $\nu = 10$.

resolution with which values of $x(\tau)$ can be determined, and we wish to examine the nature of this limit.

If there is to be a nontrivial time evolution, several states must still be available. We therefore consider a (loosely) semiclassical situation, with ν very large, taking a band of states with $m \sim \nu$ to be actually available for use. Our calculations will now be approximate, and we begin by finding an approximation for the eigenstates (45). It is convenient at this point to use the Schrödinger picture, and deal with the eigenstates $|x_j, 0\rangle\rangle$ of $X = x(0)$. When ν is large, the $\nu+1$

zeros of $H_{\nu+1}$, corresponding to the eigenvalues of $x(\tau)$, lie roughly between $-\sqrt{2\nu}$ and $\sqrt{2\nu}$ and the spacing between them is therefore of order $\nu^{-1/2}$. We will take these eigenvalues to form a continuum. For large m , the asymptotic formula for H_m [11],

$$H_m(z) \approx e^{z^2/2} \frac{\Gamma(m+1)}{\Gamma(m/2+1)} \cos[(2m+1)^{1/2}z - m\pi/2] \quad (48)$$

yields

$$[2^{4n+r}(4n+r)!]^{-1/2} H_{4n+r}(z) \approx (2\pi)^{-1/4} e^{z^2/2} n^{-1/4} c_r(\sqrt{8nz}), \quad r=0, \dots, 3, \quad (49)$$

where $c_r(\theta) = \cos(\theta - r\pi/2)$. Taking ν to be of the form $4\nu' + 3$ and $m = 4\nu'\alpha + r$ in (45), replacing the sum on α by an integral, and defining

$$\hat{\xi} = \left(\frac{\omega}{2\nu}\right)^{1/2} X, \quad (50)$$

we obtain eigenstates of $\hat{\xi}$ in the form

$$|\xi\rangle = \sqrt{\frac{\nu}{2\pi}} \int_0^1 d\alpha \alpha^{-1/4} \sum_{r=0}^3 c_r(2\nu\sqrt{\alpha}\xi) |\alpha, r\rangle, \quad (51)$$

which are orthonormal in the limit $\nu \rightarrow \infty$.

Ladder operators $A^{(1)} \equiv A$ and $A^{(-1)} \equiv A^\dagger$ are realized in this approximation by

$$A^{(\mu)} |\xi\rangle = \sqrt{\mu} \left(\xi - \frac{\mu}{2\nu} \frac{\partial}{\partial \xi} \right) |\xi\rangle, \quad (52)$$

$$A^{(\mu)} |\alpha, r\rangle = \sqrt{\nu} \left(\alpha^{1/2} - \frac{\mu}{\nu} \alpha^{1/4} \frac{\partial}{\partial \alpha} \alpha^{1/4} \right) |\alpha, r - \mu\rangle. \quad (53)$$

Because the maximal state annihilated by A^\dagger now has zero measure, these operators have the canonical commutator $[A, A^\dagger] = 1$. For the same reason, the coherent state is now an eigenstate of A . Corresponding to the rescaled variable ξ (50), we define

$$A|z\rangle = \sqrt{\nu}z|z\rangle \quad (54)$$

and obtain the usual wave function $\psi(\xi, z) = \langle \langle \xi | z \rangle \rangle \sim \exp[-\nu(\xi - z)^2]$.

Supposing that the states actually available for use are those in a narrow range of α , we introduce a window function $f(\alpha)$ and define approximate eigenstates of $\hat{\xi}$ by

$$|\xi\rangle^f = \int_0^1 d\alpha \alpha^{-1/4} f(\alpha) \sum_{r=0}^3 c_r(2\nu\sqrt{\alpha}\xi) |\alpha, r\rangle. \quad (55)$$

In the same way, we can define an approximate coherent state

$$|z\rangle^f = \int_0^1 d\alpha \alpha^{-1/4} f(\alpha) \sum_{r=0}^3 \psi_r(\alpha, z) |\alpha, r\rangle, \quad (56)$$

where

$$\psi_r(\alpha, z) = \int_{-1}^1 d\xi \psi(\xi, z) c_r(\alpha). \quad (57)$$

As in (47), the time evolution of this state is obtained by making the replacement $z \rightarrow z(\tau)$.

To obtain approximate analytical results, we define $\gamma = \sqrt{\alpha}$ and choose a Gaussian window function $f(\gamma) = f_0 \exp[-(\gamma - \gamma_0)^2/2\Delta]$, where f_0 is the appropriate normalization factor. We choose γ_0 to be a little less than the maximal value $\gamma = 1$, and assume that Δ is small enough for the limits of integration to be extended to infinity. For the uncertainty $\Delta\xi$, defined by $(\Delta\xi)^2 = \langle \langle \xi | \hat{\xi}^2 | \xi \rangle \rangle^f - (\langle \langle \xi | \hat{\xi} | \xi \rangle \rangle^f)^2$, we find

$$\Delta\xi = \frac{1}{2\nu\sqrt{2\Delta}}. \quad (58)$$

We see that, for a fixed window width Δ , it is possible to make ξ arbitrarily sharp when ν is sufficiently large. On the other hand, when Δ is made small, in effect imposing the constraint of the clockless model, it becomes impossible to construct sharp eigenstates of ξ , reflecting the fact that, without the clock, there is no gauge-invariant operator corresponding to X or $\hat{\xi}$. Given the coherent state $|z\rangle^f$, we can estimate the probability amplitude for finding the oscillator at the position ξ at time τ . Assuming that the limits of integration in (57) can be extended to infinity, which will be valid if ν is large enough and if $|\bar{\xi}| < 1$, where $\bar{\xi} = \text{Re}z$, we obtain

$$f \langle \langle \xi | z(\tau) \rangle \rangle^f \sim \exp \left[-\frac{\nu^2 \Delta}{1 + \nu \Delta} [\xi - \bar{\xi}(\tau)]^2 \right], \quad (59)$$

where \sim indicates the omission of a normalization factor and of a phase. For a fixed window width Δ , this amplitude becomes sharply peaked at the classical trajectory in the semiclassical limit $\nu \rightarrow \infty$. However, when Δ becomes very small, then (i) there is no longer a sharp peak at the classical trajectory and (ii) the probability becomes time independent. Qualitatively, at least, we recover the situation in the absence of the clock, where there is no gauge-invariant operator corresponding to ξ and all gauge-invariant amplitudes are time independent (except possibly for trivial phase factors).

It is apparent from these results that, as expected, we cannot fix the energy of the clock without losing both the observable ξ and the time dependence which the clock was intended to reveal. However, it seems that one can find situations (for example, by taking ν large, with $\Delta\nu$ fixed) in which both the energy of the clock and the eigenstates of $\hat{\xi}$ are fairly sharply defined, and in which the notion of a test clock is therefore meaningful.

IV. DISCUSSION

The absence of time-dependent observables in a time-reparametrization-invariant system can perhaps be understood by recognizing that a complete description of a closed system is inevitably from the point of view of an observer external to the system. For such an observer, a state of the $(d+1)$ -dimensional system ($d=0$ for the models considered in this paper) corresponds to what an observer internal to the

system might be supposed to regard as an entire history of a d -dimensional system. The observable quantities whose values distinguish one state of the system from another therefore characterize entire histories of the system, and cannot evolve with time in any straightforward sense.

For a classical system which is *not* time-reparametrization invariant, this does not normally present a problem. If the system contains, say, n observable objects, there will be a set of time-independent phase space coordinates $(X_1, P_1, \dots, X_n, P_n)$ which are the basic observables, since a set of values for these $2n$ quantities specifies a history of the system. From these, however, we may construct parametrized families of observables, say $x_i(t) = f_i(t)X_i + g_i(t)P_i$ which satisfy the equations of motion and may be construed as representing the time-dependent observations made by an observer internal to the system. In the case of a time-reparametrization-invariant system, the families $x_i(t)$ have no gauge-invariant meaning, since they depend on t through the undetermined lapse function $\lambda(t)$.

In elementary applications of general relativity, the time-dependent appearance of the Universe to an internal observer is accounted for in roughly the following way. Consider the reparametrization-invariant model (1) with q omitted. The solution to its equation of motion is of the form $x(t) = a \cos[\psi(t) + \gamma]$. The amplitude a is constrained to have the value ρ_0 , so there is no gauge-invariant variable, and the $(0+1)$ -dimensional system has just one state available to it. Nevertheless, it seems intuitively clear that an observer inside the $(0+1)$ -dimensional universe would perceive an oscillation of the form $X(\tau) = \rho_0 \cos(\omega\tau)$, where τ is the proper time along this observer's world line. Moreover, this assertion seems to be gauge independent, since the lapse function serves only to relate τ to an arbitrary coordinate t , which is irrelevant to the observer.

To construct gauge-invariant observables corresponding to what an internal observer might perceive, it seems necessary to incorporate in our model at least a rudimentary description of the clock from which the internal observer gains his sense of time. This has long been recognized. In special relativity, for example, the dynamics of a free particle can be formulated with an arbitrary time parametrization [12,13,3] and a physically meaningful measure of time can be provided only by a physical clock [14]. Classically, the variable q , whose behavior is exhibited in (18) mimics an instrument whose reading is linear in the geometrical proper time τ . The quantity $x(\tau)$ in (19) defines a family of gauge-invariant observables which can be construed as giving the value of x which would be perceived by an internal observer when a proper time τ has elapsed since his clock read zero. Moreover, the evolution of $x(\tau)$ with τ is governed by an equation of motion (22) of the standard Hamiltonian form. Quantum mechanically, this equation of motion translates to a Heisenberg equation (44), again of the standard form. It, thus, seems consistent with the Copenhagen interpretation of quantum mechanics to suppose that an external observer might determine the state of the "Universe" by measuring the value of $x(\tau_1)$, say with the result x_1 . The probability $|\langle\langle x_2, \tau_2 | x_1, \tau_1 \rangle\rangle|^2$ that a measurement of $x(\tau_2)$ will yield the value x_2 can then plausibly be interpreted as the probability that an internal observer, having determined the position of the oscillator as x_1 will obtain the value x_2 from a

measurement made after an interval $\tau_2 - \tau_1$ of his proper time.

To this extent, we recover precisely the usual formulation of time-dependent quantum mechanics. However, the status of the clock variable in this formulation requires further thought. Indeed, the coordinates X and P on the two-dimensional physical phase space are just sufficient to represent the position and momentum of the oscillator, and there is no further coordinate available to represent the clock as an independent dynamical object. As emphasized, for example, by Unruh [15], who has also considered the two-oscillator model, the variables a_1 and b_1 , in (25) and (26), which one would ordinarily want to treat as Schrödinger picture operators representing the oscillator and clock positions, do not correspond to operators on the physical Hilbert space.

Correspondingly, the Hamiltonian (23) describes the time evolution of a single object — the oscillator. While there exists a family of observables $q(\tau) = \sqrt{\rho_0^2 - X^2 - (P/\omega)^2} \omega \tau$, which might appear to describe the reading of the clock, the evolution of $q(\tau)$ is not governed by a Hamiltonian (or Heisenberg) equation of motion, since $\{H_0, q(\tau)\} = 0$. Quantum mechanically, the values of $x(\tau)$ and $q(\tau)$ cannot be determined simultaneously, since $[x(\tau), q(\tau)] \neq 0$. Consequently, the clock does not exist as an independent dynamical object and a statement such as $x(\tau_1) = x_1$ cannot be taken as implying that both the oscillator and the clock have been inspected, yielding the value x_1 for the position of the oscillator and the value τ_1 for the time.

Thus, τ cannot be regarded as a phenomenological time deduced by an observer from his inspection of a physical clock which reads q . Rather, the definition of $x(\tau)$ reflects a decision on the part of an external observer (or of a theoretician) to study time evolution from a particular point of view. One might, for example, imagine an observing apparatus, which performs a sequence of tasks under the control of an internal clock q . To study time evolution "from the point of view" of this apparatus, it is natural to solve the Hamiltonian constraint by eliminating the unobserved quantity q , in order to arrive at the time-dependent observable $x(\tau)$. In this sense, τ represents a "Heraclitian" time of the kind sought by Unruh and Wald [3], which "sets the conditions" for a measurement to be made. Like the coordinate time t , τ itself is not a measurable quantity.

While it is generally recognized that time parameters such as t and τ are not directly measurable, the view reached here, that the readings of a clock which provides an intrinsic notion of time are also unobservable, does not seem to be widely held. For example, interpretations of the kind advanced by DeWitt [16] and elaborated by many subsequent authors (see, in particular, Ref. [17] for a discussion of the role of quantum decoherence) consider a wave function $\Psi(x, q)$, for which the Hamiltonian constraint is realized by the Wheeler-DeWitt equation $H\Psi = 0$. Roughly speaking, solutions suitable for describing a universe evolving with time are identified (for example, through a WKB approximation) as those which peak along a classical trajectory, establishing a strong correlation between x and q , *both of which are regarded as measurable quantities*. In a suitable approximation, the Wheeler-DeWitt equation can be cast in the form of a Schrödinger equation in which q plays the role of time.

An alternative route to describing evolution with respect to a phenomenological time is described by Page and Wothers [18].

It also appears quite unnecessary for the eliminated clock variable to be linear in τ . The physical Hilbert space associated with the model (24) and the algebra of the gauge-invariant operators X and P obtained by eliminating q are independent of the parameter N which determines the frequency of the q oscillator, and for finite N , q is of course a nonlinear function of τ . The role of the clock q in passing from either (17) and (18) or (29) and (30) to the gauge-invariant observable (19) is that it distinguishes a history in which, for example, $x = \rho_0 \cos \chi \cos \alpha$ when $q = 0$ from one in which some other value of q corresponds to this value of x . In essence, the definition of a gauge-invariant observable $x(\tau)$ becomes possible when an origin $\tau = 0$ can be specified in a coordinate-independent manner, and the role of q is to provide this origin. Note, however, that for finite N , our model describes a completely periodic universe, in which this origin is repeated indefinitely. Indeed, any state constructed from the basis (45) will be periodic in τ with period $2\pi/\omega$. Since the clock oscillates with a frequency of ω/N , only one instance of the origin (at which q increases through 0) occurs within this period. For this reason, we obtain an unambiguous account of time evolution within one period of the x oscillator. The occurrence of more than one instance of this origin (corresponding to $N < 1$) would pose serious problems in defining the operators (37) and (38) from which the time-dependent observables were constructed.

The view of time evolution described here differs in several respects from that proposed by Rovelli [5–7]. In [5], Rovelli considers essentially the model (24) with $N = 1$. He defines a gauge-invariant classical observable given, in our notation, by

$$x(s) = \rho_0 \cos \chi \cos \left[\alpha + \arccos \left(\frac{s}{\rho_0 \sin \chi} \right) \right], \quad (60)$$

which classically is the value of x when $q = s$. Like our $x(\tau)$, this $x(s)$ is gauge variant at $\chi = 0$, since it depends on α , though in fact only $x(s = 0)$ is well defined. The discus-

sion given in [6] suggests that Rovelli wishes to regard $x(s)$ as representing the evolution of x relative to the time recorded by a physical clock. For the reasons given above, however, a pair of values s and $x(s)$ cannot be construed as the results of inspecting a clock and an oscillator simultaneously.

The operator corresponding to (60) is approximately self-adjoint, and obeys an approximate Heisenberg equation of motion only when restricted to a region of the physical Hilbert space which Rovelli calls the ‘‘Schrödinger regime,’’ corresponding roughly to states in which the wave function $\psi(x, q)$ is sharply peaked around the classical trajectory. For this reason, Rovelli contends that the notion of time does not exist at a fundamental level, but emerges only in a suitable semiclassical limit. By contrast, our operator $x(\tau)$ is exactly self-adjoint, and obeys the exact Heisenberg equation (44), with respect to the variable τ which classically corresponds to the geometrical proper time, and this provides a counterexample to Rovelli’s contention.

We would like, of course, to speculate that the view of time evolution proposed here can be extended to canonically quantized general relativity. In such an extension, it seems likely that the role of τ would be played by the proper time along the trajectory of an observer for whose observations we wish to account, while $x(\tau)$ would correspond to local observables defined on this trajectory. To define, say, time-dependent observables throughout a spacelike hypersurface, one would presumably need to introduce a space-filling family of observers (or, at least, their clocks). These might correspond to a reference fluid of the kind described by Brown and Kuchar [19], though the relationship of our interpretation to that proposed by these authors is not entirely clear to us.

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