Class of exact solutions in 5D gravity and its physical properties

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(Received 27 March 1995)

We derive a class of solutions of the field equations of five-dimensional general relativity that is static and spherically symmetric in ordinary three-dimensional space. In the induced-matter picture, where the extra dimension is responsible for matter in four-dimensional spacetime, the solutions represent centrally condensed clouds with density profiles similar to those of clusters of galaxies. This class of solutions could be used with data on gravitational lensing to look for a fifth dimension.

PACS number(s): 04.20.Jb, 11.10.Kk

I. INTRODUCTION

The dimensionality of the world is a subject of continuing debate, whose theoretical side involves different approaches such as Kaluza-Klein theory, superstrings, and supergravity. Recently, there has been renewed interest in the basic extension from four dimensions to five dimensions, due to certain developments that are reviewed below. It has become apparent as the result of recent work that distinctions between four- (4D) and five-dimensional (5D) physics are subtle, and that the detection of an extra dimension will require exact solutions of the 5D field equations of new kinds. To this end, we present below a class of exact solutions in 5D gravity of astrophysical importance, and evaluate its physical properties.

The reawakening of interest in 5D gravity is largely the result of recasting the field equations in a form where matter in four dimensions can be seen to be the consequence of geometry in five dimensions. This induced-matter version of 5D general relativity follows by relaxing certain conditions that unnecessarily restricted the old Kaluza-Klein theory (specifically, the so-called cylinder condition is artificial and is dropped; and compactification is not presumed *a priori*, though it may, if desired, be applied later). The resulting theory is not, therefore, a Kaluza-Klein one, but neither is it Einstein's theory with an extra dimension because there is no explicit matter source: the fifth dimension provides a consistent explanation for mechanics and matter with field equations that are purely geometrical.

This space-time matter theory has a considerable literature that has sprung up in the last three years. Various workers have proved different results, and, by way of background for the class of solutions presented below, we note the following key items. (a) Wesson and Ponce de Leon [1] showed that the 15 5D field equations for apparent vacuum can always be written in a form that includes as a subset the 10 4D Einstein equations with matter. The 5D equations, the 4D ones, and the effective or induced 4D energy-momentum tensor are given below as Eqs. (1), (2), and (3), respectively. (b) Tavakol and co-workers [2,3] used this approach to go from 4D to lower-D gravity (which might be more easily quantized than Einstein's theory) and also proved an important theorem: any analytic N-dimensional Riemannian space can be locally embedded in an $(N+1)$-dimensional Ricci flat space. This result and the preceding one together ensure that we can geometrize matter in terms of five dimensions if we so wish. (c) Kalligas, Wesson, and Everitt [4] reworked all of the classical tests of relativity using a standard class of 5D one-body solutions with curvature in the extra dimension [5–8], finding no conflict with observation [9]. The best way to detect a fifth dimension would appear to be via its indirect effects on a spinning particle or gyroscope, since these exist even when the extra part of the metric has negligible curvature [10]. (d) Mashhoon and co-workers [11,12] introduced the concept of a canonical metric and showed that when the metric is so expressed, the spacetime components of the geodesic equation in five dimensions are identical to what they are in four dimensions. Since the five coordinate degrees of freedom associated with 5D relativity can always be used to put a metric into the canonical form, this result means that there is always a coordinate system where the motion in the 4D part of the 5D theory is the same as it is in general relativity. In other words, not only 4D solutions but also their dynamics are embeddable in five dimensions. (e) Kaluza [13] and Klein [14,15] in their original work, and many workers afterwards, showed that the 4D Maxwell equations are a subset of the 5D relativity equations if the metric is expressed in a certain form, where the off-diagonal components involving the extra dimension are identified with the scalar and vec-

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tor potentials of electromagnetism. This property is also present in the modern theory (though the metric is then not written in the canonical form mentioned in the preceding comment), and the equation of motion then includes a Lorentz-type term [16]. (f) Many people have found cosmological solutions in five dimensions, but in the induced-matter picture the right-hand side of the field equations is zero, so models of the Universe are more unique. The density, pressure, and equation of state of matter are determined by the solution, and one class of such provides excellent models for the early (radiation) and late (dust) Universe [17–19]. These models have the remarkable property that, while they are curved in four dimensions, they are flat in five dimensions. They also suggest that the extra dimension, while not directly observable in terms of 4D geometry, is connected with rest mass and the existence of matter [19,20]. Further, they confirm the inference from other work [11,12], that the 4D work is recovered from the 5D one on a hypersurface, where the extra coordinate equals a constant.

The above results are significant and form a *prima facie* case for believing that 4D matter can be interpreted as 5D geometry. However, as noted, the 4D and 5D formulations often have equivalent aspects (notably as regards the equation of motion), and the only way known to date to differentiate the two involves a test such as the Stanford gyroscope experiment, which has yet to be carried out [5]. It is therefore important to find other ways of investigating a possible fifth dimension, and this means finding new kinds of 5D solutions. Below we will present the 5D field equations in a tractable form (Sec. II), derive a class of solutions with good physical properties (Sec. III), and solve the geodesic equation (Sec. IV). Before proceeding, however, a couple of notes are in order. First, the class of 5D solutions we will be concerned with is the extension of a class of 4D solutions known to be relevant to astrophysics [21–23], but the application of the solutions to (say) clusters of galaxies would carry us far outside the scope of the present paper, so we defer it to the future. Second, the class of 5D solutions we will derive, although motivated by the induced-matter picture, is of importance for any version of 5D gravity. So the results of the main part of this paper, namely, those in Secs. II, III, and IV, are useful in other contexts.

II. FIELD EQUATIONS

The field equations in five dimensions for an apparent vacuum in terms of the Ricci tensor are

$$R_{AB}=0 \quad (A,B=0\text{–}4)\ . \tag{1}$$

The field equations in four dimensions with matter in terms of the Einstein tensor and the energy-momentum tensor are

$$G_{\alpha\beta}=T_{\alpha\beta} \quad (\alpha,\beta=0\text{–}3)\ . \tag{2}$$

These 10 equations are a subset of the preceding 15 [the other five comprise one wave equation and four conservation equations and are, of course, satisfied for any solutions of (1) like those given below: see Ref. [1]]. The effective or induced energy-momentum tensor is

$$T_{\alpha\beta}=\frac{\Phi_{\alpha;\beta}}{\Phi}-\frac{\epsilon}{2\Phi^2}\left\{\frac{\overset{*}{\Phi}\overset{*}{g}_{\alpha\beta}}{\Phi}-\overset{**}{g}_{\alpha\beta}+g^{\lambda\mu}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu}\right.$$
$$\left.-\frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}}{2}+\frac{g_{\alpha\beta}}{4}[\overset{*}{g}^{\mu\nu}\overset{*}{g}_{\mu\nu}+(g^{\mu\nu}\overset{*}{g}_{\mu\nu})^2]\right\}\ . \tag{3}$$

Here we have written the 5D metric tensor as a 4D block $g_{\alpha\beta}$ plus a diagonal extra part $g_{44}=\epsilon\Phi^2$, where $\epsilon=\pm1$. Also, $\Phi_\alpha\equiv\partial\Phi/\partial x^\alpha$, a semicolon denotes the usual 4D covariant derivative, and an overasterisk denotes $\partial/\partial x^4$. [The facts that $\epsilon=\pm1$ and $\partial/\partial x^4\neq0$ in general, distinguish the induced-matter approach from the traditional Kaluza-Klein approach. However, we follow the route of the latter theory and interpret $g_{4\alpha}=0$ as meaning that the electromagnetic potentials are zero, which is an appropriate condition here as we study a neutral fluid.] Solutions of (1)–(3) are known that are of the one-body type [5–8] and of the cosmological type [17–19]. The latter admit a range of equations of state for pressure p and density ρ of the induced matter described by (3), and it is known that equations of state other than the $p=\rho/3$ of radiation require dependency of g_{AB} on x^4 (see Refs. [1,18]: We use units throughout such that $c=1$, $8\pi G=1$). We wish to find solutions that are of a new type but have some relevance to astrophysics, so we proceed to write out the components of R_{AB} for a metric that is spherically symmetric in ordinary 3D space but is otherwise unrestricted.

The line element can be written

$$ds^2=e^\nu dt^2-e^\lambda dr^2$$
$$-R^2(d\theta^2+\sin^2\theta d\phi^2)+\epsilon e^\mu d\psi^2\ . \tag{4}$$

Here the metric coefficients ν, λ, R, and μ can depend on the time (t), radius (r), and the extra coordinate (ψ) but not, of course, on the spherical angles (θ,ϕ). The nonzero components of R_{AB} for metric (4) are

$$R_{00}=-\frac{\ddot\lambda}{2}-\frac{\ddot\mu}{2}-2\frac{\ddot R}{R}+\frac{\dot\nu\dot\lambda}{4}+\frac{\dot\nu\dot\mu}{4}+\frac{\dot\nu\dot R}{R}-\frac{\dot\lambda^2}{4}-\frac{\dot\mu^2}{4}$$
$$+e^{\nu-\lambda}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'\lambda'}{4}+\frac{\nu'\mu'}{4}+\frac{\nu'R'}{R}\right)$$
$$+\epsilon e^{\nu-\mu}\left(-\frac{\overset{**}{\nu}}{2}-\frac{\overset{*}{\nu}^2}{4}+\frac{\overset{*}{\nu}\overset{*}{\mu}}{4}-\frac{\overset{*}{\nu}\overset{*}{\lambda}}{4}-\frac{\overset{*}{\nu}\overset{*}{R}}{R}\right)\ ,$$
$$R_{01}=-\frac{\dot\mu'}{2}-\frac{\dot\mu\mu'}{4}+\frac{\nu'\dot\mu}{4}+\frac{\dot\lambda\mu'}{4}+\frac{\dot\lambda R'}{R}+\frac{\nu'\dot R}{R}-\frac{2\dot R'}{R}\ ,$$
$$R_{04}=-\frac{\dot\lambda^*}{2}-\frac{\dot\lambda\overset{*}{\lambda}}{4}+\frac{\dot\lambda\overset{*}{\nu}}{4}+\frac{\overset{*}{\lambda}\dot\mu}{4}+\frac{\dot\mu\overset{*}{R}}{R}+\frac{\overset{*}{\nu}\dot R}{R}-\frac{2\dot R^*}{R}\ ,$$
$$R_{11}=-\frac{\nu''}{2}-\frac{\mu''}{2}-\frac{\nu'^2}{4}-\frac{\mu'^2}{4}+\frac{\lambda'\nu'}{4}+\frac{\lambda'\mu'}{4}+\frac{\lambda'R'}{4}$$
$$-\frac{2R''}{R}+e^{\lambda-\nu}\left(\frac{\ddot\lambda}{2}+\frac{\dot\lambda^2}{4}-\frac{\dot\lambda\dot\nu}{4}+\frac{\dot\lambda\dot\mu}{4}+\frac{\dot\lambda\dot R}{R}\right)$$
$$+\epsilon e^{\lambda-\mu}\left(\frac{\overset{**}{\lambda}}{2}+\frac{\overset{*}{\lambda}^2}{4}-\frac{\overset{*}{\lambda}\overset{*}{\nu}}{4}-\frac{\overset{*}{\lambda}\overset{*}{\mu}}{4}+\frac{\overset{*}{\lambda}\overset{*}{R}}{R}\right)\ ,$$
$$\tag{5}$$

$$R_{14} = -\frac{\overset{*}{\nu}'}{2} - \frac{\nu'\nu^{*}}{4} + \frac{\overset{*}{\lambda}\nu'}{4} + \frac{\mu'\overset{*}{\nu}}{4} + \frac{\overset{*}{\lambda}R'}{R} + \frac{\mu'\overset{*}{R}}{R} - \frac{2\overset{*}{R}'}{R} ,$$

$$R_{22} = 1 + R^2 e^{-\nu}\left[\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} - \frac{\dot{R}}{2R}(\dot{\nu} - \dot{\lambda} - \dot{\mu})\right]$$

$$-R^2 e^{-\lambda}\left[\frac{R'^2}{R^2} + \frac{R''}{R} + \frac{R'}{2R}(\nu' - \lambda' + \mu')\right]$$

$$+\epsilon R^2 e^{-\mu}\left[\frac{\overset{*}{R}^2}{R^2} + \frac{\overset{**}{R}}{R} + \frac{\overset{*}{R}}{2R}(\overset{*}{\nu} + \overset{*}{\lambda} - \overset{*}{\mu})\right] ,$$

$$R_{33} = (\sin^2\theta)R_{22} ,$$

$$R_{44} = -\frac{\overset{**}{\nu}}{2} - \frac{\overset{*}{\nu}^2}{4} - \frac{\overset{**}{\lambda}}{2} - \frac{\overset{*}{\lambda}^2}{4} + \frac{\overset{*}{\mu}\overset{*}{\nu}}{4} + \frac{\overset{*}{\mu}\overset{*}{\lambda}}{4} + \frac{\overset{*}{\mu}\overset{*}{R}}{R} - \frac{2\overset{**}{R}}{R}$$
$$-\epsilon e^{\mu-\nu}\left(\frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} - \frac{\dot{\mu}\dot{\nu}}{4} + \frac{\dot{\mu}\dot{\lambda}}{4} + \frac{\dot{\mu}\dot{R}}{R}\right)$$
$$+\epsilon e^{\mu-\lambda}\left(\frac{\mu''}{2} + \frac{\mu'^2}{4} + \frac{\mu'\nu'}{4} - \frac{\mu'\lambda'}{4} + \frac{\mu'R'}{R}\right) .$$

Here an overdot denotes $\partial/\partial t$ and a prime denotes $\partial/\partial r$. These relations are general in the sense that solutions of them such as presented below are relevant to any interpretation of 5D gravity.

In the induced-matter interpretation, we can obtain the components of the effective 4D energy-momentum tensor either by sorting the terms in (5) into parts that correspond to the left- and right-hand sides of (2), or by computing them from (3). Either way, they are

$$T_0^0 = -e^{-\nu}\left(\frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\dot{R}\dot{\mu}}{R}\right) + e^{-\lambda}\left(\frac{R'\mu'}{R} - \frac{\lambda'\mu'}{4} + \frac{\mu''}{2} + \frac{\mu'^2}{4}\right) - \epsilon e^{-\mu}\left(\frac{\overset{**}{\lambda}}{2} + \frac{\overset{*}{\lambda}^2}{4} - \frac{\overset{*}{\mu}\overset{*}{\lambda}}{4} + \frac{\overset{*}{R}\overset{*}{\lambda}}{R} - \frac{\overset{*}{R}\overset{*}{\mu}}{R} + \frac{\overset{*}{R}^2}{R^2} + \frac{2\overset{**}{R}}{R}\right) ,$$
$$T_0^1 = -e^{-\lambda}\left(\frac{\dot{\mu}'}{2} + \frac{\dot{\mu}\mu'}{4} - \frac{\nu'\dot{\mu}}{4} - \frac{\dot{\lambda}\mu'}{4}\right) ,$$
$$T_1^1 = -e^{-\nu}\left(\frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} - \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{R}\dot{\mu}}{R}\right) + e^{-\lambda}\left(\frac{R'\mu'}{R} + \frac{\nu'\mu'}{4}\right) - \epsilon e^{-\mu}\left(\frac{\overset{*}{R}^2}{R^2} + \frac{2\overset{**}{R}}{R} + \frac{\overset{*}{R}\overset{*}{\nu}}{R} - \frac{\overset{*}{R}\overset{*}{\mu}}{R} + \frac{\overset{*}{\nu}^{*}}{2} + \frac{\overset{*}{\nu}^2}{4} - \frac{\overset{*}{\nu}\overset{*}{\mu}}{4}\right) ,$$
$$T_2^2 = -e^{-\nu}\left(\frac{\dot{R}\dot{\mu}}{2R} - \frac{\dot{\nu}\dot{\mu}}{4} + \frac{\dot{\lambda}\dot{\mu}}{4} + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4}\right) + e^{-\lambda}\left(\frac{R'\mu'}{2R} + \frac{\mu''}{2} + \frac{\mu'^2}{4} - \frac{\lambda'\mu'}{4} + \frac{\nu'\mu'}{4}\right)$$
$$-\epsilon e^{-\mu}\left(\frac{\overset{**}{R}}{R} + \frac{\overset{*}{R}\overset{*}{\nu}}{2R} + \frac{\overset{*}{R}\overset{*}{\lambda}}{2R} - \frac{\overset{*}{R}\overset{*}{\mu}}{2R} + \frac{\overset{*}{\nu}^{*}}{2} + \frac{\overset{*}{\nu}^2}{4} + \frac{\overset{**}{\lambda}}{2} + \frac{\overset{*}{\lambda}^2}{4} + \frac{\overset{*}{\nu}\overset{*}{\lambda}}{4} - \frac{\overset{*}{\nu}\overset{*}{\mu}}{4} - \frac{\overset{*}{\mu}\overset{*}{\lambda}}{4}\right) ,$$
$$T_3^3 = T_2^2 . \tag{6}$$

We see that, in general, the fluid is anisotropic ($T_1^1 \neq T_2^2$) and that in the time-dependent case there is a radial flow of energy ($T_0^1 \neq 0$). In the future, we believe it would be useful to look for time-dependent solutions of (5) that would be the 5D analogues of known 4D solutions ([21]; presently there is only one time-dependent soliton-type solution known, given in Ref. [24]). However, it is apparent from (5) that considerable simplification occurs if we restrict ourselves to the time-independent case, which we now do.

III. A CLASS OF SOLUTIONS AND ITS PROPERTIES

As in four dimensions, there are an infinite number of solutions of the general 5D field equations (1), and numerous solutions where the 3D part of the metric (4) is restricted to spherical symmetry. In the latter case, in four dimensions the solutions are relatively easy to classify and possess certain generic properties [21], but in five dimensions they are not because of the algebraic complexity introduced by the addition of a dimension. (For example, the one-body or soliton solutions [5–8] are a class in five dimensions because Birkhoff's theorem breaks down, and even the class is not known to be unique.) Thus we present here a one-parameter class of solutions that is static but whose other properties we need to investigate without 4D preconceptions. Such a class of solutions may be confirmed by substitution or computer to be given by

$$ds^2 = \left(\frac{r}{r_0}\right)^{2(\alpha+1)}\psi^{2(\alpha+3)/\alpha}dt^2 - (3-\alpha^2)\psi^2 dr^2$$
$$-\psi^2 r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$+3(3\alpha^{-2} - 1)r^2 d\psi^2 . \tag{7}$$

Here r_0 is a constant and α is a parameter related to the properties of matter (see below). In this and other regards, (7) is similar to the 5D cosmological models [17–19]. However, (7) represents a cloud of matter that is centrally condensed, since on substitution into (6) we find

$$
\begin{aligned}
T_0^0 &= \frac{(2-\alpha^2)}{(3-\alpha^2)\psi^2 r^2} , \\
T_0^1 &= 0 , \\
T_1^1 &= \frac{-(\alpha^2+2\alpha)}{(3-\alpha^2)\psi^2 r^2} , \\
T_2^2 = T_3^3 &= \frac{-(\alpha^2+2\alpha+1)}{(3-\alpha^2)\psi^2 r^2} .
\end{aligned} \tag{8}
$$

Using the method of [8], we associate the density ρ with T_0^0 and the pressure p with $-(T_1^1+T_2^2+T_3^3)/3$ to obtain

$$\rho = \frac{(2-\alpha^2)}{(3-\alpha^2)\psi^2 r^2} , \tag{9}$$

$$p = \frac{(\alpha^2+2\alpha+2/3)}{(3-\alpha^2)\psi^2 r^2} . \tag{10}$$

We see that the equation of state is isothermal.

The fact that ρ and p are both proportional to r^{-2} implies that this class is the 5D analogue of the 4D one studied by Henriksen and Wesson [22,23], which describes inhomogeneous spheres of matter in static isothermal equilibrium. In fact, along hypersurfaces $x^4 = \psi = \text{const}$, (7) is very similar to their solutions:

$$
\begin{aligned}
ds^2 = \left(\frac{r}{r_0}\right)^{4P_0/(P_0+\eta_0)} dt^2 - \frac{dr^2}{(1-\eta_0)} \\
-r^2(d\theta^2 + \sin^2\theta\, d\phi^2) .
\end{aligned} \tag{11}
$$

Here $p = P_0/r^2$, $\rho = \eta_0/r^2$, and stability implies $4P_0(1-\eta_0) = (P_0+\eta_0)^2$. However, although the two classes of solutions are similar, (7) has some notable differences from (11). First of all, in the 4D situation, as $\eta_0 \to 0$ so does P_0, so that there is no situation where a negative pressure exists for positive density. In the 5D case, the density is positive for $\alpha^2 \leq 2$ (which physically restricts the parameter α). However, it may be verified that for $-2 \leq \alpha < (-1+1/\sqrt{3})$ the pressure is negative. Furthermore, at $\alpha = -1+1/\sqrt{3}$ ($\simeq -0.42$) the pressure is zero (dust) although the density is not. This is unlike the 4D case where the dust solution exists only for $\rho = 0$, and (11) is just a Minkowski spacetime. Another major difference between the 4D solutions and the 5D ones is that the former admit the radiation equation of state via the choice $P_0 = \eta_0/3$ (which with the stability condition implies $\eta_0 = 3/7$), whereas the latter do not. The reason for this bears comment. For $\alpha = 0$, there is a divergence in the first and last terms of the 5D metric (7). However, the characteristic 5D invariant (Kretschmann scalar) is

$$R^{ABCD}R_{ABCD} = \frac{8(2\alpha^4 - 9\alpha^2 + 18)}{3(3-\alpha^2)^2\psi^4 r^4} , \tag{12}$$

which is well behaved at $\alpha = 0$ (though not at $\alpha = \pm\sqrt{3}$, $\psi = 0$, and $r = 0$). We have investigated this and find that the situation is analagous to what happens with the cosmological solutions [17–19], namely, that a radiation equation of state can be approached ($\alpha \to 0$), but that the precise case ($\alpha = 0$) takes us outside of the class of solutions. This can be confirmed by calculating the mass of the fluid described by (8). For this, we can use the standard 4D definition of the gravitational mass of a 3D volume of fluid [25,26] to obtain

$$
\begin{aligned}
M_g &= \int (T_0^0 - T_1^1 - T_2^2 - T_3^3)\sqrt{-g_4}\, dV_3 \\
&= \frac{8\pi(\alpha+1)}{\sqrt{3-\alpha^2}} \psi^{2+3/\alpha} \left(\frac{r}{r_0}\right)^{2+\alpha} r_0 .
\end{aligned} \tag{13}
$$

Here g_4 is the determinant of the 4D part of the metric. It is apparent that $\alpha = 0$ is not allowed. Also, $M_g < 0$ for $\alpha < -1$, and so α is, furthermore, restricted to the range $[-1, \sqrt{2}]$. With such a restriction, we see that $p \geq -\rho/3$. In general, a variety of equations of state can be obtained for $\alpha^2 \leq 2$, ranging from that for stiff matter ($p = \rho, \alpha = \sqrt{11/12} - 1/2 \simeq 0.46$) through various softer forms. The latter are relevant to various situations in astrophysics, but the density (9) and pressure (11) have forms similar to those seen in clusters in galaxies [27–30] or protogalaxies, which may therefore be a realization of this class of solutions.

IV. GEODESIC MOTION

The five components of the velocity of a test particle moving in the 5D manifold (7) may be obtained by solving the geodesic equation

$$\frac{d^2x^C}{ds^2} + \Gamma^C_{AB}\frac{dx^A}{ds}\frac{dx^B}{ds} = 0 .$$

When we substitute into this for the nonzero Christoffel symbols listed in the Appendix, we obtain

$$\ddot{t} = -\frac{2(\alpha+1)}{r}\dot{t}\dot{r} - \frac{2(1+3/\alpha)}{\psi}\dot{t}\dot{\psi} , \tag{14a}$$

$$
\begin{aligned}
\ddot{r} = &-\frac{(\alpha+1)}{(3-\alpha^2)r\psi^2}\left(\frac{r}{r_0}\right)^{2(\alpha+1)} \psi^{2(1+3/\alpha)}\dot{t}^2 \\
&+\frac{r}{(3-\alpha^2)}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \\
&-\frac{2\dot{r}\dot{\psi}}{\psi} - \left(\frac{3}{\alpha^2}\right)\frac{r\dot{\psi}^2}{\psi^2} ,
\end{aligned} \tag{14b}
$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} - \frac{2\dot{\psi}\dot{\theta}}{\psi} + \sin\theta\cos\theta\,\dot{\phi}^2 , \tag{14c}$$

$$\ddot{\phi} = -\frac{2\dot{r}\dot{\phi}}{r} - \frac{2\dot{\psi}\dot{\phi}}{\psi} - \frac{2\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} , \tag{14d}$$

$$
\begin{aligned}
\ddot{\psi} = {} & \left(\frac{\alpha^2}{3}\right)\frac{(1+3/\alpha)}{(3-\alpha^2)r^2\psi}\left(\frac{r}{r_0}\right)^{2(1+\alpha)}\psi^{2(1+3/\alpha)}\dot{t}^2 \\
& -\left(\frac{\alpha^2}{3}\right)\frac{\psi}{(3-\alpha^2)}(\dot{\theta}^2+\sin^2\dot{\phi}^2) \\
& -\frac{2\dot{r}\dot{\psi}}{r}-\frac{\alpha^2}{3}\frac{\psi\dot{r}^2}{r^2}\,.
\end{aligned} \tag{14e}
$$

Here an overdot denotes d/ds, with s as the proper time for particles and as some other affine parameter for photons.

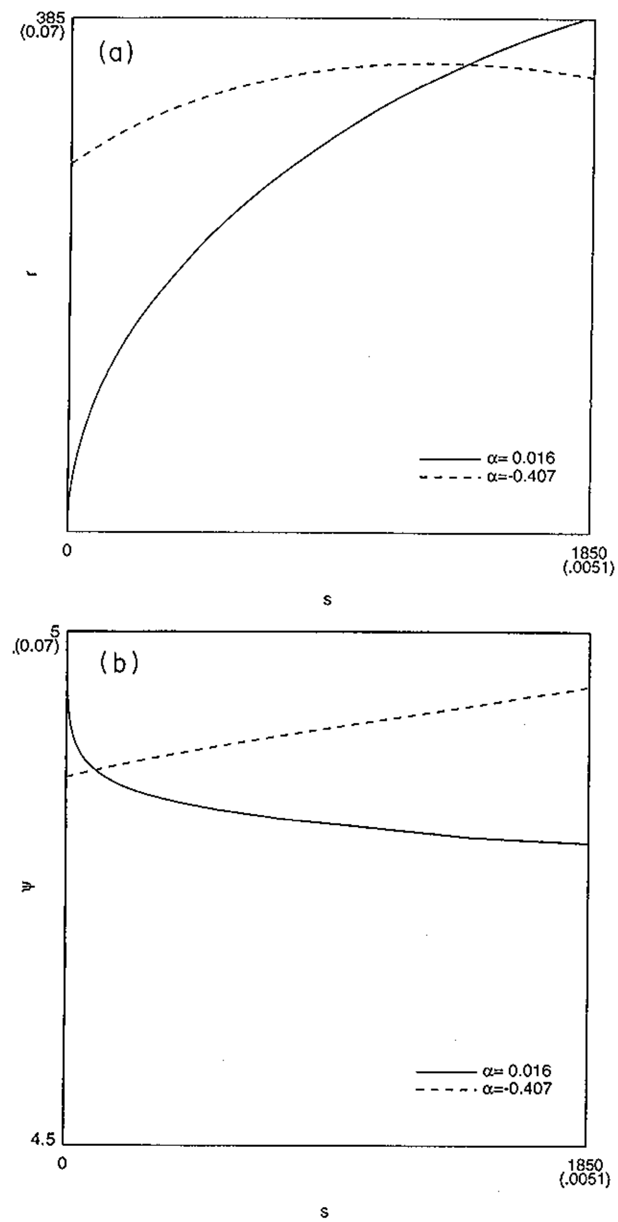

FIG. 1. Simultaneous numerical solutions of (18) and (19) for two values of α. For $\alpha = 0.016$, by (9) and (10), $p = 0.35\rho$. For $\alpha = -0.407$, $p = 0.01\rho$ (the numbers on the axes in parentheses are for this case). Other constants in (18)–(20) are set to $E = 1$, $L = 0$, $\Xi = 1$, $r_0 = 1$ to describe the radial motion of a massive particle in the metric (7). For $\alpha = 0.016$, the initial values of the integration are $r_i = 5$, $\psi_i = 5$, $\dot{r}_i = 10$, $\dot{\psi}_i = 0.09$. For $\alpha = -0.407$, they are $r_i = 0.05$, $\psi_i = 0.05$, $\dot{r}_i = 10$, $\dot{\psi}_i = 3.6$.

It may easily be verified that

$$\dot{t} = E\left(\frac{r}{r_0}\right)^{-2(1+\alpha)}\psi^{-2(1+3/\alpha)}\,, \tag{15}$$

$$\dot{\theta} = \frac{L}{r^2\psi^2}\sqrt{W - csc^2\theta}\,, \tag{16}$$

$$\dot{\phi} = \frac{L}{r^2\psi^2\sin^2\theta}\,, \tag{17}$$

satisfy (14a), (14c), and (14d) above. The constant L is the angular momentum (per unit rest mass for massive particles), and E is an energy (per unit rest mass for massive particles). As in the Schwarzschild solution, we see that if the particle is in the $\theta = \pi/2$ plane with $\dot{\theta} = 0$, then it will remain there. Hence, we can set the constant in (16) to $W = 1$ for this case. The other two equations, namely, (14b) and (14e) are harder to deal with, but it may be verified that they are satisfied by

$$\dot{r} = \pm\frac{(\sqrt{A}+\alpha\sqrt{B}/\sqrt{3})}{(3-\alpha^2)\psi^2 r}\,, \tag{18}$$

$$\dot{\psi} = \mp\left(\frac{\alpha}{\sqrt{3}}\right)\frac{(\alpha\sqrt{A}/\sqrt{3}+\alpha\sqrt{B})}{(3-\alpha^2)\psi r^2}\,, \tag{19}$$

where

$$
\begin{aligned}
A &\equiv k_1^2 + \Xi k_2 - 3\psi^2 r^2 \Xi\,, \\
B &\equiv k_1^2 + \Xi k_2 - \frac{3E^2 r_0^2}{\psi^{6/\alpha}(r/r_0)^{2\alpha}} + 3WL^2\,,
\end{aligned} \tag{20}
$$

and k_1, k_2 are arbitrary constants. The other constant Ξ is 0 for photons and 1 for massive particles. Representative solutions of (18) and (19) are illustrated in Fig. 1. Equations (18) and (19) could be analyzed in more detail, notably in regard to gravitational lensing. Previously, gravitational lensing has been studied in five dimensions mainly in relation to solitons [4], but a preliminary investigation applied to clusters of galaxies [31] indicates that the soliton metric is not a good approximation in this case. Conversely, the solutions (7) have density and pressure profiles closer to those of real clusters, so the geodesics (18) and (19) could be studied to see if there is evidence from gravitational lensing for a fifth dimension.

V. CONCLUSION

We have derived a class of solutions of the apparently empty field equations of 5D gravity (7) that in the induced-matter picture where the extra dimension manifests itself as matter in spacetime has physically reasonable density and pressure (9),(10). These solutions appear to be the analogues of the known astrophysically relevant 4D ones (22),(23). By solving the geodesic equation, we have obtained the five components of the velocity of a particle moving in the 5D space, (15)–(19). The class of 5D solutions derived here has numerous astrophysical implications, since it extends the class of 4D solutions. The latter have a density profile and equation of state relevant to observed (rich or Abell) clusters of galaxies and might also be applicable to the condensations from which galaxies formed. The 5D solutions have the same density profile and equation of state, but the freedom to choose the constant that defines the class means that the latter can describe a wider range of physical situations. In combination with the geodesics, we have noted that the 5D solutions can in the future be used with gravitational lens data to see if there is astrophysical evidence for an extra dimension.

ACKNOWLEDGMENTS

For discussions we thank C. Alvarez and J. Ponce de Leon, and for financial support NSERC and NASA.

APPENDIX: NONZERO CHRISTOFFEL SYMBOLS FOR METRIC (7)

We have

$$\Gamma^t_{rt} = \frac{(1+\alpha)}{r}, \quad \Gamma^t_{\psi t} = \frac{(1+3/\alpha)}{\psi} ,$$

$$\Gamma^r_{tt} = \frac{(1+\alpha)(r/r_0)^{2(\alpha+1)}\psi^{2(1+3/\alpha)}}{(3-\alpha^2)\psi^2 r} ,$$

$$\Gamma^r_{\theta\theta} = -\frac{r}{(3-\alpha^2)}, \quad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{(3-\alpha^2)} ,$$

$$\Gamma^r_{r\psi} = \frac{1}{\psi}, \quad \Gamma^r_{\psi\psi} = \left(\frac{3}{\alpha^2}\right)\frac{r}{\psi^2} ,$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r}, \quad \Gamma^\theta_{\psi\theta} = \frac{1}{\psi}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta ,$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{\psi\phi} = \frac{1}{\psi}, \quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} ,$$

$$\Gamma^\psi_{tt} = -\left(\frac{\alpha^2}{3}\right)\frac{(1+3/\alpha)(r/r_0)^{2(\alpha+1)}\psi^{2(1+3/\alpha)}}{(3-\alpha^2)\psi r^2} ,$$

$$\Gamma^\psi_{\theta\theta} = \left(\frac{\alpha^2}{3}\right)\frac{\psi}{(3-\alpha^2)}, \quad \Gamma^\psi_{\phi\phi} = \left(\frac{\alpha^2}{3}\right)\frac{\psi\sin^2\theta}{(3-\alpha^2)} ,$$

$$\Gamma^\psi_{r\psi} = \frac{1}{r}, \quad \Gamma^\psi_{rr} = \left(\frac{\alpha^2}{3}\right)\frac{\psi}{r^2} .$$

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[20] That the extra coordinate x^4 in induced-matter theory is related to rest mass is indicated by several arguments. (a) All of mechanics depends on base units of length, time, and mass. So if the former two can be treated as coordinates then maybe the last should also. Dimensionally, $x^4 = \mathrm{G}m/\mathrm{c}^2$ allows us to treat the rest mass m of a particle as a length coordinate, in analogy with $x^0 = ct$. (b) Metrics, such as those for solitons [5–8], that do not depend on x^4 yield only the equation of state for photons, while metrics, such as those for cosmologies [17–19], that do depend on x^4 are necessary to obtain equations of state for fluids composed of massive particles. (c) The metrics $ds^2 = dT^2 - d\sigma^2 - d\Psi^2$ and $ds^2 = \psi^2 dt^2 - d\sigma^2 - t^2 d\psi^2$ are related by the coordinate transformation $T = t^2\psi^2/4 + \ln[t^{1/2}\psi^{-1/2}]$, $\Psi = t^2\psi^2/4 - \ln[t^{1/2}\psi^{-1/2}]$. The former metric is flat, while the latter for a particle at rest in 3-space ($d\sigma/ds = 0$) and viewed on a hypersurface ($\psi =$ const) gives an action principle $\delta\int\psi\, dt = 0$ that is formally the same as that of particle physics if $\psi = m$ in the local, low-velocity limit. The same is true of cosmological metrics [17–19]. This agrees with the view from quantum field theory that the rest masses of elementary particles are generated spontaneously in a conformally invariant theory that includes a scalar dilaton field or Nambu-Goldstone boson in Minkowski space. A corollary of the belief that x^4 is not merely a length (or time) is that the extra part of the metric can have either sign without running afoul of closed timelike paths and causality problems. These top-

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